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RECENT DEVELOPMENTS OF THE CAMPANATO THEORY OF NEAR OPERATORS

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Dedicated to Professor Sergio Campanato on his 70th birthday

1. Introduction.

The concept of *nearness* between operators introduced by Campanato is defined as follows:

Definition 1.1. *Let $\mathcal{X}$ be a set, $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$, A and B be two operators such that $A, B : \mathcal{X} \to \mathcal{B}$. We say that A is near B, if there are two positive constants, α, k, with $0 < k < 1$, such for every $x_1, x_2 \in \mathcal{X}$ we have:*

$$\big\|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\big\| \le k\big\|B(x_1) - B(x_2)\big\|. \tag{1}$$

The starting point of the *theory of near operators* is the following theorem which was demonstrated by Campanato, firstly in the case of two Hilbert spaces (see [4]), and then in the following form (see [9]).

Theorem 1.2. *Let $\mathcal{X}$ be a set, $\mathcal{B}$ a Banach space with norm $\|\cdot\|$, A, B be two operators such that: $A, B : \mathcal{X} \to \mathcal{B}$, and let A be near B. Under these hypotheses, if B is a bijection between $\mathcal{X}$ and $\mathcal{B}$, A is also a bijection between $\mathcal{X}$ and $\mathcal{B}$.*

2. A short history.

The idea of introducing the above defined concept of nearness between operators was generated from the problem of showing the existence and uniqueness of solutions of nonvariational elliptic problems of the type:

$$(2)\qquad \begin{cases} u \in H^2 \cap H_0^1(\Omega) \\ a(x, H(u)) = f, \end{cases}$$

where: $f \in L^2(\Omega)$, $H(u) = \{D_{ij}u\}_{i,j=1,\dots,n}$, Ω is a bounded open set of $\mathbb{R}^n$, $a : \Omega \times \mathbb{R}^{n^2} \to \mathbb{R}$ is a measurable function which is bounded with respect to the first variable, continuous with respect to the second variable and verifies a particular condition of ellipticity, denominated *Condition A* and considered in the following. We remember that, even in the linear case where $a(x, H(u)) = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u$, Problem (2) is not well posed in general, under the sole hypothesis of uniform ellipticity i.e. there exists $\nu > 0$ such that

$$(3)\qquad \sum_{i,j=1}^{n} a_{ij}\eta_i\eta_j \geq \nu\|\eta\|_n^2 , \quad \forall \eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n.$$

Hypotheses that are more restrictive than (3) are needed to prove existence and uniqueness of the solution for Problem (2). One of these is the above mentioned *Condition A*, which has been suggested by one of the quite natural modes in which these problems are usually solved, i.e. the classic fixed point theorem of contractions. We consider in fact the equation

$$(4)\qquad \Delta u = \alpha f + \Delta w - \alpha a(x, H(w))$$

and define a map $\mathcal{T} : H^2 \cap H_0^1(\Omega) \to H^2 \cap H_0^1(\Omega)$ which associates to each $w \in H^2 \cap H_0^1(\Omega)$ the solution $u \in H^2 \cap H_0^1(\Omega)$ of equation (4).

It can be proved that $\mathcal{T}$ is a contraction in the above space if $a(x, \cdot)$ verifies the following algebraic condition (see [4]):

Condition A 2.1. *Three positive constants α, γ, δ exist, with $\gamma + \delta < 1$, such that for all $\xi, \tau \in \mathbb{R}^{n^2}$, and for all $x \in \Omega$ we have*

$$(5)\qquad \left| \sum_{i=1}^{n} \xi_{ii} - \alpha\left[a(x, \xi + \tau) - a(x, \tau)\right] \right| \leq \gamma \|\xi\|_{n^2} + \delta \left| \sum_{i=1}^{n} \xi_{ii} \right|.$$

Basically, from these considerations, it can be deduced that the operator $u \mapsto a(\cdot, H(u))$ is a bijection between $H^2 \cap H_0^1(\Omega)$ and $L^2(\Omega)$ as a consequence of the following facts:

(i) $a(., H(u))$ has a certain relationship with Δ

(ii) Δu is a *bjiection* between $H^2 \cap H_0^1(\Omega)$ and $L^2(\Omega)$.

