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MULTI-VARIABLE GOULD-HOPPER AND LAGUERRE POLYNOMIALS

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The *monomiality principle* was introduced by G. Dattoli, in order to derive the properties of special or generalized polynomials starting from the corresponding ones of monomials. In this article we show a general technique to extend the monomiality approach to multi-index polynomials in several variables. Application to the case of Hermite, Laguerre-type and mixed-type (i.e. between Laguerre and Hermite) are derived.

1. Introduction

The idea of monomiality traces back to the early forties of the last century, when J.F. Steffensen, in a largely unnoticed paper [25], suggested the concept of *poweroid*. A new interest in this subject was by the work of G. Dattoli and his collaborators [15], [16]

It turns out that all polynomial families, and in particular all special polynomials, are essentially the same, since it is possible to obtain each of them transforming a basic monomial set by means of suitable operators (called the *derivative* and *multiplication* operator of the considered family). This was

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shown by a theoretical proof in refs. [3], [4], and can be viewed as the basis of the umbral calculus [24] - a term invented by Sylvester - since the exponent, for example in x^n, is transformed into his "shadow" in $p_n(x)$.

Unfortunately the derivative and multiplication operators for a general set of polynomials are given by formal series of the ordinary derivative operator. It is therefore impossible to obtain sufficiently simple formulas to work with. However, for particular polynomials sets, related to suitable classes of generating functions, the above mentioned formal series reduce to finite sums, so that the relevant properties can be easily derived. The leading set in this field is given by the Hermite-Kampé de Fériet (shortly H-KdF) also called Gould-Hopper polynomials [1], [2].

Many sets of multi-variable or multi-index polynomials have been constructed starting from this important polynomial set [18], [10], [5], [6].

The main results are obtained by using operational techniques [29], [15], which have wide applications in the solution of BVP for PDE (see e.g. [28], [11], [12], [13], [14], [20], [21], [22]).

In this article we give a survey of results connected with this subject.

2. The monomiality principle

Recalling the leading work of G. Dattoli [16] we put the following definition

Definition 2.1. The polynomial set $\{p_n(x)\}_{n\in\mathbf{N}}$ is *quasi-monomial*, if there exist two operators $\hat{P}$ and $\hat{M}$, called respectively *derivative operator* and *multiplication operator*, verifying $(\forall n \in \mathbf{N})$ the identities

$$\begin{aligned} \hat{P}(p_n(x)) &= np_{n-1}(x) \\ \hat{M}(p_n(x)) &= p_{n+1}(x)\,. \end{aligned} \tag{1}$$

The $\hat{P}$ and $\hat{M}$ operators are shown to satisfy the commutation property

$$[\hat{P},\hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \tag{2}$$

and thus display a Weyl group structure.

If the considered polynomial set $\{p_n(x)\}$ is quasi-monomial, its properties can be easily derived from those of the $\hat{P}$ and $\hat{M}$ operators. In fact

i) if $\hat{P}$ and $\hat{M}$ have a differential realization, then the polynomial $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}(p_n(x)) = np_n(x)\,. \tag{3}$$

ii) Assuming here and in the following $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \hat{M}^n(1)\,. \tag{4}$$

iii) The last identity implies that the exponential generating function of $p_n(x)$ can be cast in the form

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{(t\hat{M})^n}{n!}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\hat{M}^n(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x)\,, \tag{5}$$

and therefore

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x)\,. \tag{6}$$

3. Laguerre-type exponentials

An important role in this framework is played by the so called *Laguerre derivative*, defined by $D_L := DxD$. According to a result by O.V. Viskov [27], it turns out that for every integer n

$$(DxD)^n = D^n x^n D^n\,. \tag{1}$$

Consider the first *Laguerre-type exponential* defined by

$$e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\,. \tag{2}$$

The following result holds

Theorem 3.1. *The function $e_1(ax)$ is an eigenfunction of the Laguerre derivative operator, i.e.*

$$D_L\, e_1(ax) = a e_1(ax) \tag{3}$$

Proof. Note that

$$D_L = D + xD^2, \tag{4}$$

and consequently

$$D_L\, e_1(ax) = \left(D + xD^2\right) \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} = \tag{5}$$

$$= \sum_{k=1}^{\infty} (k + k(k-1))\; a^k \frac{x^{k-1}}{(k!)^2} = \sum_{k=1}^{\infty} k^2\, a^k \frac{x^{k-1}}{(k!)^2} =$$

$$= a \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} = a e_1(ax).$$

$\square$

The above result can be iterated by considering the n^{th} Laguerre-type exponential

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}} , \tag{6}$$

and the n^{th} Laguerre-derivative, i.e. the operator (containing $n+1$ derivatives):

$$\begin{aligned} D_{nL} := Dx \cdots DxDxD = \\ = S(n+1,1)D + S(n+1,2)xD^2 + \cdots + S(n+1,n+1)x^n D^{n+1}, \end{aligned} \tag{7}$$

where $S(n+1,1), S(n+1,2), \ldots, S(n+1,n+1)$ denote Stirling numbers of the second kind [23]. In fact, in [17], the following theorem is proved:

