LE MATEMATICHE
Vol. LIV (1999) – Fasc. I, pp. 3–47

SOME NEW RESULTS ON A LAVRENTIEFF PHENOMENON FOR PROBLEMS OF HOMOGENIZATION WITH CONSTRAINTS ON THE GRADIENT

C. D'APICE - T. DURANTE - A. GAUDIELLO

In this paper we analyze, in the context of a Lavrentieff phenomenon, the process of homogenization for Dirichlet problems of the following type:

$$m_h^p(\Omega, \beta) = \inf \left\{ \int_\Omega f(hx, Du)\, dx + \right.$$
$$+ \int_\Omega \beta u\, dx : \quad u \in W^{1,p}(\Omega)\ (u \in C^1(\Omega) \text{ if } p = 'c1'),$$
$$\left. u = 0 \text{ on } \partial\Omega,\ |Du(x)| \leq \varphi(hx) \text{ for a.e. } x \text{ in } \Omega \right\},$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $\beta \in L^1(\Omega)$, $p \in]n, +\infty]$ or $p = 'c1'$ and under suitable hypothesis on f and φ. This problem has been considered in [20] under different hypothesis on f and φ.

0. Introduction.

The mathematical models of problems concerning nonhomogeneous materials use equations or functionals with periodical coefficients or integrands with small period. A good approximation of the macroscopic behaviour of such materials can be found letting the parameter ϵ that describes the microstructure go to zero. This procedure is called homogenization.

Entrato in Redazione il 30 Ottobre 1998.

A sequence of equations or functionals is considered and, using an appropriate convergence, the limit equation or functional that describes the macroscopic properties is found. This allows to replace a highly nonhomogeneous medium with an equivalent homogeneous material.

On the other hand it's very important in the study of physical problems schematizable by minimization of a functional of Calculus of Variations the choice of the class of functions. Infact since 1926 it was pointed out an unexpected phenomenon concerning an integral functional of Calculus of Variations.

The considered functional was naturally defined and lower semicontinuous (with respect the L^1 topology) on the set of the absolutely continuous functions defined on the interval [0,1]; moreover on the set of the Lipschitz functions u of this kind and such that $u(0) = 0$ and $u(1) = 1$ a minimum value was attained. This value surprisingly enough was strictly lower than the infimum value of the same functional computed on the set of Lipschitz functions with the same boundary conditions (Lavrentieff phenomenon); this fact implies that, for example, this minimum value cannot be approximated by finite elements method. Other examples of the same phenomenon concerning much simpler functionals were shown in [38].

In this paper we study the homogenization of variational problems for integral functionals defined on functions subject to oscillating constraints on the gradient that can describe some phenomena in elastic-plastic torsion and elastatics. These problems can show the presence of a Lavrentieff phenomenon and, being the integral functional suitable for a process of homogenization, its persistence after this process. Precisely we analyze the homogenization for Dirichlet problems of following type:

$$
\begin{aligned}
m_h^p(\Omega, \beta) = \inf \Big\{ & \int_\Omega f(hx, Du)\, dx + \\
& + \int_\Omega \beta u\, dx : u \in W^{1,p}(\Omega)\ (u \in C^1(\Omega) \text{ if } p = {}'c1'), \\
& u = 0 \text{ on } \partial\Omega,\ |Du(x)| \leq \varphi(hx) \text{ for a.e. } x \text{ in } \Omega \Big\},
\end{aligned} \tag{0.1}
$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $\beta \in L^1(\Omega)$, $p \in]n, +\infty]$ or $p = {}'c1'$, and f, φ are functions satisfying the following conditions (here and in the sequel $Y =]0, 1[^n$):

$$
\begin{cases}
f : (x, z) \in \mathbb{R}^n \times \mathbb{R}^n \to f(x, z) \in [0, +\infty[, \\
f \text{ measurable and } Y\text{-periodic in the } x \text{ variable, convex in the } z \text{ one}, \\
f(\cdot, z) \in L^1(Y) \quad \forall z \in \mathbb{R}^n, \\
\varphi : x \in \mathbb{R}^n \to \varphi(x) \in [0, +\infty[, \\
\varphi\ Y\text{-periodic}.
\end{cases} \tag{0.2}
$$

A classical conjecture (see for instance [3] and [8]), suggests

$$(0.3)\quad \lim_{h\to+\infty}\{m_h^p(\Omega,\beta)\}_{h\in N} = m_{\text{hom}}^p(\Omega,\beta) = \inf\Big\{\int_\Omega f_{\text{hom}}^p(Du)\,dx + \\ + \int_\Omega \beta u\,dx : u\in W^{1,p}(\Omega)\ (u\in C^1(\Omega) \text{ if } p='c1'),\ u=0 \text{ on } \partial\Omega\Big\},$$

where f_{hom}^p is the convex function from $\mathbb{R}^n$ to $[0,+\infty]$ defined by

$$(0.4)\quad f_{\text{hom}}^p(z) = \inf\Big\{\int_Y f(y, z+Du)\,dy : u\in W^{1,p}(Y)\ (u\in C^1(Y) \\ \text{if } p='c1'),\, u\ Y\text{-periodic},\ |z+Du(y)|\le\varphi(y) \text{ a.e. in } Y\Big\}\quad z\in\mathbb{R}^n$$

(in (0.4) it is assumed that $\inf\emptyset=+\infty$).

It is possible to verify, with some examples, that the function f_{hom}^p really depends on p (see Remarks 1.12 and 1.13).

Out of the context of the Lavrentieff phenomenon, convergence as in (0.3) has been verified in many papers under different assumptions (see [1], [8]–[16], [18], [22]–[26]).

In the context of the Lavrentieff phenomenon, convergence (0.3) has been analyzed in [19] and [20]. Precisely in [20] it has been proved that if

$$(0.5)\quad \text{there exist } \vartheta\in[0,\tfrac12[\text{ and } m>0 \text{ such that } 0<m\le\varphi(y) \\ \text{for a.e. } y \text{ in }]0,1[^n\setminus]\tfrac12-\vartheta,\tfrac12+\vartheta[^n$$

and if (0.2) together with one of the following conditions is satisfied:

$$(0.6)\quad k|z|^q \le f(x,z)\quad \text{a.e. } x \text{ in } \mathbb{R}^n,\quad z \text{ in } \mathbb{R}^n,\ k>0, q>n$$

or

$$(0.7)\quad \varphi\in L^q(Y)\quad q\in]n,+\infty],$$

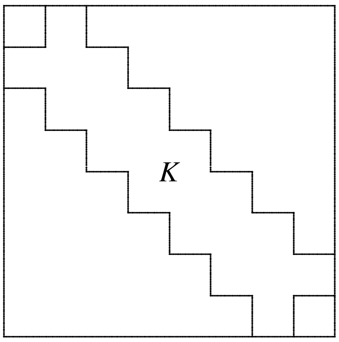
then (0.3) holds.

In this paper we prove (0.3) with the hypothesis (0.5) replaced by

$$(0.8)\quad \exists\alpha\in\mathbb{R}_+ : \int_Y f(y,\pm\sqrt{n}\alpha\varphi(y)e_j)\,dy<+\infty\quad \forall j\in\{1,\dots,n\},$$

where $\{e_1,\dots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. More precisely if f and φ satisfy (0.2), (0.7), (0.8), then (0.3) holds. Moreover an analysis of the convergence of the subsequence of the minimum points of the relaxed problems in (0.1) is performed.

Observe that the assumption (0.8) allows to analyse some type of constraints on the gradient that are not examined by assumption (0.5). For example, if φ is bounded assumption (0.8) but not necessarily (0.5) is satisfied (see fig.1).



$y = \chi_{y-k}$

fig. 1

We make use of the Γ-convergence introduced by E. De Giorgi and of the identification techniques of Γ-limits contained in [18], [20] and [26].

1. Notations and preliminaries.

We recall the definition and the main properties of Γ^- convergence (see also [28]).