This is the starting point to apply the scheme of the theory, i.e. Definition 1.1 and Theorem 1.2.

In fact we can write

$$\begin{aligned} Bu &= \Delta u \\ A(u) &= a(x, H(u)) \\ \mathcal{X} &= H^2 \cap H_0^1(\Omega) \\ \mathcal{B} &= L^2(\Omega). \end{aligned}$$

From *Condition A* it can be obtained that A is *near* B, while from Theorem 1.2, since B is a *bjiection* between these spaces, it can be deduced that A is a *bjiection* between them (see [5]).

3. Some developments of the theory.

The following results have been added subsequently to Theorem 1.2 (see [4], [9] and [13]):

Theorem 3.1. *The map $A : \mathcal{X} \to \mathcal{B}$ is injective (surjective) if, and only if, it is near a map $B : \mathcal{X} \to \mathcal{B}$ which is injective (surjective).*

Theorem 3.2. (Open range). *Let A be near B. If $B(\mathcal{X})$ is open in $\mathcal{B}$, then $A(\mathcal{X})$ is also open in $\mathcal{B}$.*

Theorem 3.3. (Dense range). *Let A be near B. If $B(\mathcal{X})$ is dense in $\mathcal{B}$, then $A(\mathcal{X})$ is also dense in $\mathcal{B}$.*

One of the ways to demonstrate that $A(\mathcal{X})$ is open (or dense) in $\mathcal{B}$ is therefore linked to finding an operator B such that $B(\mathcal{X})$ is open (or dense) in $\mathcal{B}$ and such that A is *near* B.

Moreover we can also show the following theorem (see also [8]).

Theorem 3.4. (Compact range). *Let A be near B. If $B(\mathcal{X})$ is compact in $\mathcal{B}$, then $A(\mathcal{X})$ is also compact in $\mathcal{B}$.*

Proof. Let $\{y_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $A(\mathcal{X})$. We denote with:

$$X_n = \{x \in \mathcal{X} : y_n = A(x)\},$$

and we define by $\{z_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X})$ the unique point such that $z_n = B(X_n)$. The uniqueness is obtained from the following inequality, that is deduced from (1) of Definition 1.1:

$$\|B(x') - B(x'')\| \leq \frac{\alpha}{k-1}\|A(x') - A(x'')\|, \quad \forall x', x'' \in \mathcal{X}. \tag{6}$$

From (6) it follows that $\{z_n\}_{n\in\mathbb{N}}$ is a bounded sequence. $B(\mathcal{X})$ is compact so that if $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ is a monotonic increasing sequence such that $z_{h_n} \to z \in B(\mathcal{X})$, we consider $y_{h_n} = A(X_{h_n})$, $X = \{x \in \mathcal{X} : B(x) = z\}$ and $y = A(X)$. In this case the uniqueness of y is obtained from the following inequality (deduced from (1) of Definition 1.1)

$$\|A(x') - A(x'')\| \leq \frac{k+1}{\alpha}\|B(x') - B(x'')\|, \quad \forall x', x'' \in \mathcal{X}. \tag{7}$$

We claim that $y_{h_n} \to y$. In fact:

$$\|y_{h_n} - y\| = \|A(x_{h_n}) - A(x)\| \leq \frac{k+1}{\alpha}\|B(x_{h_n}) - B(x)\| =$$

$$= \frac{k+1}{\alpha}\|z_{h_n} - z\|, \quad x_{h_n} \in X_{h_n}, x \in X.$$

This proves that $A(\mathcal{X})$ is compact.

However we remark that the theory of near operators is not limited to differential operators which satisfy some algebraic condition such as *Condition A*, but it has a wider field of applications. Moreover Definition 1.1 and Theorem 1.2 can be generalized to complete metric spaces (see [12]).