Theorem 3.2. *Let a be an arbitrary real or complex number. The n^{th} Laguerre-type exponential $e_n(ax)$ is an eigenfunction of the operator D_{nL} i.e.*

$$D_{nL}\, e_n(ax) = a e_n(ax). \tag{8}$$

For $n = 0$ $D_{0L} := D$, and therefore equation (3.8) gives back the classical property of the exponential function

$$D\, e^{ax} = a e^{ax}.$$

It is worth noting that $\forall n$, the nL-exponential function satisfies $e_n(0) = 1$, and is an increasing convex function when $x \geq 0$; furthermore,

$$e^x = e_0(x) > e_1(x) > e_2(x) > \cdots > e_n(x) > \ldots, \qquad \forall x > 0.$$

4. The isomorphism $\mathscr{T}_x$

Consider the differential isomorphism, denoted by the symbol $\mathscr{T} := \mathscr{T}_x$, acting onto the space $\mathscr{A} := \mathscr{A}_x$ of analytic functions of the x variable by means of the correspondence:

$$D := \frac{d}{dx} \to \hat{D}_L := DxD; \qquad x\cdot \to \hat{D}_x^{-1}, \tag{1}$$

where

$$\hat{D}_x^{-1} f(x) := \int_0^x f(\xi)d\xi, \tag{2}$$

$$\hat{D}_x^{-n} f(x) := \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi) d\xi, \tag{3}$$

so that

$$\mathscr{T}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} d\xi = \frac{x^n}{n!} \,. \tag{4}$$

Note that:

$$\mathscr{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathscr{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x)$$

$$\mathscr{T}_x^2(e^x) = \sum_{k=0}^{\infty} \frac{\mathscr{T}_x(x^k)}{(k!)^2} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3} = e_2(x),$$

and so on.

A straightforward computation gives:

$$\hat{D}_L \frac{x^n}{n!} = DxD \frac{x^n}{n!} = n \frac{x^{n-1}}{(n-1)!} \,,$$

$$\hat{D}_x^{-1} \frac{x^n}{n!} = \frac{x^{n+1}}{(n+1)!} \,,$$

i.e. the Laguerre derivative is a “derivative” operator for the polynomial family $p_n(x) := \dfrac{x^n}{n!}$, and the anti-derivative $\hat{D}_x^{-1}$ is the corresponding “multiplication” operator.

5. Iterations of the isomorphism $\mathscr{T}_x$

The isomorphism $\mathscr{T} := \mathscr{T}_x$ can be iterated producing a set of generalized Laguerre derivatives as follows.

By definition of $\mathscr{T}_x$, we find:

$$\begin{aligned} \mathscr{T}_x \hat{D}_L &= \mathscr{T}_x(DxD) = DxD\hat{D}_x^{-1} DxD = DxDxD =: \hat{D}_{2L} \\ \mathscr{T}_x \hat{D}_{2L} &= \mathscr{T}_x(DxDxD) = DxDxDxD =: \hat{D}_{3L}, \end{aligned} \tag{1}$$

and, in general, by induction:

$$\mathscr{T}_x^{s-1} \hat{D}_L = \mathscr{T}_x^{s-1}(DxD) = DxDxD \cdots xD =: \hat{D}_{sL} \,, \tag{2}$$

where the last operator contains $s+1$ ordinary derivatives, denoted by D. It is convenient, in the following, to introduce a suitable notation regarding the iterations of the isomorphism $\mathscr{T}_x$, showing their action on powers, and consequently on all functions belonging to $\mathscr{A} := \mathscr{A}_x$. In fact, according to the above definition we can write:

$$\hat{D}_x^{-n}(1) = \frac{x^n}{n!}\,, \tag{3}$$

$$\mathscr{T}_x \hat{D}_x^{-1}(1) = \hat{D}_{\mathscr{T}_x}^{-1}(1) \qquad \text{i.e.} \qquad \hat{D}_{\mathscr{T}_x}^{-n}(1) = \frac{x^n}{(n!)^2}\,, \tag{4}$$

and, by induction:

$$\mathscr{T}_x^{s-1} \hat{D}_x^{-1}(1) = \hat{D}_{\mathscr{T}_x^{s-1}}^{-1}(1) \qquad \text{i.e.} \qquad \hat{D}_{\mathscr{T}_x^{s-1}}^{-n}(1) = \frac{x^n}{(n!)^s}\,. \tag{5}$$

6. A quasi-monomiality criterium

In the following the trivial dependence of the considered operators on the variable x is always omitted.

Let $\{p_n(x)\}_{n\in\mathbf{N}}$, with $p_0 = 1$, be a quasi-monomial family, and denote by $\hat{P}_0$ the corresponding derivative operator.