Let (U, τ) be a topological space satisfying the first countability axiom and denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

Definition 1.1. *Let F_h, $h \in \mathbb{N}$, F' and F'' be functionals from U to $\overline{\mathbb{R}}$.*

We say that F' is the $\Gamma^-(\tau)$-lower limit of $\{F_h\}_{h\in N}$ and we write

$$F'(u) = \Gamma^-(\tau) \liminf_{h\to+\infty} F_h(u) \quad \forall u \in U, \tag{1.1}$$

if the following conditions are satisfied:

$$u \in U, \{u_h\}_{h\in N} \subseteq U, u_h \xrightarrow{\tau} u \Rightarrow F'(u) \leq \liminf_{h\to+\infty} F_h(u_h), \tag{1.2}$$

$$\forall u \in U, \exists \{u_h\}_{h\in N} \subseteq U : u_h \xrightarrow{\tau} u,\ F'(u) \geq \liminf_{h\to+\infty} F_h(u_h). \tag{1.3}$$

We say that F'' is the $\Gamma^-(\tau)$-upper limit of $\{F_h\}_{h\in N}$ and we write

$$F''(u) = \Gamma^-(\tau) \limsup_{h\to+\infty} F_h(u) \quad \forall u \in U, \tag{1.4}$$

if (1.2) *and* (1.3) *hold with the operator* "lim inf" *replaced by* "lim sup".

When $F' = F''$, we say that $\{F_h\}_{h\in N}$ $\Gamma^-(\tau)$-converges to $F'(= F'')$ on U and we write

$$F'(u) = F''(u) = \Gamma^-(\tau) \lim_{h\to+\infty} F_h(u) \quad \forall u \in U. \tag{1.5}$$

Remark 1.2. *Since (U, τ) satisfies the first countability axiom, for every u in U the subsets of $\overline{\mathbb{R}}$:*

$$\left\{ \liminf_{h\to+\infty} F_h(u_h) : \{u_h\}_{h\in N} \subseteq U \text{ and } u_h \xrightarrow{\tau} u \right\}$$
$$\text{and} \left\{ \limsup_{h\to+\infty} F_h(u_h) : \{u_h\}_{h\in N} \subseteq U \text{ and } u_h \xrightarrow{\tau} u \right\}$$

have minima. Consequently, the limits in (1.1) *and* (1.4) *exist and are given by*

$$F'(u) = \min \left\{ \liminf_{h\to+\infty} F_h(u_h) : \{u_h\}_{h\in N} \subseteq U \text{ and } u_h \xrightarrow{\tau} u \right\}, \tag{1.6}$$

$$F''(u) = \min \left\{ \limsup_{h\to+\infty} F_h(u_h) : \{u_h\}_{h\in N} \subseteq U \text{ and } u_h \xrightarrow{\tau} u \right\}. \tag{1.7}$$

Recall the following properties of Γ^--convergence proved in [28].

Proposition 1.3. [28]. *Let $\{F_h\}_{h\in N}$ be a sequence of functionals from U to $\overline{\mathbb{R}}$.*

Then the functionals $\Gamma^-(\tau) \liminf_{h\to+\infty} F_h$ and $\Gamma^-(\tau) \limsup_{h\to+\infty} F_h$ are τ-lower semicontinuous on U.

Moreover, if $\{h_k\}_{k\in N}$ is an increasing sequence of integer numbers, it results

$$\Gamma^-(\tau) \liminf_{h\to+\infty} F_h(u) \leq \Gamma^-(\tau) \liminf_{k\to+\infty} F_{h_k}(u) \leq$$
$$\leq \Gamma^-(\tau) \limsup_{k\to+\infty} F_{h_k}(u) \leq \Gamma^-(\tau) \limsup_{h\to+\infty} F_h(u), \quad \forall u \in U.$$

Definition 1.4. [28]. *Let $\{F_h\}_{h\in N}$ be a sequence of functionals from U to $\overline{\mathbb{R}}$.*

We say that the functionals F_h are equicoercive, if for every real number c there exists a compact set K_c in U such that

$$\{u \in U : F_h(u) \leq c\} \subseteq K_c , \quad \forall h \in \mathbb{N}.$$

If F is a functional from U to $\overline{\mathbb{R}}$, $sc^{-}(\tau)F$ denotes the greatest τ-lower semicontinuous functional on U less than or equal to F.

Theorem 1.5. [28]. *Let $F_h (h \in \mathbb{N})$ and G be functionals from U to $\overline{\mathbb{R}}$. Assume that there exists*

$$F(u) = \Gamma^{-}(\tau) \lim_{h\to+\infty} F_h(u) \quad \forall u \in U, \tag{1.8}$$

that G is a τ-continuous functional and the functionals F_h+G are equicoercive. Then the functional $F + G$ attains its minimum on U and

$$\min\Big\{F(v) + G(v) : v \in U\Big\} = \lim_{h\to+\infty} \inf\Big\{F_h(v) + G(v) : v \in U\Big\}.$$

Moreover, if $u_h \in U$ is a solution of

$$\min\Big\{sc^{-}(\tau)F_h(v) + G(v) : v \in U\Big\} \quad h \in \mathbb{N}$$

and

$$u_h \xrightarrow{\tau} u,$$

then u is a solution of

$$\min\Big\{F(v) + G(v) : v \in U\Big\}.$$

Introduce, now, some notations.

If A and B are two bounded open subsets of $\mathbb{R}^n$ such that $\overline{A} \subseteq B$, write $A \subset\subset B$.

A $\overline{\mathbb{R}}$ valued function G, defined on the set of the bounded open subsets of $\mathbb{R}^n$, is increasing if

$$\Omega_1 \subseteq \Omega_2 \Rightarrow G(\Omega_1) \leq G(\Omega_2).$$

For an increasing function G, the inner regular envelope G_{-} of G (see [29]) on an open subset Ω of $\mathbb{R}^n$ is defined by

$$G_{-}(\Omega) = \sup_{A\subset\subset\Omega} G(A). \tag{1.9}$$

For every bounded open subset Ω of $\mathbb{R}^n$, $C^\circ(\Omega)$ and $C^\circ_\circ(\Omega)$ denote the topologies induced on $C^\circ(\mathbb{R}^n)$ respectively by the extended metrics

$$d(u, v) = \|u - v\|_{C^\circ(\Omega)} = \sup_{x\in\Omega} |u(x) - v(x)|,$$

$$\delta(u, v) = \begin{cases} d(u, v) & \text{if } u = v \text{ on } \partial\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

For every p in $[1, +\infty[$, $W^{1,p}_{\text{per}}(Y)$ denotes the set of the functions u in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ with u Y-periodic. For every bounded open subset Ω of $\mathbb{R}^n$, $W^{1,p}_\circ(\Omega)$ denotes the set of the functions u in $W^{1,p}(\Omega)$ with $u = 0$ on $\partial\Omega$. $C^1_{\text{per}}(Y)$, $\text{Lip}_{\text{per}}(Y)$, $C^1_\circ(\Omega)$, $\text{Lip}_\circ(\Omega)$ are introduced in a similar way.

For any z in $\mathbb{R}^n$, u_z denotes the function defined by

$$u_z(x) = z \cdot x \quad \forall x \in \mathbb{R}^n.$$

Recall that a subset of $\mathbb{R}^n$ is said to be a polyhedron if it is intersection of a finite number of half spaces and call a function u on $\mathbb{R}^n$ a piecewise affine function if it is $C^\circ(\mathbb{R}^n)$ and can be expressed as

$$u(x) = \sum_{j=1}^m (u_{z_j}(x) + s_j)\chi_{\overset{\circ}{P}_j}(x) \quad \forall x \in \bigcup_{j=1}^m \overset{\circ}{P}_j,$$

where $z_1, \ldots, z_m \in \mathbb{R}^n$, $s_1, \ldots, s_m \in \mathbb{R}$ and $P_1, \ldots, P_m$ are piecewise disjoint polyhedrons with nonempty interiors such that $\bigcup\limits_{j=1}^m P_j = \mathbb{R}^n$.