A first example can be given by showing, using the theory of *near operators*, one of the possible generalizations of the Lax-Milgram Theorem (see also [7]). Therefore, this theory, which was created with the scope of solving non-variational elliptic problems, has interesting applications in variational elliptic problems too.

Theorem 3.5. *Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a function, with the properties:*

(1) $v \to a(u, v)$ *is linear* $\forall u \in H$,

(2) $|a(u_1, v) - a(u_2, v)| \leq M\|u_1 - u_2\|_H \|v\|_H \quad \forall v \in H,$
(3) $\exists \nu > 0 : \; a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \geq \nu \|u_1 - u_2\|_H^2, \; \forall u_1, u_2 \in H.$

(If $u \to a(u, v)$ is a linear map, condition (3) reduces to the well known coercivity condition).

Then, for all $F \in H^*$ there exists one and only one $u \in H$ which solves the equation

$$a(u, v) = F(v), \quad \forall v \in H. \tag{8}$$

Proof. We denote with $\mathcal{A}$ the map between H and H^* defined by: $\mathcal{A}(u)(v) = a(u, v)$. We are going to show that $\mathcal{A}$ is a bijection between H and H^*, i.e. for all $F \in H^*$ it there exists one and only one $u \in H$ such that

$$\mathcal{A}(u)(v) = F(v), \quad \forall v \in H.$$

This is equivalent to the thesis of the theorem: there exists one and only one solution $u \in H$ of the equation

$$a(u, v) = \mathcal{A}(u)(v) = F(v), \quad \forall v \in H.$$

By Theorem 1.2, it is sufficient to show that $\mathcal{A}$ is *near* to the operator $\mathcal{J} : H \to H^*$ defined by:

$$\mathcal{J}(u)(v) = (u, v)_H.$$

In particular we remark that $\|\mathcal{J}(u)\|_{H^*} = \|u\|_H$. Moreover, consider the Riesz operator $\mathcal{R} : H^* \to H$ defined by $\mathcal{R}(F) = w, \; F \in H^*, w \in H$, where $F(v) = (w, v)_H, \; \forall v \in H$ and $\|w\|_H = \|F\|_{H^*}$. Then, in particular, $(\mathcal{R}(\mathcal{A}(u)), v)_H = \mathcal{A}(u)(v) = a(u, v)$, and $\mathcal{R} = \mathcal{J}^{-1}$, so that $\mathcal{J}$ is a bijection between H and H^*.

We obtain the thesis of thé theorem by showing the inequality (1), for the operator $\mathcal{J}$ and $\mathcal{A}$, i.e. showing that two positive constants α and $k \in (0, 1)$ exist such that:

$$\left\|\mathcal{J}(u_1) - \mathcal{J}(u_2) - \alpha[\mathcal{A}(u_1) - \mathcal{A}(u_2)]\right\|_{H^*} \leq k \left\|\mathcal{J}(u_1) - \mathcal{J}(u_2)\right\|_{H^*}.$$

We remark that :

$$\left\|\mathcal{J}(u_1) - \mathcal{J}(u_2) - \alpha[\mathcal{A}(u_1) - \mathcal{A}(u_2)]\right\|_{H^*}^2 =$$

$$= \left\|u_1 - u_2 - \alpha[\mathcal{R}(\mathcal{A}(u_1)) - \mathcal{R}(\mathcal{A}(u_2))]\right\|_H^2 =$$

$$= \left\|u_1 - u_2\right\|_H^2 + \alpha^2 \left\|\mathcal{R}(\mathcal{A}(u_1)) - \mathcal{R}(\mathcal{A}(u_2))\right\|_H^2 -$$

$$- 2\alpha(\mathcal{R}(\mathcal{A}(u_1)) - \mathcal{R}(\mathcal{A}(u_2)), u_1 - u_2)_H =$$
$$= \|u_1 - u_2\|_H^2 + \alpha^2 \|\mathcal{R}(\mathcal{A}(u_1)) - \mathcal{R}(\mathcal{A}(u_2))\|_H^2 -$$
$$- 2\alpha[a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2)] \leq$$