The the following result holds

Theorem 6.1. *Suppose that there exists an operator $\hat{\Phi}$ commuting with $\hat{P}_0$ such that:*

$$e^{y\hat{\Phi}}(p_n(x)) =: Q_n(x,y), \tag{1}$$

and furthermore, for a suitable operator $\hat{M}_1(y)$, it results:

$$e^{y\hat{\Phi}}(p_n(x)) =: Q_n(x,y) = (\hat{M}_1(y))^n(1), \tag{2}$$

where $\forall y : [\hat{P}_0, \hat{M}_1(y)] = 1$.

Then the polynomial family $\{Q_n(x,y)\}_{n\in\mathbf{N}}$ is quasi-monomial with respect to the operators $\hat{P}_1 \equiv \hat{P}_0$ and $\hat{M}_1(y)$.

Proof. In fact, since $\hat{P}_0$ is commuting with $\hat{\Phi}$, it is also commuting with $e^{y\hat{\Phi}}$, so that

$$\hat{P}_0(Q_n(x,y)) = \hat{P}_0 e^{y\hat{\Phi}}(p_n(x)) = e^{y\hat{\Phi}}\hat{P}_0(p_n(x)) =$$

$$= n e^{y\hat{\Phi}}(p_{n-1}(x)) = nQ_{n-1}(x,y),$$

and the first monomiality condition is satisfied by the operator $\hat{P}_1 \equiv \hat{P}_0$.

Furthermore, we obviously have:

$$\hat{M}_1(y)(Q_n(x,y)) = \hat{M}_1(y)(\hat{M}_1(y))^n(1) = (\hat{M}_1(y))^{n+1}(1) = Q_{n+1}(x,y),$$

and the second condition holds too, assuming $\hat{M}_1(y)$ as the multiplication operator. □

Example 6.2. The Hermite–Kampé de Fériet [26] (or Gould–Hopper polynomials, see [19]) $H_n^{(m)}(x,y)$ satisfy the conditions of the quasi-monomiality criterium, assuming $p_n(x) := x^n$, and moreover:

$$\begin{aligned} \hat{P}_1 \equiv \hat{P}_0 &:= \frac{\partial}{\partial x} = D_x \\ \hat{\Phi} &:= \frac{\partial^m}{\partial x^m} \\ \hat{M}_1(y) &:= x + my\frac{\partial^{m-1}}{\partial x^{m-1}} , \end{aligned}$$

since

$$\exp\left(y\,\frac{\partial^m}{\partial x^m}\right)x^n = H_n^{(m)}(x,y) = \left(x + my\,\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n (1)\ .$$

Example 6.3. The two variable higher order Laguerre polynomials:

$$L_n^{(m)}(x,y) = n!\,\sum_{k=0}^{n} \frac{y^{n-k}\,x^k}{(n-k)!\,(k!)^{m+1}}$$

satisfy the above mentioned conditions assuming $p_n(x) := x^n/n!$, and moreover:

$$\begin{aligned} \hat{P}_1 \equiv \hat{P}_0 &:= \hat{D}_L := D_x x D_x \\ \hat{\Phi} &:= \hat{D}_L^m = D_x^m x^m D_x^m \\ \hat{M}_1(y) &:= \hat{D}_x^{-1} + my\hat{D}_L^{m-1} , \end{aligned}$$

where

$$\hat{D}_x^{-1} f(x) := \int_0^x f(\xi)d\xi\ .$$

In fact we have:

$$\exp\left(y\,\hat{D}_L^m\right)\frac{x^n}{n!} = L_n^{(m)}(x,y) = \left(\hat{D}_x^{-1} + my\hat{D}_L^{m-1}\right)^n (1)\ .$$

We recall that for every integer m it results $\hat{D}_L^m = D_x^m x^m D_x^m$ according to a general result in [27].

In [7], the link between the above Examples is explained in terms of the above mentioned differential isomorphism $\mathscr{T}_x$, connecting the two polynomial families.

In fact the Laguerre polynomials $L_n(x,y)$ correspond to the Gould-Hopper ones $H_n^{(1)}(x,y) := (x+y)^n$ under the action of the isomorphism $\mathscr{T}_x$, since:

$$\mathscr{T}_x H_n^{(1)}(x,y) \equiv L_n(x,y)\,.$$

7. Finding the multiplication operator

The construction of the multiplication operator $M_1(y)$ is based on the Hausdorff identity.

Theorem 7.1. *Let $\hat{A}$ and $\hat{B}$, be operators independent of the parameter y. Then the Hausdorff identity holds:*

$$e^{y\hat{A}}\hat{B}e^{-y\hat{A}} = \hat{B} + y[\hat{A},\hat{B}] + \frac{y^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{y^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots \tag{1}$$

This identity is obviously interesting when $\hat{A}$ and $\hat{B}$ does not commute, and can be usefully applied when the above series reduces to a finite sum.