For every subset P of $\mathbb{R}^n$ and $\epsilon > 0$, P^+_ϵ and P^-_ϵ denote the open sets defined by

$$P^+_\epsilon = \{x \in \mathbb{R}^n : \text{dist}\,(x, P) < \epsilon\}, \ P^-_\epsilon = \{x \in P : \ \text{dist}\,(x, \partial P) > \epsilon\}.$$

If $x_\circ$ belongs to $\mathbb{R}^n$ and r is in $]0, +\infty[$, denote

$$B_r(x_\circ) = \Big\{x \in \mathbb{R}^n : |x - x_\circ| < r\Big\}.$$

Let α be a nonnegative function in $C^\infty_\circ(\mathbb{R}^n)$, whit support contained in $B_1(0)$, such that $\int_{\mathbb{R}^n} \alpha(y)\,dy = 1$ and let u be in $L^1_{\text{loc}}(\mathbb{R}^n)$. For every $\epsilon > 0$ set $\alpha^{(\epsilon)}(y) = \epsilon^{-n}\alpha(y/\epsilon)$, y in $\mathbb{R}^n$, and define the ϵ-regularization u_ϵ of u by

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \alpha^{(\epsilon)}(x - y)u(y)\,dy, \quad x \in \mathbb{R}^n. \tag{1.10}$$

If φ satisfies (0.2), set

$$\varphi_h(x) = \varphi(hx) \qquad \forall x \in \mathbb{R}^n, \quad \forall h \in \mathbb{N}.$$

By virtue of the Sobolev embedding theorem, $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \subset C^\circ(\mathbb{R}^n)$ with continuous injection, for $p \in]n, +\infty]$. Then, for every bounded open subset Ω of $\mathbb{R}^n$, $h \in \mathbb{N}$, and p in $]n, +\infty]$, consider the following functionals on $C^\circ(\mathbb{R}^n)$:

$$(1.11) \qquad F^p_h(\Omega, u) = \begin{cases} \displaystyle\int_\Omega f(hx, Du)\,dx \quad \text{if } u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n), \\ \qquad\qquad |Du(x)| \le \varphi_h(x) \text{ for a.e. } x \text{ in } \Omega, \\ +\infty \quad \text{otherwise on } C^\circ(\mathbb{R}^n), \end{cases}$$

$$(1.12) \qquad F^{c1}_h(\Omega, u) = \begin{cases} \displaystyle\int_\Omega f(hx, Du)\,dx \quad \text{if } u \in C^1(\mathbb{R}^n), \\ \qquad\qquad |Du(x)| \le \varphi_h(x) \text{ for a.e. } x \text{ in } \Omega, \\ +\infty \quad \text{otherwise on } C^\circ(\mathbb{R}^n). \end{cases}$$

For every bounded open subset Ω of $\mathbb{R}^n$ and for every p in $]n, +\infty]$ or $p = 'c1'$ set

$$(1.13) \qquad \begin{cases} F'^p(\Omega, u) = \Gamma^-(C^\circ(\Omega)) \liminf\limits_{h\to+\infty} F^p_h(\Omega, u), \quad \forall u \in C^\circ(\mathbb{R}^n), \\ F''^p(\Omega, u) = \Gamma^-(C^\circ(\Omega)) \limsup\limits_{h\to+\infty} F^p_h(\Omega, u), \quad \forall u \in C^\circ(\mathbb{R}^n). \end{cases}$$

If

$$F'^p(\Omega, u) = F''^p(\Omega, u) \quad \forall u \in C^\circ(\mathbb{R}^n),$$

set

$$(1.14) \qquad F^p(\Omega, u) = \Gamma^-(C^\circ(\Omega)) \lim_{h\to+\infty} F^p_h(\Omega, u) \quad \forall u \in C^\circ(\mathbb{R}^n).$$

Moreover, set

$$(1.15) \qquad \begin{cases} F'^p_\circ(\Omega, u) = \Gamma^-(C^\circ_\circ(\Omega)) \liminf\limits_{h\to+\infty} F^p_h(\Omega, u), \quad \forall u \in C^\circ(\mathbb{R}^n), \\ F''^p_\circ(\Omega, u) = \Gamma^-(C^\circ_\circ(\Omega)) \limsup\limits_{h\to+\infty} F^p_h(\Omega, u), \quad \forall u \in C^\circ(\mathbb{R}^n). \end{cases}$$

If

$$F'^p_\circ(\Omega, u) = F''^p_\circ(\Omega, u) \qquad \forall u \in C^\circ(\mathbb{R}^n),$$

set

$$F^p_{\circ}(\Omega, u) = \Gamma^-(C^\circ_\circ(\Omega)) \lim_{h\to+\infty} F^p_h(\Omega, u), \quad \forall u \in C^\circ(\mathbb{R}^n). \tag{1.16}$$

By virtue of (0.2), for every u in $C^\circ(\mathbb{R}^n)$ and for every p in $]n, +\infty]$ or $p = 'c1'$, the above set functions $F'^p(\cdot, u)$ and $F''^p(\cdot, u)$ are increasing. Consequently define $F\_'^p(\Omega, u)$ and $F\_''^p(\Omega, u)$ by (1.9) written with $G = F'^p(\cdot, u)$ and $G = F''^p(\cdot, u)$ respectively.

For every p in $]n, +\infty]$ or $p = 'c1'$, the following properties hold:

$$F'^p(\Omega, u) \le F''^p(\Omega, u) \text{ for every bounded open subset } \Omega \text{ of } \mathbb{R}^n \text{ and } u \text{ in } C^\circ(\mathbb{R}^n). \tag{1.17}$$

$$F'^p(\Omega, u + c) = F'^p(\Omega, u) \text{ and } F''^p(\Omega, u + c) = F''^p(\Omega, u) \text{ for every bounded open subset } \Omega \text{ of } \mathbb{R}^n,\ u \text{ in } C^\circ(\mathbb{R}^n) \text{ and } c \text{ in } \mathbb{R}. \tag{1.18}$$

$$F'^p(\Omega, u_1) = F'^p(\Omega, u_2) \text{ and } F''^p(\Omega, u_1) = F''^p(\Omega, u_2) \tag{1.19}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and u_1, u_2 in $C^\circ(\mathbb{R}^n)$ with $u_1 = u_2$ in Ω.

For every function $g : \mathbb{R}^n \to]-\infty, +\infty]$ we set $\operatorname{dom} g = \{z \in \mathbb{R}^n : g(z) < +\infty\}$.

The following results yield some properties of the function f^p_{hom} defined in (0.4).

Proposition 1.6. *Let f and φ be functions satisfying* (0.2), (0.8), *α be the constant given in* (0.8) *and, for p in $]n, +\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4).

Then

i) $\operatorname{dom} f^p_{\text{hom}}$ *is a convex subset of* $\mathbb{R}^n$*;*
ii) f^p_{hom} *is a convex function on* $\mathbb{R}^n$*;*
iii) $0 \in \operatorname{dom} f^p_{\text{hom}}$*;*
iv) 0 *belongs to* $(\operatorname{dom} f^p_{\text{hom}})^\circ$ *if* $(\operatorname{dom} f^p_{\text{hom}})^\circ$ *is not empty;*
v) $\operatorname{dom} f^p_{\text{hom}}$ *is bounded if f and φ satisfy* (0.7) *too;*
vi) f^p_{hom} *is bounded on* $B_{\alpha r}(0)$*, if* $B_r(0)$ *is included in* $\operatorname{dom} f^p_{\text{hom}}$*;*
vii)
$$f^p_{\text{hom}}(z) = \inf\left\{ \int_Y f(hx, z + Dv)\, dx : v \in W^{1,p}_{\text{per}}(Y)\ (C^1_{\text{per}}(Y) \text{ if } p =' c1'),\ |z + Dv| \le \varphi_h \text{ a.e. in } Y \right\} \quad \forall z \in \mathbb{R}^n,\ h \in \mathbb{N}.$$

Proof. The statements i), ii) and iii) immediately follow from Definition (0.4).

Prove now iv) for p in $]n, +\infty[$. The proof is similar in the other cases.