(by hypotheses (2) and (3))

$$\leq \|u_1 - u_2\|_H^2 + \alpha^2 M^2 \|u_1 - u_2\|_H^2 - 2\alpha\nu \|u_1 - u_2\|_H^2 =$$
$$= [1 + \alpha^2 M^2 - 2\alpha\nu] \|u_1 - u_2\|_H^2 = k \|\mathcal{J}(u_1) - \mathcal{J}(u_2)\|_{H^*}^2.$$

Here we have set : $k = 1 + \alpha^2 M^2 - 2\alpha\nu$. It holds $0 < k < 1$ provided $0 < 1 + \alpha^2 M^2 - 2\alpha\nu < 1$, that is:

$$0 < \alpha < \frac{2\nu}{M^2}.$$

As an application of Theorem 3.5 we can solve a simple classic variational elliptic problem (see for example [16], Section 26.5).

Let us consider a bounded open set Ω in $\mathbb{R}^n$, with a sufficiently regular boundary, and the form

$$a(u, v) = \sum_{i=1}^{n} \int_{\Omega} a_i(x, Du) D_i v \, dx\,,$$

where $Du = (D_1 u, \ldots, D_n u)$, $u \in H_0^1(\Omega)$. On $a(\cdot, \cdot)$ we make the following hypotheses:

(a) $a_i(x, p)$ is measurable in x, and continuous in $p \in \mathbb{R}^n$.

(b) $\exists\, \nu > 0$ such that $\forall\, p, \overline{p} \in \mathbb{R}^n$, $\forall\, x \in \Omega$:

$$\sum_{i=1}^{n} [a_i(x, p) - a_i(x, \overline{p})](p_i - \overline{p}_i) \geq \nu \|p - \overline{p}\|_n^2.$$

(c) $\exists\, M > 0$ such that $\forall\, p, \overline{p} \in \mathbb{R}^n$, $\forall\, x \in \Omega$:

$$\sum_{i=1}^{n} [a_i(x, p) - a_i(x, \overline{p})]^2 \leq M \|p - \overline{p}\|_n^2.$$

Under these hypotheses , for each $f \in H^{-1}(\Omega)$ a unique solution $u \in H_0^1(\Omega)$ exists for the equation:

$$\sum_{i=1}^{n} \int_{\Omega} a_i(x, Du) D_i v \, dx = < f, v > \qquad \forall\, v \in H_0^1(\Omega).$$

The theory of *near operators* is also applicable in more general situations than those considered up to now, as it can be seen from the following theorem.

Theorem 3.6. *Let X and Y be Banach spaces, $\Omega \subset X$ be an open set, and let $A : \Omega \to Y$, $A \in C^1(\Omega)$, $x_0 \in \Omega$.*

If the Fréchet differential $dA(x_0)$ is a bijection between X and Y, then a constant $\sigma > 0$ exists such that the restriction of A to the ball $S(x_0, \sigma)$ is near $dA(x_0)$.

As a consequence of this theorem and Theorem 3.2, we obtain the classic theorem of local invertibility of differentiable functions.

More generally, if A is not differentiable, the question rises whether it is possible to find a smooth operator B, such that A is near B, and under what hypotheses on A, $\mathcal{X}$, $\mathcal{B}$ does this occurs. For the moment, a positive answer has been given to this question in the case $A : \Omega \to \mathbb{R}^m$, where Ω is an open set in $\mathbb{R}^m$ (see [15]). In this case, we succeed in finding a *good* operator B, such that A *is near* B provided A is *injective* and *continuous* on Ω and if a particular hypothesis of monotonicity is valid for A. More precisely, if A *is injective and continuous* on Ω and if $\exists c \in (0, 1)$ and $r > 0$ such that $\forall x, y \in \Omega$ and $\forall z \in S(0, r)$ the condition

$$(A(x + z) - A(y + z)|A(x) - A(y))_{\mathbb{R}^m} \geq c\|A(x) - A(y)\|^2_{\mathbb{R}^m}$$

is satisfied, it can be shown that, for a fixed $x_0 \in \Omega$, a neighbourhood $U(x_0) \subset \Omega$ and an operator $B : U(x_0) \to \mathbb{R}^m$ exist such that:

(a) B is differentiable in x_0,
(b) $dB(x_0)$ is invertible,
(c) A is near B in $U(x_0)$.