Therefore the following theorem holds [8], [9]:

Theorem 7.2. *Consider the polynomial set $\{p_n(x)\}_{n\in\mathbf{N}}$, with $p_0 = 1$, and suppose that this family is quasi-monomial with respect to the operators $\hat{P}_0$ and $\hat{M}_0$. Consider an operator $\hat{\Phi}$ satisfying $[\hat{\Phi},\hat{P}_0] = 0$, and $e^{y\hat{\Phi}}(1) = 1$. Define again*

$$Q_n(x,y) := e^{y\hat{\Phi}}(p_n(x)).$$

Then the polynomial family $\{Q_n(x,y)\}_{n\in\mathbf{N}}$, has the "derivative operator" $\hat{P}_1 \equiv \hat{P}_0$. Moreover, the "multiplication operator" $\hat{M}_1(y)$ is given by

$$\hat{M}_1(y) = \hat{M}_0 + y[\hat{\Phi},\hat{M}_0] + \frac{y^2}{2!}[\hat{\Phi},[\hat{\Phi},\hat{M}_0]] + \dots \tag{2}$$

8. The multi-dimensional case

Theorem 8.1. *Consider the polynomial set $\{p_n(x,y_1,\dots,y_r)\}_{n\in\mathbf{N}}$, with $p_0 = 1$, and suppose that this family is quasi-monomial with respect to the operators $\hat{P}_r$ and $\hat{M}_r := \hat{M}_r(y_1,\dots,y_r)$. Consider an operator $\hat{\Psi}$ satisfying $[\hat{\Psi},\hat{P}_r] = 0$, and*

$e^{y_{r+1}\hat{\Psi}}(1) = 1.$
Define again

$$Q_n(x, y_1, \dots, y_{r+1}) := e^{y_{r+1}\hat{\Psi}} p_n(x, y_1, \dots, y_r) \ .$$

Then the polynomial family $\{Q_n(x, y_1, \dots, y_{r+1})\}_{n \in \mathbf{N}}$, *has the "derivative operator"* $\hat{P}_{r+1} \equiv \hat{P}_r$.
Moreover, the "multiplication operator" $\hat{M}_{r+1} := \hat{M}_{r+1}(y_1, \dots, y_{r+1})$ *is given by*

$$\hat{M}_{r+1}(y_1, \dots, y_{r+1}) = \hat{M}_r + y_r[\hat{\Psi}, \hat{M}_r] + \frac{y_r^2}{2!}[\hat{\Psi}, [\hat{\Psi}, \hat{M}_r]] + \dots \tag{1}$$

9. Multi-variable Gould-Hopper polynomials

Putting $D_x := \dfrac{d}{dx}$ we find:

$$[D_x^m, x] = m\, D_x^{m-1} \ ,$$

so that the action of the above commutator is equivalent to a formal derivative on the symbol D_x^m.

The r-variables Gould-Hopper polynomials $H_n^{(1)}(x_1, x_2, \dots, x_r)$ are defined as follows.

Assuming $p_n(x_1) = x_1^n$, and putting

$$\begin{aligned} \hat{P}_0 &:= \frac{\partial}{\partial x_1} = D_{x_1} \\ \hat{\Phi} &:= x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial^2}{\partial x_1^2} + \dots + x_r \frac{\partial^{r-1}}{\partial x_1^{r-1}} \\ \hat{M}_0 &:= x_1 \cdot \ , \end{aligned}$$

we find:

$$\begin{gathered} \exp\left(\hat{\Phi}\right) x_1 = x_1 + \left[\hat{\Phi}, x_1\right] + \frac{1}{2!}\left[\hat{\Phi}, \left[\hat{\Phi}, x_1\right]\right] + \dots = \\ = x_1 + \left[x_2 D_{x_1} + x_3 D_{x_1}^2 + \dots + x_r D_{x_1}^{r-1}, x_1\right] = \\ = x_1 + x_2 + 2x_3 D_{x_1} + \dots + (r-1) x_r D_{x_1}^{r-2} \ , \end{gathered}$$

$$\hat{M} \equiv \hat{M}(x_2, x_3, \dots, x_r) = x_1 + x_2 + 2x_3 D_{x_1} + \dots + (r-1) x_r D_{x_1}^{r-2}$$

and consequently

$$H_n^{(1)}(x_1, x_2, \dots, x_r) = \left(x_1 + x_2 + 2x_3 D_{x_1} + \dots + (r-1) x_r D_{x_1}^{r-2}\right)^n$$

Remark 9.1. Note that the above polynomials include the $H_n^{(j)}(x,y)$, for every integer $j \geq 1$.