Since $(\mathrm{dom} f^p_{\mathrm{hom}})^\circ$ is not empty, there exist $z_\circ$ in $\mathbb{R}^n$ and r in $]0, +\infty[$ such that

$$B_r(z_\circ) \subseteq \mathrm{dom} f^p_{\mathrm{hom}}. \tag{1.20}$$

Let α be the constant given in (0.8) (observe that it is not restrictive to assume α in $]0, 1[$) and verify that

$$B_{\alpha r}(\alpha z_\circ) \subseteq \mathrm{dom} f^p_{\mathrm{hom}}, \qquad B_{\alpha r}(-\alpha z_\circ) \subseteq \mathrm{dom} f^p_{\mathrm{hom}}. \tag{1.21}$$

Let z be in $B_{\alpha r}(\alpha z_\circ)$ (resp. $B_{\alpha r}$ $(-\alpha z_\circ)$), then $\frac{1}{\alpha}z$ belongs to $B_r(z_\circ)$ (resp. $-\frac{1}{\alpha}z$ belongs to $B_r(z_\circ)$). Consequently, by virtue of (1.21),

$$\exists\, w_1 \in W^{1,p}_{\mathrm{per}}(Y) : |z + Dw_1| \le \alpha\varphi \text{ a.e. in } Y \text{ (resp. } \exists\, w_2 \in W^{1,p}_{\mathrm{per}}(Y) :$$
$$|z + D(-w_2)| = |-z + Dw_2| \le \alpha\varphi \text{ a.e. in } Y),$$

from which it follows that, $z+Dw_1(y)$ (resp. $z-Dw_2(y)$) belongs to the convex envelope of the set $\left\{\pm\sqrt{n}\alpha\varphi(y)e_j\right\}_{j=1,\dots,n}$ for a.e. y in Y. Consequently, since α is in $]0, 1[$, Definition (0.4) and assumptions (0.2), (0.8) provide that

$$f^p_{\mathrm{hom}}(z) \le \int_Y f(y, z + Dw_1)\, dy \le \sum_{j=1}^n \Big\{ \int_Y f(y, +\sqrt{n}\alpha\varphi(y)e_j)\, dy +$$
$$+ \int_Y f(y, -\sqrt{n}\alpha\varphi(y)e_j)\, dy \Big\} < +\infty,$$

$$f^p_{\mathrm{hom}}(z) \le \int_Y f(y, z - Dw_2)\, dy \le \sum_{j=1}^n \Big\{ \int_Y f(y, +\sqrt{n}\alpha\varphi(y)e_j)\, dy +$$
$$+ \int_Y f(y, -\sqrt{n}\alpha\varphi(y)e_j)\, dy \Big\} < +\infty,$$

i.e. (1.21). By virtue of i), the statement iv) follows from (1.21).

Regarding the proof of v), first observe that (0.7) and (0.8) provide that φ is in $L^1(Y)$. If z is in $\mathrm{dom} f^p_{\mathrm{hom}}$, there exists u in $W^{1,p}_{\mathrm{per}}(Y)(C^1_{\mathrm{per}}(Y)$ if $p = 'c1')$ such that

$$|z + Du| \le \varphi \text{ a.e. in } Y, \qquad \int_Y Du\, dy = 0. \tag{1.22}$$

Consequently

$$|z| = \Big| \int_Y (z + Du)\, dy \Big| \leq \int_Y |z + Du|\, dy \leq \int_Y \varphi\, dy < +\infty,$$

that is

$$\operatorname{dom} f^p_{\text{hom}} \subseteq \Big\{ z \in \mathbb{R}^n : |z| \leq \int_Y \varphi\, dy \Big\}.$$

To prove vi) observe that if z belong to $B_{\alpha r}(0)$, then $\frac{1}{\alpha} z$ is in $B_r(0)$. Then by virtue of our assumption,

$$\exists\, w \in W^{1,p}_{\text{per}}(Y)\ (C^1_{\text{per}}(Y) \text{ if } p = {}'c1') : |z + Dw| \leq \alpha\varphi \qquad \text{a.e. in } Y,$$

from which it follows that $z + Dw$ belong to the convex envelope of the set $\Big\{ \pm \sqrt{n}\alpha\varphi(y)e_j \Big\}_{j=1,\dots,n}$ for a.e. y in Y.
Consequently, since we can assume α less than 1, Definition (0.4) and assumptions (0.2), (0.8) provide that

$$\begin{aligned} f^p_{\text{hom}}(z) &\leq \int_Y f(y, z + Dw)\, dy \leq \\ &\leq \sum_{j=1}^n \int_Y f(y, \pm\sqrt{n}\alpha\varphi(y)e_j)\, dy < +\infty \qquad \forall z \in B_{\alpha r}(0). \end{aligned}$$

The proof of vii) is achieved arguing in the same way of Theorem 7.6 in [17]. □

For every x in $\mathbb{R}^n$, p in $]n, +\infty]$ or $p = {}'c1'$, define

$$K'^p(x) = \Big\{ z \in \mathbb{R}^n : F'^p(I_x, u_z) < +\infty \text{ for some neighbourhood } I_x \text{ of } x \Big\}, \tag{1.23}$$

$$K''^p(x) = \Big\{ z \in \mathbb{R}^n : F''^p(I_x, u_z) < +\infty \text{ for some neighbourhood } I_x \text{ of } x \Big\} \tag{1.24}$$

where, for a fixed z in $\mathbb{R}^n$, $u_z : x \in \mathbb{R}^n \to zx \in \mathbb{R}$.

The following result is proved in Lemma 2.1 of [18]:

Lemma 1.7. [18]. *Let f and φ be functions satisfying* (0.2), (0.8), *for p in* $]n, +\infty]$ *or $p = 'c1'$, let $f^p_{\hom}$ be the function defined in* (0.4) *and, for every x in $\mathbb{R}^n$, $K^{'p}(x)$, $K^{''p}(x)$, be the sets defined in* (1.23) *and* (1.24). *Then*

$$\mathrm{dom} f^p_{\hom} = K^{'p}(x) = K^{''p}(x) \qquad \forall x \in \mathbb{R}^n. \tag{1.25}$$

Let $g : \mathbb{R}^n \to]-\infty, +\infty]$ be a convex function such that 0 belongs to $\mathrm{dom}\, g$. Then the limit

$$\overline{g}(z) = \lim_{t \to 1^-} g(tz) \quad z \in \mathbb{R}^n \tag{1.26}$$

exists for every z in $\mathbb{R}^n$.

The following result is proved in [26]:

Lemma 1.8. [26]. *Let $g : \mathbb{R}^n \to]-\infty, +\infty]$ be a convex function such that 0 belongs to* $\mathrm{dom}\, g$ *and let $\overline{g}$ be the function defined in* (1.26).

Then $\overline{g}$ is convex and

$$\overline{g}(z) \leq g(z) \qquad \forall z \in \mathbb{R}^n, \tag{1.27}$$

$$\overline{g}(z) = g(z) \qquad \forall z \in \mathbb{R}^n \setminus \partial \mathrm{dom}\, g. \tag{1.28}$$

Moreover, if 0 belongs to $(\mathrm{dom}\, g)^\circ$, *$\overline{g}$ is lower semicontinuous on $\mathbb{R}^n$.*

Definition 1.9. *Let f and φ be functions satisfying* (0.2), (0.8) *and, for p in* $]n, +\infty]$ *or $p = 'c1'$, let $f^p_{\hom}$ be the function defined in* (0.4). *Since $f^p_{\hom}$ is a convex function and 0 belongs to* $\mathrm{dom} f^p_{\hom}$ *[see i) and iii) of Proposition 1.6], define $\overline{f}^p_{\hom}$ by*

$$\overline{f}^p_{\hom}(z) = \lim_{t \to 1^-} f^p_{\hom}(tz) \quad z \in \mathbb{R}^n. \tag{1.29}$$

Lemma 1.10. *Let f and φ be functions satisfying* (0.2), (0.8), *α be the constant given in* (0.8), *Ω be a bounded measurable subset of $\mathbb{R}^n$ and $\{m_h\}_{h \in N}$ be a sequence of measurable vectorial functions on Ω such that*

$$\exists h_\circ \in \mathbb{N} : \ \forall h (\in \mathbb{N}) > h_\circ \ \ |m_h(x)| \leq \alpha \varphi_h(x) \ \ \textit{a.e. in } \Omega. \tag{1.30}$$