4. A theorem on implicit functions.

In view of the strict link existing between the open mapping theorem and the theorem of implicit functions, a result can also be found in the theory of *near operators* that concerns implicit functions (see [14]).

Theorem 4.1. *Let X be a topological space, Y be a set, Z be a Banach space with norm $\|\cdot\|$; moreover let $F : X \times Y \to Z$, and $B : Y \to Z$ be prescribed functions such that:*

(1) $\exists (x_0, y_0) \in X \times Y$ *such that* $F(x_0, y_0) = 0$;
(2) $x \to F(x, y_0)$ *is continuous in* $x = x_0$;
(3) *there are* $\alpha > 0$, $k \in (0, 1)$ *and a neighbourhood of* $U(x_0)$ *of* x_0 *in X such that* $\forall y_1, y_2 \in Y$ *and* $\forall x \in U(x_0)$ *we have*

(9) $$\left\| B(y_1) - B(y_2) - \alpha[F(x, y_1) - F(x, y_2)] \right\| \leq k \left\| B(y_1) - B(y_2) \right\|;$$

(4) *B is injective;*
(5) *$B(Y)$ is a neighbourhood of $z_0 = B(y_0)$.*

Under these hypotheses there is a ball $S(z_0, \sigma) \subset B(Y)$ and a neighbourhood $V(x_0) \subset U(x_0)$ such that a unique function $f : V(x_0) \to B^{-1}(S(z_0, \sigma))$ exists which is the solution of the problem:

(10) $$\begin{cases} F(x, f(x)) = 0 \\ f(x_0) = y_0 . \end{cases}$$

We observe that the hypotheses (3), (4) and (5) replace the hypotheses of differentiability with respect to the y variable and the invertibility of the differential in the classic Hildebrandt-Graves theorem.

From Theorem 4.1 the following theorem can be deduced (see [15]).

Theorem 4.2. *Let the function $F : \Omega \to \mathbb{R}^n$ be continuous in the open set Ω in $\mathbb{R}^m \times \mathbb{R}^n$. Assume that:*

(1) *$\forall x, \quad y \to F(x, y)$ is injective;*
(2) *there are $c \in (0, 1)$ and $r > 0$ such that $\forall (x, y_1), (x, y_2) \in \Omega$ and $\forall z \in S(0, r)$*

$$(F(x, y_1 + z) - F(x, y_2 + z) | F(x, y_1) - F(x, y_2))_{\mathbb{R}^n} \geq$$

$$\geq c \left\| F(x, y_1) - F(x, y_2) \right\|^2_{\mathbb{R}^n}.$$

Finally, let $(x_0, y_0) \in \Omega$ be such that

$$F(x_0, y_0) = 0.$$

Then there are two neighbourhoods $U(x_0)$ and $V(y_0)$ such that there is a unique function $f : U \to V$ which solve

(11) $$\begin{cases} F(x, f(x)) = 0 \\ f(x_0) = y_0 . \end{cases}$$

5. A topology on the set of operators.

The term *near*, introduced by means of Definition 1.1, is not purely formal, but in fact indicates a relation of topological nearness between operators, in the following sense.

Let $\mathcal{X}$ be a set and $\mathcal{B}$ be a Banach space with the norm $\|\cdot\|$. Let us denote with $\mathcal{A}$ and $\mathcal{H}$ the sets:

$$\mathcal{A} = \{B,\, B : \mathcal{X} \to \mathcal{B}\},$$
$$\mathcal{H} = \{\Phi,\, \Phi : \mathcal{X} \to \mathcal{B},\, \Phi \text{ is a bijection between } \mathcal{X} \text{ and } \mathcal{B}\}.$$

We want to define a topology τ on the set $\mathcal{A}$ such that $\mathcal{H}$ is open in $\mathcal{A}$ with the topology τ.