In fact, we can write:

$$H_n^{(2)}(\xi_1,\xi_2,\ldots,\xi_r) = H_n^{(1)}(x_1,0,x_3,\ldots,x_{r+1}) \ ,$$

where

$$\xi_1 = x_1, \ \xi_2 = x_3, \ \ldots, \ \xi_r = x_{r+1} \ .$$

and, in general:

$$H_n^{(m)}(\xi_1,\xi_2,\ldots,\xi_r) = H_n^{(1)}(x_1,0,\ldots,0,x_{m+1},\ldots,x_{r+m-1}) \ ,$$

where

$$\xi_1 = x_1, \ \xi_2 = x_{m+1}, \ \ldots, \ \xi_r = x_{r+m-1} \ .$$

9.1. Properties

By using the standard monomiality technique, we find the properties described below.

- Differential equation

$$\hat{M}\hat{P}H_n^{(1)} = n\,H_n^{(1)}$$

i.e.

$$\left(x_1 + x_2 + 2x_3 D_{x_1} + \cdots + (r-1)x_r D_{x_1}^{r-2}\right) D_{x_1} H_n^{(1)} = n\,H_n^{(1)}$$

and therefore

$$\left((x_1+x_2)D_{x_1} + 2x_3 D_{x_1}^2 + \cdots + (r-1)x_r D_{x_1}^{r-1}\right) H_n^{(1)} = n\,H_n^{(1)} \tag{1}$$

- Generating function

Theorem 9.2. *The generating function of the r-variable Gould-Hopper polynomials is given by*

$$\begin{aligned} G(t;x_1,x_2,\ldots,x_r) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(1)}(x_1,\ldots,x_r) = \\ &= \exp\left[t\left(x_1 + x_2 + 2x_3 D_{x_1} + \cdots + (r-1)x_r D_{x_1}^{r-2}\right)\right](1) = \\ &= e^{x_1 t} \cdot \exp\left(t x_2 + t^2 x_3 + \cdots + t^{r-1} x_r\right) \ , \end{aligned} \tag{2}$$

and therefore the $H_n^{(1)}(x_1,x_2,\ldots,x_r)$ polynomials belong to the class of Appell polynomials.

- Explicit form

Theorem 9.3. *The r-variable Gould-Hopper polynomials are explicitly given by*

$$H_n^{(1)}(x_1,x_2,\ldots,x_r) = n! \sum_{\pi_k(n|r-1)} \frac{(x_1+x_2)^{h_1}}{h_1!} \frac{x_3^{h_2}}{h_2!} \cdots \frac{x_r^{h_{r-1}}}{h_{r-1}!}\,, \tag{3}$$

where $k := h_1 + h_2 + \cdots + h_{r-1}$, $n := h_1 + 2h_2 + \cdots + (r-1)h_{r-1}$, and the sum runs over all the restricted partitions $\pi_k(n|r-1)$ (containing at most $r-1$ sizes) of the integer n, k denoting the number of parts of the partition and h_i the number of parts of size i. Note that, using the ordinary notation for the partitions of n, i.e. $n = h_1 + 2h_2 + \cdots + nh_n$, we have to assume $h_r = h_{r+1} = \cdots = h_n = 0$.

Remark 9.4. – Note that the polynomials $H_n^{(1)}(x_1,x_2,\ldots,x_r)$ are Bell polynomials corresponding to a composite function of the type $\exp(g(t))$. Namely, by using the notation of Riordan's book, we have:

$$H_n^{(1)}(x_1,x_2,\ldots,x_r) = Y_n(x_1+x_2,x_3,\ldots,x_r,0,0,\ldots,0)\,.$$

10. Multi-variable Laguerre polynomials

Introducing the notation $D_L := D_x x D_x$, we find:

$$\left[D_L^m, D_x^{-1}\right] = m\, D_L^{m-1}\,.$$

The r-variables Laguerre polynomials $L_n^{(1)}(x_1,x_2,\ldots,x_r)$ are defined as follows.

Assuming $p_n(x_1) = \dfrac{x_1^n}{n!}$, and putting

$$\begin{aligned}
\hat{P}_0 \quad &:= \quad \frac{\partial}{\partial x_1} x_1 \frac{\partial}{\partial x_1} = D_{L_{x_1}} \\
\hat{\Phi} \quad &:= \quad x_2\, D_{L_{x_1}} + x_3\, D^2_{L_{x_1}} + \cdots + x_r\, D^{r-1}_{L_{x_1}} \\
\hat{M}_0 \quad &:= \quad D_{x_1}^{-1}\,,
\end{aligned}$$

we find:

$$\begin{aligned}
\exp\left(\hat{\Phi}\right) D_{x_1}^{-1} &= D_{x_1}^{-1} + \left[\hat{\Phi}, D_{x_1}^{-1}\right] + \frac{1}{2!}\left[\hat{\Phi},\left[\hat{\Phi}, D_{x_1}^{-1}\right]\right] + \cdots = \\
&= D_{x_1}^{-1} + \left[x_2 D_{L_{x_1}} + x_3 D^2_{L_{x_1}} + \cdots + x_r D^{r-1}_{L_{x_1}}\,, D_{x_1}^{-1}\right] = \\
&= D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \cdots + (r-1) x_r D^{r-2}_{L_{x_1}}\,,
\end{aligned}$$

since all the subsequent commutators vanish.