Then we have

$$\limsup_{h \to +\infty} \int_\Omega f(hx, m_h(x))\, dx \leq c|\Omega|,$$

where

$$c = \sum_{i=1}^n \Big\{ \int_Y f(y, \sqrt{n}\alpha\varphi(y)e_i)\, dy + \\ + \int_Y f(y, -\sqrt{n}\alpha\varphi(y)e_i)\, dy \Big\} < +\infty. \tag{1.31}$$

Proof. By virtue of (1.30), for every integer number $h > h_\circ$ and for a.e. x in Ω, $m_h(x)$ belongs to the convex envelope of the set $\left\{ \pm \sqrt{n}\alpha\varphi_h(x)e_i \right\}_{i=1,\dots,n}$. Consequently assumptions (0.2) and (0.8) provide that

$$\limsup_{h\to+\infty} \int_\Omega f(hx, m_h(x))\, dx \leq$$
$$\leq \sum_{i=1}^n \left\{ \limsup_{h\to+\infty} \int_\Omega f(hx, \sqrt{n}\alpha\varphi_h(x)e_i)\, dx + \right.$$
$$\left. + \limsup_{h\to+\infty} \int_\Omega f(hx, -\sqrt{n}\alpha\varphi_h(x)e_i)\, dx \right\} =$$
$$= |\Omega| \sum_{i=1}^n \left\{ \int_Y f(y, \sqrt{n}\alpha\varphi(y)e_i)\, dy + \int_Y f(y, -\sqrt{n}\alpha\varphi(y)e_i)\, dy \right\}. \qquad \square$$

The following result is proved in Lemma 1.3 of [20].

Lemma 1.11. [20]. *Let f and φ be functions satisfying* (0.2)*, let Ω be a bounded open subset of $\mathbb{R}^n$ and, for p in $]n, +\infty]$ or $p = 'c1'$, let F'^p be defined in* (1.13)*. Then,*

$$F'^p(\Omega, tu) \leq tF'^p(\Omega, u) + |\Omega|(1-t) \int_Y f(y, 0)\, dy \tag{1.32}$$

for every u in $C^\circ(\mathbb{R}^n)$ and t in $[0, 1]$.

Similar inequalities hold for F'^p_-, F''^p, F''^p_- in place of F'^p.

We prove through the following examples that the function $f^p_{\hom}$ really depends on p.

Remark 1.12. *Let $n = 1$, f be a function satisfying* (0.2) *and K be a closed set such that $K \subseteq Y$, $|K| = \frac{1}{2}$, $\overset{\circ}{K} = \phi$ and $\varphi = \chi_K$.*

For p in $]n, +\infty]$ or $p = 'c1'$, let $f^p_{\hom}$ be the function defined in (0.4)*.*

Then, in [20] *it is proved that there exists $c \in \mathbb{R}_+$ depending on f such that*

$$\overline{f}^p_{\hom}(z) \leq f^p_{\hom}(z) \leq c < \overline{f}^{c1}_{\hom}(z) = f^{c1}_{\hom}(z) = +\infty$$

for every $z \in \mathbb{R}$ such that $0 < |z| < \frac{1}{2}$ *and for every p in* $]n, +\infty]$.

Remark 1.13. *Let w be the function on $\mathbb{R}^2$ defined by*

$$w(x_1, x_2) = \frac{x_1}{\max\{|x_1|, |x_2|\}} \text{ a.e. in } P =]-1, 1[^2,\ w - x_1\ P\text{-periodic.}$$

Let q in $]1, 2[$ and define the function

$$f : (x, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \to f(x, z) = |\det[Dw(x), z]| + |z|^q.$$

For p in $[1, +\infty]$, set

$$\hat{f}^p_{\text{hom}}(z) = \inf\left\{\frac{1}{\text{meas}(P)} \int_P f(y, z + Dv) : v \in W^{1,p}_{\text{per}}(P)\right\}, \quad z \in \mathbb{R}^n.$$

In [19], Section 6, it is proved that

$$\hat{f}^p_{\text{hom}}(z) \geq |z_2| + |z|^q,\ \forall z = (z_1, z_2) \in \mathbb{R}^2,\ p > 2, \tag{1.33}$$

$$\hat{f}^p_{\text{hom}}(z) \leq c_q|z|^q,\ \forall z = (z_1, z_2) \in \mathbb{R}^2,\ 1 \leq p \leq 2, \tag{1.34}$$

where c_q is a positive constant only depending on q.

Fix now $\overline{z}$ in the non empty open set $A = \left\{z \in \mathbb{R}^2 : |z_2| + |z|^q > c_q|z|^q\right\}$ and consider the function

$$\varphi(x_1, x_2) = \frac{\overline{z}x}{\max\{|x_1|, |x_2|\}} \qquad \text{a.e. in } P =]-1, 1[^2, \varphi - u_z\ P\text{-periodic.}$$

For p in $[1, +\infty]$, set

$$f^p_{\text{hom}}(z) = \inf\left\{\frac{1}{\text{meas}(P)} \int_P f(y, z + Dv) : v \in W^{1,p}_{\text{per}}(P),\ |z + Dv| \leq |D\varphi|\right\},$$
$$z \in \mathbb{R}^n.$$

From (1.33) it follows that

$$f^p_{\text{hom}}(z) \geq \hat{f}^p_{\text{hom}}(z) \geq |z_2| + |z|^q, \quad \forall z = (z_1, z_2) \in \mathbb{R}^2,\ p > 2. \tag{1.35}$$

On the other hand, arguing as in (6.12) in [19], it results that

$$f^{p'}_{\text{hom}}(\overline{z}) \leq c_q|\overline{z}|^q < |\overline{z}_2| + |\overline{z}|^q,\ 1 \leq p' \leq 2. \tag{1.36}$$

Combining (1.35) with (1.36), it follows that

$$f^{p'}_{\text{hom}}(\overline{z}) < f^p_{\text{hom}}(\overline{z}), \quad 1 \leq p' \leq 2 < p \leq +\infty.$$

2. Some properties of Γ-limits.

In this section we adapt some results on the sub-additivity of Γ-limits proved in [18].

Let f and φ be function satisfying (0.2), (0.8), α is the constant given in (0.8) and for p in $]n, +\infty]$ or $p = {}'c1'$, let f^p_{hom} be the function defined in (0.4). If $(\text{dom} f^p_{\text{hom}})^\circ \neq \emptyset$, iv) of Proposition 1.6 provides that

$$\exists\, \delta \in (0,1) : B_{(r/\alpha)\delta}(0) \subseteq \ \text{dom} f^p_{\text{hom}}. \tag{2.1}$$

Obviously δ depends on p.

By arguing as in Lemma 2.3 in [18], it is easy to prove the following result:

Lemma 2.1. *Let f and φ be functions satisfying* (0.2), (0.8) *and, for p in $]n, +\infty]$ or $p = {}'c1'$, let f^p_{hom} be the function defined in* (0.4). *Assume that $(\text{dom} f^p_{\text{hom}})^\circ \neq \emptyset$ and let δ be the constant given in* (2.1).

Then there exists a constant M dependent only on n, and φ such that for every bounded open subset Ω of $\mathbb{R}^n$ and for every compact subset B of $\mathbb{R}^n$ included in Ω there exist a sequence $\{\psi_h\}_{h\in N}$ in $W^{1,p}_\circ(\Omega)(Lip_\circ(\Omega)$ if $p = +\infty, C^1_\circ(\Omega)$ if $p = {}'c1')$ and ψ in $W^{1,p}_\circ(\Omega)(Lip_\circ(\Omega)$ if $p = +\infty$ or if $p = {}'c1')$ with

$$0 \le \psi_h \le 1 \qquad \textit{in } \Omega, \forall h \in \mathbb{N}; \tag{2.2}$$

$$\psi_h = 1 \qquad \textit{in } B, \forall h \in \mathbb{N}; \tag{2.3}$$

$$\psi_h \to \psi \qquad \textit{strongly in } L^\infty(\Omega), \ \textit{as } h \to +\infty; \tag{2.4}$$

$$|D\psi_h| \le \frac{M}{\delta\, \text{dist}(B, \partial\Omega)}\varphi_h \qquad \textit{a.e. in } \Omega, \quad \forall h \in \mathbb{N}. \tag{2.5}$$

The proofs of Proposition 2.2 and Proposition 2.3 essentially follow the same outlines of the proofs of Proposition 2.4 and Proposition 2.5 in [18]. In our case, to complete proofs we make use of (2.5), of Lemma 2.1 and of Lemma 1.10.