The topology τ can be identified by selecting a neighbourhoods base on $\mathcal{A}$, defined in the following way:

$$\mathcal{U}(B) = \{U_k(B),\, k \in (0,1)\},$$

where for each $B \in \mathcal{A}$ and $k \in (0,1)$ we set

$$U_k(B) = \Big\{A : \mathcal{X} \to \mathcal{B} \quad \text{such that} \quad \forall x_1, x_2 \in \mathcal{X} \quad \text{we have}$$
$$\big\|B(x_1) - B(x_2) - [A(x_1) - A(x_2)]\big\| \le k \big\|B(x_1) - B(x_2)\big\|\Big\}.$$

It is not difficult to verify that $\mathcal{U}(B)$ satisfies the three properties required for the identification of a neighbourhoods base.

(i) The intersection of a finite family of elements of $\mathcal{U}(B)$ is itself an element of $\mathcal{U}(B)$: in fact for all $h, k \in (0,1)$ and for all $A \in U_k(B) \cap U_h(B)$ we have:

$$\big\|t[B(x_1) - B(x_2)] - t[A(x_1) - A(x_2)]\big\| \le k \big\|t[B(x_1) - B(x_2)]\big\|$$

$$\big\|(1-t)[B(x_1) - B(x_2)] - (1-t)[A(x_1) - A(x_2)]\big\| \le h \big\|(1-t)[B(x_1) - B(x_2)]\big\|.$$

Hence, $U_k(B) \cap U_h(B) \in \mathcal{U}(B)$ if $t \in (0,1)$ because:

$$\big\|B(x_1) - B(x_2) - [A(x_1) - A(x_2)]\big\| =$$
$$= \big\|[t + (1-t)][B(x_1) - B(x_2)] - [t + (1-t)][A(x_1) - A(x_2)]\big\| \le$$
$$\le \big\|t[B(x_1) - B(x_2)] - t[A(x_1) - A(x_2)]\big\| +$$
$$+ \big\|(1-t)[B(x_1) - B(x_2)] - (1-t)[A(x_1) - A(x_2)]\big\| \le$$

$$\leq [tk + (1-t)h]\big\|B(x_1) - B(x_2)\big\|.$$

(ii) If $U_k(B) \in \mathcal{U}(B)$, then $B \in U_k(B)$: indeed

$$0 = \big\|B(x_1) - B(x_2) - [B(x_1) - B(x_2)]\big\| \leq k\big\|B(x_1) - B(x_2)\big\|.$$

(iii) If $U_k(B) \in \mathcal{U}(B)$, then there exists $V_h(B) \in \mathcal{U}(B)$ with the property: for all $C \in V_h(B)$ $\exists W_l(C) \in \mathcal{U}(C)$ such that $W_l(C) \subset U_k(B)$. In fact, let $C \in V_h(B)$, i.e.

$$\big\|B(x_1) - B(x_2) - [C(x_1) - C(x_2)]\big\| \leq h\big\|B(x_1) - B(x_2)\big\|.$$

We consider

$$W_l(C) = \Big\{A : \mathcal{X} \to \mathcal{B} : \forall x_1, x_2 \in \mathcal{X} \quad \text{we have}$$
$$\big\|C(x_1) - C(x_2) - [A(x_1) - A(x_2)]\big\| \leq l\big\|C(x_1) - C(x_2)\big\|\Big\}.$$