Therefore (the trivial dependence on the first variable is again omitted)

$$\hat{M}(x_2, x_3, \ldots, x_r) = D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \cdots + (r-1)x_r D_{L_{x_1}}^{r-2}$$

and consequently

$$L_n^{(1)}(x_1, x_2, \ldots, x_r) = \left(D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \cdots + (r-1)x_r D_{L_{x_1}}^{r-2}\right)^n$$

Remark 10.1. Note that even in this case the above polynomials include the $L_n^{(j)}(x,y)$, introduced for every integer $j \geq 1$, since:

$$L_n^{(j)}(\xi_1, \xi_2, \ldots, \xi_r) := L_n^{(1)}(x_1, 0, \ldots, 0, x_{j+1}, \ldots, x_{r+j-1}) ,$$

where

$$\xi_1 = x_1,\ \xi_2 = x_{j+1},\ \ldots,\ \xi_r = x_{r+j-1} .$$

10.1. Properties

- Differential equation

$$\hat{M}\hat{P}L_n^{(1)} = n L_n^{(1)}$$

i.e.

$$\left[\left(\hat{D}_{x_1}^{-1} + x_2\right) D_{L_{x_1}} + 2x_3 D_{L_{x_1}}^2 + \cdots + (r-1)x_r D_{L_{x_1}}^{r-1}\right] L_n^{(1)} = n L_n^{(1)} \tag{1}$$

- Generating function

Theorem 10.2. *The generating function of the r-variable Laguerre polynomials is given by*

$$\begin{aligned} F_1(t; x_1, x_2, \ldots, x_r) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(1)}(x_1, \ldots, x_r) = \\ &= \exp\left[t\left(D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \cdots + (r-1)x_r D_{L_{x_1}}^{r-2}\right)\right](1) = \\ &= e_1(tx_1) \cdot \exp\left(tx_2 + t^2 x_3 + \cdots + t^{r-1} x_r\right) , \end{aligned} \tag{2}$$

where $e_1(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$ *is the Laguerre-type exponential function, and therefore the* $L_n^{(1)}(x_1, x_2, \ldots, x_r)$ *polynomials belong to the class of Laguerre-type Appell polynomials.*

- Explicit form

Theorem 10.3. *The r-variable Laguerre polynomials are explicitly given by*

$$L_n^{(1)}(x_1,x_2,\ldots,x_r) = n! \sum_{\pi_k(n|r-1)} \frac{\left(D_{x_1}^{-1}+x_2\right)^{h_1}}{h_1!} \frac{x_3^{h_2}}{h_2!} \cdots \frac{x_r^{h_{r-1}}}{h_{r-1}!}\,, \tag{3}$$

where $k := h_1 + h_2 + \cdots + h_{r-1}$, $n := h_1 + 2h_2 + \cdots + (r-1)h_{r-1}$, *and the sum runs over all the restricted partitions* $\pi_k(n|r-1)$ *(containing at most* $r-1$ *sizes) of the integer* n, k *denoting the number of parts of the partition and* h_i *the number of parts of size* i.

11. Higher order multi-variable Laguerre polynomials

We recall that, in preceding articles, general forms of two variable Laguerre polynomials was introduced, by using the above mentioned differential isomorphism $\mathscr{T}$ connecting the Laguerre polynomials with the Hermite-Kampé de Fériet (or Gould-Hopper) ones. These polynomials represent the Laguerrian counterpart of the Gould-Hopper sets.

We are now in condition to generalize these polynomials to the multi-variable case.

Note that

$$\left[D_{sL_x}^m, D_{\mathscr{T}_x^s}^{-1}\right] = m\, D_{sL_x}^{m-1}\,.$$

Let us define the s-order and r-variable Laguerre polynomials as

$$L_n^{(1;s)}(x_1,x_2,\ldots,x_r) := \left(D_{\mathscr{T}_{x_1}^s}^{-1} + x_2 + 2x_3 D_{sL_{x_1}} + \cdots + (r-1)x_r D_{sL_{x_1}}^{r-2}\right)^n .$$

Then, by iterative applications of the isomorphism $\mathscr{T}_{x_1}$, over the corresponding equations of Section 4, we find the following results

- Differential equation

$$\left[\left(D_{\mathscr{T}_{x_1}^s}^{-1} + x_2\right) D_{sL_{x_1}} + 2x_3 D_{sL_{x_1}}^2 + \cdots + (r-1)x_r D_{sL_{x_1}}^{r-1}\right] L_n^{(1;s)} = n L_n^{(1;s)} \tag{1}$$

- Generating function

Theorem 11.1. *The generating function of the s-order and r-variable Laguerre polynomials is given by*

$$F_s(t;x_1,x_2,\ldots,x_r) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(1;s)}(x_1,\ldots,x_r) =$$

$$= \exp\left[t\left(D_{\mathscr{T}_{x_1}^s}^{-1} + x_2 + 2x_3 D_{sL_{x_1}} + \cdots + (r-1)x_r D_{sL_{x_1}}^{r-2}\right)\right](1) = \tag{2}$$

$$= e_s(tx_1)\cdot \exp\left(tx_2 + t^2x_3 + \cdots + t^{r-1}x_r\right)\ ,$$

where $e_s(x)$ is the Laguerre-type exponential function, and therefore the $L_n^{(1;s)}(x_1,\ldots,x_r)$ polynomials belong to the class of Laguerre-type Appell polynomials.