Proposition 2.2. *Let f and φ be functions satisfying* (0.2), (0.8) *and, for p in $]n, +\infty]$ or $p = {}'c1'$, let f^p_{hom} be the function defined in* (0.4), *F'^p, F''^p, $F'^p_\circ$, $F''^p_\circ$ be the functionals defined in* (1.13), (1.15). *Assume that $(\text{dom} f^p_{\text{hom}})^\circ \neq \emptyset$.*

Then

$$\begin{aligned} F'^p(\Omega, u) &= F'^p_-(\Omega, u) = F'^p_\circ(\Omega, u), \\ F''^p(\Omega, u) &= F''^p_-(\Omega, u) = F''^p_\circ(\Omega, u), \end{aligned} \tag{2.6}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every u in $C^\circ(\mathbb{R}^n)$ such that $u = 0$ on $\partial\Omega$.

Proof. The proof is performed only for p in $]n, +\infty[$. In the other cases the proof is similar.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and u in $C^\circ(\mathbb{R}^n)$ such that $u = 0$ on $\partial\Omega$.

Prove (2.6) for the functionals $F^{''p}$ and $F_\circ^{''p}$. The proof for $F^{'p}$ and $F_\circ^{'p}$ is analogous.

Let $\{\epsilon_k\}_{k\in N}$ be a sequence of positive numbers such that $\epsilon_k \to 0^+$ as $k \to +\infty$ and, for every k in $\mathbb{N}$, let χ_k be the real function defined by

$$\chi_k(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \epsilon_k], \\ \frac{1}{2\epsilon_k}(t-\epsilon_k)^2 & \text{if} \quad t \in]\epsilon_k, 2\epsilon_k], \\ t - \frac{3}{2}\epsilon_k & \text{if} \quad t \in [2\epsilon_k, +\infty[, \\ -\chi_k(-t) & \text{if} \quad t \in]-\infty, 0[. \end{cases} \tag{2.7}$$

For every k in $\mathbb{N}$ let $\Omega_k \subset\subset \Omega$ be such that

$$\sup_{x\in\Omega\setminus\Omega_k} |u(x)| < \frac{\epsilon_k}{2}, \quad |\Omega \setminus \Omega_k| < \epsilon_k. \tag{2.8}$$

Prove that

$$F_\circ^{''p}(\Omega, u) \le F_-^{''p}(\Omega, u). \tag{2.9}$$

To this aim assume that $F_-^{''p}(\Omega, u) < +\infty$. Then for every k in $\mathbb{N}$ there exists a sequence $\{u_h^k\}_{h\in N}$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ such that $u_h^k \to u$ in $C^\circ(\Omega_k)$ as $h \to +\infty$ and there exists r_k in $\mathbb{N}$ such that $r_k \ge k$,

$$|Du_h^k| \le \varphi_h \quad \text{a.e. in } \Omega_k, \quad \forall h \ge r_k \tag{2.10}$$

and

$$F^{''p}(\Omega_k, u) \ge \limsup_{h\to+\infty} \int_{\Omega_k} f(hx, Du_h^k)\,dx. \tag{2.11}$$

For every k in $\mathbb{N}$, let be s_k in $\mathbb{N}$ such that $s_k \ge r_k$,

$$F^{''p}(\Omega_k, u)) + \frac{1}{k} \ge \int_{\Omega_k} f(hx, Du_h^k)\,dx \quad \forall h > s_k, \tag{2.12}$$

and

$$\|u - u_h^k\|_{C^\circ(\Omega_k)} \le \frac{\epsilon_k}{2} \quad \forall h > s_k. \tag{2.13}$$

For h sufficiently large, set $k_h = \max\{k \in \mathbb{N} : s_k \leq h\}$ and define the functions u_h and $\overline{u}_h$ by

$$u_h(x) = u_h^{k_h}(x) \qquad \overline{u}_h(x) = \chi_{k_h}(u_h(x)) \quad x \in \mathbb{R}^n. \tag{2.14}$$

Since $h \geq r_{k_h}$, from (2.10) it follows that

$$|Du_h| \leq \varphi_h \text{ a.e. in } \Omega_{k_h} \text{ for } h \text{ sufficiently large,} \tag{2.15}$$

moreover from (2.13) and (2.8) it follows that

$$\overline{u}_h = 0 \text{ on } \partial\Omega_{k_h} \text{ for } h \text{ sufficiently large.} \tag{2.16}$$

Still denote by $\overline{u}_h$ the function defined by

$$\overline{u}_h = \begin{cases} \overline{u}_h & \text{in} \quad \Omega_{k_h} \\ 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega_{k_h}. \end{cases}$$

Then by virtue of (2.7), (2.13) and (2.14) it results that

$$\begin{aligned} |\overline{u}_h(x) - u(x)| &\leq |\overline{u}_h(x) - u_h(x)| + |u_h(x) - u(x)| \leq \\ &\leq 3\epsilon_{k_h} + \frac{1}{2}\epsilon_{k_h} = \frac{7}{2}\epsilon_{k_h} \text{ in } \Omega_{k_h}, \text{ for } h \text{ sufficiently large,} \end{aligned} \tag{2.17}$$

and by (2.8) that

$$|\overline{u}_h(x) - u(x)| = |u(x)| \leq \frac{1}{2}\epsilon_{k_h} \text{ in } \Omega \setminus \Omega_{k_h}, \text{ for } h \text{ sufficiently large.} \tag{2.18}$$

Consequently, from (2.17) and (2.18) it follows that

$$\overline{u}_h \to u \text{ in } C^{\circ}_{\circ}(\Omega) \tag{2.19}$$

and, from (2.7) and (2.15), that

$$|D\overline{u}_h| \leq \varphi_h \text{ a.e. in } \Omega, \text{ for } h \text{ sufficiently large.} \tag{2.20}$$

Let B_1 and B_2 be two open subsets of $\mathbb{R}^n$ such that $B_1 \subset\subset B_2 \subset\subset \Omega_{k_h}$ for h large enough, let $\{\psi_h\}_h \subseteq W^{1,p}_{\circ}(B_2)$ be the sequence given by Lemma 2.1 whit $B = \overline{B}_1$ and set

$$w_h = \psi_h u_h + (1 - \psi_h)\overline{u}_h \qquad \forall x \in \Omega. \tag{2.21}$$

Obviously $w_h \to u$ in $C^\circ_\circ(\Omega)$. Moreover, for every t in $[0,1[$, by virtue of (2.15), (2.20), Lemma 2.1, and (2.19) it results

$$
\begin{aligned}
(2.22)\qquad t|Dw_h| &\le t|\psi_h Du_h + (1-\psi_h)D\overline{u}_h + D\psi_h(u_h - \overline{u}_h)| \le \\
&\le t\Big(\psi_h\varphi_h + (1-\psi_h)\varphi_h + |D\psi_h||u_h - \overline{u}_h|\Big) \le \\
&\le t\Big(\varphi_h + \frac{M}{\delta\,\mathrm{dist}(B_1,\partial B_2)}\varphi_h\|u_h - \overline{u}_h\|_{C^\circ(\Omega)}\Big) \le \varphi_h \\
&\text{a.e. in } \Omega, \text{ for } h \text{ sufficiently large.}
\end{aligned}
$$

By using the convexity of $f(x,\cdot)$, it results

$$
\begin{aligned}
(2.23)\qquad &\int_\Omega f(hx, tDw_h)\,dx \le t\Big(\int_\Omega \psi_h f(hx, Du_h)\,dx + \\
&+\int_\Omega (1-\psi_h)f(hx, D\overline{u}_h)\,dx\Big) + (1-t)\int_\Omega f\Big(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h\Big)\,dx \le \\
&\le t\Big(\int_{\Omega_{k_h}} \psi_h f(hx, Du_h)\,dx + \int_{\Omega_{k_h}} (1-\psi_h)f(hx, D\overline{u}_h)\,dx + \\
&+\int_{\Omega\setminus\Omega_{k_h}} (1-\psi_h)f(hx,0)\,dx\Big) + (1-t)\int_\Omega f\Big(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h\Big)\,dx, \\
&\forall t \in [0,1[\text{ for } h \text{ sufficiently large.}
\end{aligned}
$$