Let us show that there exists $l \in (0, 1)$ such that if $A \in W_l(C)$, then $A \in U_k(B)$:

$$\begin{aligned}
&\big\|B(x_1) - B(x_2) - [A(x_1) - A(x_2)]\big\| \leq \\
&\leq \big\|B(x_1) - B(x_2) - [C(x_1) - C(x_2)]\big\| + \\
&+ \big\|C(x_1) - C(x_2) - [A(x_1) - A(x_2)]\big\| \leq \\
&\leq h\big\|B(x_1) - B(x_2)\big\| + l\big\|C(x_1) - C(x_2)\big\| \leq \\
&\leq h\big\|B(x_1) - B(x_2)\big\| + l\big\|B(x_1) - B(x_2) - [C(x_1) - C(x_2)]\big\| + \\
&+ l\big\|B(x_1) - B(x_2)\big\| \leq (h + lh + l)\big\|B(x_1) - B(x_2)\big\|,
\end{aligned}$$

we obtain the thesis by choosing $l < \frac{k-h}{1+h}$, with $0 < h < k$.

Moreover, it is also simple to verify that the set $\mathcal{H}$ is open in $\mathcal{A}$ with respect to the topology τ defined above.

In fact, if $B \in \mathcal{H}$ then $U_k(B)$ (with $0 < k < 1$) cannot intersect $\mathcal{A} \setminus \mathcal{H}$: indeed, if $A \in U_k(B)$ then, according to Theorem 1.2, A is a bijection and therefore $A \notin \mathcal{A} \setminus \mathcal{H}$.

Remark 1. In the topology of the uniform convergence $\mathcal{H}$ is not open in $\mathcal{A}$. Indeed, it is enough to consider as an example $\mathcal{X} = [-1, 1]$, $\mathcal{B} = \mathbb{R}$ and the functions $f, f_n : \mathcal{X} \to \mathcal{B}$ defined in the following way : $f(x) = x^3$, $f_n(x) = x^3 - \frac{2}{n}x^2 + \frac{1}{n^2}x$. It is evident that $f_n \to f$ uniformly on $\mathcal{X}$ but the functions f_n cannot be inverted.

Remark 2. The above defined topology is not a Hausdorff topology on $\mathcal{A}$. Indeed, $U_k(B) = U_k(B + T_y)$, $\forall y \in \mathcal{B}$, (where $T_y(x) = y, \forall x \in \mathcal{X}$) so that B and $B + T_y$ cannot be separated.

On the contrary, all operators taking the same value at an assigned point can be separated. For example, consider $\mathcal{A}_0 = \{B : \mathcal{X} \to \mathcal{B} : B(x_0) = 0\}$ with fixed $x_0 \in \mathcal{X}$: then τ is a Hausdorff topology on $\mathcal{A}_0$.

Remark 3. $U_k(B)$ is convex.

In fact, if we take $D, C \in U_k(B)$ and consider $\forall \lambda \in (0, 1)$ the operator $A_\lambda = \lambda D + (1 - \lambda)C$, then $A_\lambda \in U_k(B)$ since:

$$\big\|B(x_1) - B(x_2) - [A_\lambda(x_1) - A_\lambda(x_2)]\big\| =$$

$$\begin{aligned}
&\big\|B(x_1) - B(x_2) - [\lambda D(x_1) + (1-\lambda)C(x_1) - \lambda D(x_2) - (1-\lambda)C(x_2)]\big\| \le \\
&\le \big\|\lambda[B(x_1) - B(x_2)] - \lambda[D(x_1) - D(x_2)]\big\| + \\
&+ \big\|(1-\lambda)[B(x_1) - B(x_2)] - (1-\lambda)[C(x_1) - C(x_2)]\big\| \le \\
&\le k\lambda\big\|B(x_1) - B(x_2)\big\| + k(1-\lambda)\big\|B(x_1) - B(x_2)\big\| = \\
&= k\big\|B(x_1) - B(x_2)\big\|.
\end{aligned}$$

6. Conclusive considerations.

The theory of near operators allows us to obtain:

(1) a generalization of some important results;
(2) the possibility of demonstrating some already known results in a "simpler manner".

Estimates similar to the *nearness Condition* (1) can be found, for example, in [1] and in [11].

In particular, some open mapping theorems in metric spaces are reported in [11].

Concerning the theorem of implicit functions, a wide bibliography of its several generalizations is given in [2].

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