- Explicit form

Theorem 11.2. *The s-order and r-variable Laguerre polynomials are explicitly given by*

$$L_n^{(1;s)}(x_1,x_2,\ldots,x_r) = n! \sum_{\pi_k(n|r-1)} \frac{\left(D_{\mathscr{T}_{x_1}^s}^{-1} + x_2\right)^{h_1}}{h_1!} \frac{x_3^{h_2}}{h_2!} \cdots \frac{x_r^{h_{r-1}}}{h_{r-1}!}\ , \tag{3}$$

where $k := h_1 + h_2 + \cdots + h_{r-1}$, $n := h_1 + 2h_2 + \cdots + (r-1)h_{r-1}$, and the sum runs over all the restricted partitions $\pi_k(n|r-1)$ (containing at most $r-1$ sizes) of the integer n, k denoting the number of parts of the partition and h_i the number of parts of size i.

12. The r-variable monomiality principle

We give in this section the natural generalization of the above technique. Proofs and more general results can be found in [5], [6].

Definition 12.1. An r-variable r-index polynomial family $\{p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\}$, is said to be *quasi-monomial* if $2r$ operators $\hat{P}_{x_1},\ldots,\hat{P}_{x_r}, \hat{M}_{x_1},\ldots,\hat{M}_{x_r}$ exist in such a way that

$$\left\{\begin{array}{l} \hat{P}_{x_1}\, p_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = n_1\, p_{n_1-1,n_2,\ldots,n_r}(x_1,\ldots,x_r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{P}_{x_r}\, p_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = n_r\, p_{n_1,\ldots,n_{r-1},n_r-1}(x_1,\ldots,x_r)\,, \end{array}\right. \tag{1}$$

$$\left\{\begin{array}{l} \hat{M}_{x_1}\, p_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = p_{n_1+1,n_2,\ldots,n_r}(x_1,\ldots,x_r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{M}_{x_r}\, p_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = p_{n_1,\ldots,n_{r-1},n_r+1}(x_1,\ldots,x_r)\,. \end{array}\right. \tag{2}$$

From the above formulas it follows:

$$\left[\hat{P}_{x_1},\hat{M}_{x_1}\right]=1,\ \ldots\ ,\left[\hat{P}_{x_r},\hat{M}_{x_r}\right]=1\,. \tag{3}$$

Under the above hypotheses, the main properties of the polynomial family under consideration can be easily derived, since

- If the derivative and multiplication operators have a differential realization, then

$$\begin{array}{l} \hat{M}_{x_1}\hat{P}_{x_1}\,p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)=n_1\,p_{n_1,\ldots,n_r}(x_1,\ldots,x_r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{M}_{x_r}\hat{P}_{x_r}\,p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)=n_r\,p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\,, \end{array} \tag{4}$$

  i.e. we find r (independent) differential equations satisfied by the polynomial family.

- Assuming $p_{0,\ldots,0}(x_1,\ldots,x_r)\equiv 1$, the explicit expression of $\{p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\}$ is given by

$$p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)=\hat{M}_{x_1}^{n_1}\cdots\hat{M}_{x_r}^{n_r}(1)\,. \tag{5}$$

- The exponential generating function of $\{p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\}$, assuming again $p_{0,\ldots,0}(x_1,\ \ldots,x_r)\equiv 1$, is given by

$$\begin{aligned} & e^{t_1\hat{M}_{x_1}+\cdots+t_r\hat{M}_{x_r}}(1)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\frac{(t_1\hat{M}_{x_1})^{n_1}\cdots(t_r\hat{M}_{x_r})^{n_r}}{n_1!\cdots n_r!}= \\ & =\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\,p_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\,. \end{aligned} \tag{6}$$

Theorem 12.2. *Let $\hat{A}_1,\ldots,\hat{A}_r$, be commuting operators* (i.e. $[\hat{A}_i,\hat{A}_j]=0,\ \forall i,j$) *independent of the parameters $t_1,\ldots,t_r$. Then the Hausdorff identity holds:*

$$\begin{aligned} & e^{t_1\hat{A}_1+\cdots+t_r\hat{A}_r}\,\hat{C}\,e^{-t_1\hat{A}_1-\cdots-t_r\hat{A}_r}= \\ & =\hat{C}+\left(\sum_{i=0}^{r}t_i[\hat{A}_i,\hat{C}]\right)+\frac{1}{2!}\left(\sum_{i,j=0}^{r}t_it_j[\hat{A}_i,[\hat{A}_j,\hat{C}]]\right)+ \\ & +\frac{1}{3!}\left(\sum_{i,j,k=0}^{r}t_it_jt_k[\hat{A}_i,[\hat{A}_j,[\hat{A}_k,\hat{C}]]]\right)+\ldots \end{aligned} \tag{7}$$