Hence, by definitions (2.14),

$$
\begin{aligned}
(2.24)\qquad &\int_\Omega f(hx, tDw_h)\,dx \le \\
&\le t\Big(\int_{\Omega_{k_h}} \psi_h f(hx, Du_h)\,dx + \int_{\Omega_{k_h}} (1-\psi_h)\chi'_{k_h}(u_h) f(hx, Du_h)\,dx + \\
&+\int_{\Omega_{k_h}} (1-\psi_h)(1-\chi'_{k_h}(u_h)) f(hx,0)\,dx + \int_{\Omega\setminus\Omega_{k_h}} f(hx,0)\,dx\Big) + \\
&+(1-t)\int_\Omega f\Big(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h\Big)\,dx, \\
&\forall t \in [0,1[\text{ for } h \text{ sufficiently large,}
\end{aligned}
$$

from which, by virtue of the properties of $\{\psi_h\}_{h\in N}$, $\{\chi_{k_h}\}_{k_h\in N}$ and (2.11), it follows that

$$
(2.25)\qquad \int_\Omega f(hx, tDw_h)\,dx \le
$$

$$\leq t\Big(\int_{\Omega_{k_h}} f(hx, Du_h)\,dx + 2\int_{\Omega\setminus B_1} f(hx,0)\,dx\Big) +$$
$$+ (1-t)\int_{\Omega} f(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h)\,dx \leq$$
$$\leq t\Big(F^{''p}(\Omega_{k_h}, u) + \frac{1}{k_h} + 2\int_{\Omega\setminus B_1} f(hx,0)\,dx\Big) +$$
$$+ (1-t)\Big(\int_{B_2} f(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h)\,dx + \int_{\Omega\setminus B_2} f(hx,0)\,dx\Big),$$
$$\forall t \in [0,1[, \text{ for } h \text{ sufficiently large.}$$

Since
$$u_h - \overline{u}_h \to 0 \text{ in } C^\circ(B_2) \text{ as } h \to +\infty,$$
by virtue of (2.5) of Lemma 2.1 $\forall t \in [0,1[\ \exists h_t \in \mathbb{N} : \forall h(\in \mathbb{N}) > h_t$

$$\Big|\frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h\Big| \leq \frac{t}{1-t}||u_h - \overline{u}_h||_{C^\circ(B_2)} \frac{M}{\delta\,\mathrm{dist}(B_1, \partial B_2)}\varphi_h \leq$$
$$\leq \alpha\varphi_h \text{ a.e. in } \Omega,$$

where α is the constant given in (0.8). Consequently, Lemma 1.10 provides that

$$\limsup_{h\to+\infty}\int_{\Omega} f(hx, \frac{t}{1-t}(u_h - \overline{u}_h)D\psi_h)\,dx \leq c|B_2| \quad \forall t \in [0,1[, \tag{2.26}$$

where c is the constant defined in (1.31). Hence by (2.25) and (2.26), as $h \to +\infty$,

$$F^{''p}_{\circ}(\Omega, tu) \leq tF^{''p}_{-}(\Omega, u) + 2t|\Omega\setminus B_1|\int_Y f(y,0)\,dy + (1-t)|\Omega|\cdot \tag{2.27}$$
$$\cdot\Big(c + \int_Y f(y,0)\,dy\Big), \quad \forall t \in [0,1[.$$

Since $u = 0$ on $\partial\Omega$ it follows that $tu \to u$ in $C^\circ_\circ(\Omega)$.Therefore by (2.27) as $t \to 1^-$ and as B_1 converges to Ω we deduce that

$$F^{''p}_{\circ}(\Omega, u) \leq F^{''p}_{-}(\Omega, u). \tag{2.28}$$

On the other side,since always

$$F^{''p}_{-}(\Omega, u) \leq F^{''p}(\Omega, u) \leq F^{''p}_{\circ}(\Omega, u), \tag{2.29}$$

the conclusion follows from (2.28) and (2.29). □

Proposition 2.3. *Let f and φ be functions satisfying* (0.2), (0.8), *for p in $]n, +\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4), *F'^p,F''^p be the functionals defined in* (1.13) *and let Ω, Ω_1, Ω_2 be bounded open subsets of $\mathbb{R}^n$.*

If $\Omega_1 \cap \Omega_2 = \phi$ and $\Omega_1 \cup \Omega_2 \subseteq \Omega$, then

$$F'^p_-(\Omega, u) \geq F'^p_-(\Omega_1, u) + F'^p_-(\Omega_2, u) \quad \forall u \in C^\circ(\mathbb{R}^n). \tag{2.30}$$

If $\Omega \subseteq \Omega_1 \cup \Omega_2$ and $(\mathrm{dom}\, f^p_{\mathrm{hom}})^\circ \neq \phi$*, then*

$$F''^p_-(\Omega, u) \leq F''^p_-(\Omega_1, u) + F''^p_-(\Omega_2, u) \quad \forall u \in C^\circ(\mathbb{R}^n). \tag{2.31}$$

Proof. The proof is performed only for p in $]n, +\infty[$. In the other cases the proof is similar.

Inequality (2.30) follows directly from the definition of F'^p_-.

To prove (2.31), it suffices to consider the case in which $\Omega \subset\subset \Omega_1 \cup \Omega_2$ and prove that

$$F''^p(\Omega, u) \leq F''^p(\Omega_1, u) + F''^p(\Omega_2, u) \quad \forall u \in C^\circ(\mathbb{R}^n). \tag{2.32}$$

Fix u in $C^\circ(\mathbb{R}^n)$ and assume that the right hand side of (2.32) is finite. Consequently, for $i = 1, 2$ there exists a sequence $\{u_h^{(i)}\}_{h\in N}$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, such that $u_h^{(i)} \to u$ in $C^\circ(\Omega_i)$ as $h \to \infty$, $|Du_h^{(i)}| \leq \varphi_h$ a.e. in Ω_i for h sufficiently large and

$$F''^p(\Omega_i, u) \geq \limsup_{h\to+\infty} \int_{\Omega_i} f(hx, Du_h^{(i)})\, dx. \tag{2.33}$$

Since $\Omega \subset\subset \Omega_1 \cup \Omega_2$, for ϵ small enough it result that $\Omega \subset\subset \Omega^-_{1,\epsilon} \cup \Omega_2$.

Let $\{\psi_h\}_{h\in N}$ be a sequence in $W^{1,p}_\circ(\Omega_1)$ satisfying Lemma 2.1 with $B = c1(\Omega^-_{1,\epsilon})$ and set

$$w_h = \psi_h u_h^{(1)} + (1 - \Psi_h) u_h^{(2)}. \tag{2.34}$$

Observe that $w_h \to u$ in $C^\circ(\Omega)$. Moreover, as in (2.22), for every t in $[0, 1[$ it results $t|Dw_h| \leq \varphi_h$ a.e. in Ω, for h sufficiently large.