Theorem 12.3. *Consider r operators $\hat{\Phi}_{x_1}, \dots, \hat{\Phi}_{x_r}$ commuting respectively with $\hat{P}_{x_1}, \dots, \hat{P}_{x_r}$, and put*

$$Q_{n_1,\dots,n_r}(x_1,\dots,x_r;t_1,\dots,t_r) := e^{t_1 \hat{\Phi}_{x_1} + \cdots + t_r \hat{\Phi}_{x_r}} p_{n_1,\dots,n_r}(x_1,\dots,x_r). \tag{8}$$

Suppose that there exist r operators $\hat{M}_{1,x_1}(t_1,\dots,t_r), \dots, \hat{M}_{1,x_r}(t_1,\dots,t_r)$ such that:

$$\begin{aligned} Q_{n_1,\dots,n_r}(x_1,\dots,x_r;t_1,\dots,t_r) = \left(\hat{M}_{1,x_1}(t_1,\dots,t_r)\right)^{n_1} \times \cdots \\ \cdots \times \left(\hat{M}_{1,x_r}(t_1,\dots,t_r)\right)^{n_r}(1), \end{aligned} \tag{9}$$

and furthermore that, $\forall\, t_1,\dots,t_r$:

$$\left[\hat{P}_{x_1}, \hat{M}_{1,x_1}(t_1,\dots,t_r)\right] = \cdots = \left[\hat{P}_{x_r}, \hat{M}_{1,x_r}(t_1,\dots,t_r)\right] = 1. \tag{10}$$

Then the polynomial family $Q_{n_1,\dots,n_r}(x_1,\dots,x_r;t_1,\dots,t_r)$ is quasi-monomial with respect to the operators

$$\hat{P}_{x_1}, \dots \hat{P}_{x_r}, \hat{M}_{1,x_1}(t_1,\dots,t_r), \dots, \hat{M}_{1,x_r}(t_1,\dots,t_r).$$

Theorem 12.4. *Consider the polynomial family $\{p_{n_1,\dots,n_r}(x_1,\dots,x_r)\}$, put again $p_{0,\dots,0}(x_1,\dots,x_r) \equiv 1$, and suppose that it is quasi-monomial with respect to the operators $\hat{P}_{x_1},\dots,\hat{P}_{x_r}$, $\hat{M}_{x_1},\dots,\hat{M}_{x_r}$. Let $\hat{\Phi}_{x_1},\dots,\hat{\Phi}_{x_r}$, be operators, independent of the parameters $t_1,\dots,t_r$, such that:*

$$\left[\hat{\Phi}_{x_1}, \hat{P}_{x_1}\right] = \cdots = \left[\hat{\Phi}_{x_r}, \hat{P}_{x_r}\right] = 0, \qquad e^{t_1 \hat{\Phi}_{x_1} + \cdots + t_r \hat{\Psi}_{x_r}}(1) = 1. \tag{11}$$

Define again the polynomial set

$$Q_{n_1,\dots,n_r}(x_1,\dots,x_r;t_1,\dots,t_r) := e^{t_1 \hat{\Phi}_{x_1} + \cdots + t_r \hat{\Psi}_{x_r}} p_{n_1,\dots,n_r}(x_1,\dots,x_r) \tag{12}$$

Then the multiplication operators are given by the Hausdorff expansions

$$\begin{aligned} \hat{M}_{1,x_1} &= \hat{M}_{x_1} + \left\{t_1\left[\hat{\Phi}_{x_1}, \hat{M}_{x_1}\right] + \cdots + t_r\left[\hat{\Phi}_{x_r}, \hat{M}_{x_1}\right]\right\} + \\ &+ \frac{1}{2!}\left(\sum_{i,j=0}^{r} t_i t_j \left[\hat{\Phi}_{x_i}, \left[\Phi_{x_j}, \hat{M}_{x_1}\right]\right]\right) + \dots \end{aligned} \tag{13}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} \hat{M}_{1,x_r} &= \hat{M}_{x_r} + \left\{t_1\left[\hat{\Phi}_{x_1}, \hat{M}_{x_r}\right] + \cdots + t_r\left[\hat{\Phi}_{x_r}, \hat{M}_{x_r}\right]\right\} + \\ &+ \frac{1}{2!}\left(\sum_{i,j=0}^{r} t_i t_j \left[\hat{\Phi}_{x_i}, \left[\Phi_{x_j}, \hat{M}_{x_r}\right]\right]\right) + \dots \end{aligned} \tag{14}$$

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