By making use of the convexity of $f(x,\cdot)$ we have

$$F''^p(\Omega, tu) \le \limsup_{h\to+\infty}\int_\Omega f(hx, tDw_h)\,dx \le \tag{2.35}$$

$$\le t\limsup_{h\to+\infty}\int_\Omega \psi_h f(hx, Du_h^{(1)})\,dx + t\limsup_{h\to+\infty}\int_\Omega (1-\psi_h) f(hx, Du_h^{(2)})\,dx +$$

$$+(1-t)\limsup_{h\to+\infty}\int_\Omega f(hx, \frac{t}{1-t}(u_h^{(1)} - u_h^{(2)} D\psi_h)\,dx \le$$

$$\le t\limsup_{h\to+\infty}\int_{\Omega 1} f(hx, Du_h^{(1)}\,dx + t\limsup_{h\to+\infty}\int_{\Omega\setminus\Omega^-_{1,\epsilon}} f(hx, Du_h^{(2)})\,dx +$$

$$+ (1-t)\limsup_{h\to+\infty}\int_{\Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon})} f\big(hx, \frac{t}{1-t}(u_h^{(1)} - u_h^{(2)})D\psi_h\big)\,dx +$$

$$+ (1-t)\limsup_{h\to+\infty}\int_{\Omega\setminus(\Omega_1\setminus\Omega^-_{1,\epsilon})} f(hx, 0)\,dx, \quad \forall t \in [0,1[.$$

Since $\Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon}) \subseteq \Omega_1\cap\Omega_2$,

$$u_h^{(1)} - u_h^{(2)} \to 0 \quad \text{in } C^\circ(\Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon})) \text{ as } h\to+\infty.$$

Then, by virtue of (2.5) of Lemma 2.1, it results that

$$\forall t\in[0,1[,\ \exists h_t\in\mathbb{N}:\ \forall h(\in\mathbb{N}) > h_t$$

$$\Big|\frac{t}{1-t}(u_h^{(1)} - u_h^{(2)})D\psi_h\Big| \le \frac{t}{1-t}||u_h^{(1)} - u_h^{(2)}||_{C^\circ(\Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon}))}\frac{M}{\delta\epsilon}\varphi_h \le \alpha\varphi_h$$

$$\text{a.e. in } \Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon}),$$

where α is the constant given in (0.8). Consequently, Lemma 1.10 provides that

$$\limsup_{h\to+\infty}\int_{\Omega\cap(\Omega_1\setminus\Omega^-_{1,\epsilon})} f(hx, \frac{t}{1-t}(u_h^{(1)} - u_h^{(2)})D\psi_h)\,dx \le \tag{2.36}$$

$$\le c|\Omega| \quad \forall t\in[0,1[,$$

where c is the constant defined in (1.31).

Combining (2.33) with (2.35) and (2.36) it results

$$F''^p(\Omega, tu) \le tF''^p(\Omega_1, u) + tF''^p(\Omega_2, u) + (1-t)c|\Omega| + \tag{2.37}$$

$$+ (1-t)|\Omega|\int_Y f(y,0)\,dy \qquad \forall t\in[0,1[.$$

Finally, passing to the limit, as $t\to 1^-$, in (2.37), inequality (2.32) is proved. $\square$

3. Finiteness conditions.

Let f and φ be functions satisfying (0.2), (0.8) and, for p in $]n, +\infty]$ or $p = 'c1'$, let F''^p be the functional defined in (1.13). In this section, following the some outlines of cap.3 in [18], we give sufficient conditions on Ω and u in order to get finiteness of $F''^p(\Omega, u)$. To this purpose we first prove some lemmas.

Lemma 3.1. *Let f and φ be functions satisfying* (0.2), (0.8) *and for p in $]n, +\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4).

Let $\vartheta > 0$, z be in $\operatorname{dom} f^p_{\text{hom}}$ *and v in $W^{1,p}_{\text{per}}(Y)$ (in $C^1_{\text{per}}(Y)$ if $p = 'c1'$) such that $|z + Dv| \le \varphi$ a.e. in Y and $f^p_{\text{hom}}(z) + \vartheta > \int_Y f(y, z + Dv)\, dy$ and, for every h in $\mathbb{N}$ and x in $\mathbb{R}^n$, set $v_h(x) = \frac{1}{h} v(hx)$.*

Then

$$|z + Dv_h| \le \varphi_h \text{ a.e. in } \mathbb{R}^n, \forall h \in \mathbb{N} \tag{3.1}$$

and

$$|\Omega|(f^p_{\text{hom}}(z) + \vartheta) > \lim_{h \to +\infty} \int_\Omega f(hx, z + Dv_h)\, dx \tag{3.2}$$

for every bounded open subset Ω of $\mathbb{R}^n$.

Proof. Inequality (3.1) follows immediately by the assumptions on v and φ.

Fix a bounded open subset Ω of $\mathbb{R}^n$. By the periodicity of $f(\cdot, Dv(\cdot))$, the limit in (3.2) exists and

$$\begin{aligned} \lim_{h \to +\infty} \int_\Omega f(hx, z + Dv_h)\, dx &= \lim_{h \to +\infty} \int_\Omega f(hx, z + (Dv)(hx))\, dx = \\ &= |\Omega| \int_Y f(y, z + Dv)\, dy < |\Omega|(f^p_{\text{hom}}(z) + \vartheta). \qquad \square \end{aligned} \tag{3.3}$$

Let $\epsilon > 0$ and let $P_1, \dots, P_m$ be subsets of $\mathbb{R}^n$. Denote by $\nu_\epsilon(P_1, \dots, P_m)$ the function defined by

$$\begin{aligned} \nu_\epsilon(P_1, \dots, P_m)(x) = {} & \text{cardinality of the set} \\ & \{P \in \{P_1, \dots, P_m\} : \operatorname{dist}(x, P) < \epsilon\}, \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.4}$$

By making use of Lemma 2.1 and by arguing as in Lemma 3.2 in [18], it is easy to prove the following result:

Lemma 3.2. *Let f and φ be functions satisfying* (0.2), (0.8) *and, for p in $]n, +\infty]$ or $p = 'c1'$, let $f^P_{\hom}$ be the function defined in* (0.4). *Assume* $(\mathrm{dom}\, f^p_{\hom})^\circ \neq \phi$. *Let δ be the constant fixed in* (2.1) *and M be the constant given in Lemma 2.1.*

Then for every finite family $\{\Omega_1, \ldots, \Omega_m\}$ of bounded disjoint open subsets of $\mathbb{R}^n$, $\epsilon > 0$ sufficiently small and j in $\{1, \ldots, m\}$, there exist a sequence $\{\gamma_h^{\epsilon,j}\}_{h\in N} \subseteq W^{1,p}_\circ(\Omega^+_{j,\epsilon})(Lip_\circ(\Omega^+_{j,\epsilon})$ if $p = +\infty$, $C^1_\circ(\Omega^+_{j,\epsilon})$ if $p = 'c1')$ and γ^ϵ_j in $W^{1,p}_\circ(\Omega^+_{j,\epsilon})(Lip_\circ(\Omega^+_{j,\epsilon})$ if $p = +\infty$ or $p = 'c1')$ such that

$$0 \le \gamma_h^{\epsilon,j} \le 1 \quad in \quad \bigcup_{i=1}^m \overline{\Omega}_i\,, \qquad \forall h \in \mathbb{N}; \tag{3.5}$$

$$\gamma_h^{\epsilon,j} = 1 \quad in \quad \Omega^-_{j,\epsilon}, \qquad \forall h \in \mathbb{N}; \tag{3.6}$$

$$\sum_{i=1}^m \gamma_h^{\epsilon,i}(x) = 1 \quad in \quad \bigcup_{i=1}^m \overline{\Omega}_i, \qquad \forall h \in \mathbb{N}; \tag{3.7}$$

$$\gamma_h^{\epsilon,j} \to \gamma_i^\epsilon \quad \textit{strongly in} \quad L^\infty\Big(\bigcup_{i=1}^m \overline{\Omega}_i\Big),\ \textit{as}\ h \to +\infty; \tag{3.8}$$

$$|D\gamma_h^{\epsilon,j}(x)| \le \frac{M}{\delta\epsilon} \sup_{y\in R^n} \nu_\epsilon(\Omega_1, \ldots, \Omega_m)(y)\varphi_h(x) \qquad \textit{a.e. in} \quad \bigcup_{i=1}^m \overline{\Omega}_i\,, \quad \forall h \in \mathbb{N}. \tag{3.9}$$

For every function $u = \sum_{j=1}^m (u_{z_j} + s_j)\chi_{P_j}$ in $PA(\mathbb{R}^n)$ and $\epsilon > 0$, set

$$\sigma_\epsilon(u) = \sup_{x\in\mathbb{R}^n} \nu_\epsilon(P_1, \ldots, P_m)(x),$$

where $\nu_\epsilon(P_1, \ldots, P_m)(x)$ is defined in (3.4). Observe that there exist $\epsilon(u) > 0$ and $\sigma(u)$ in $\mathbb{N}$ such that

$$\sigma_\epsilon(u) = \sigma(u) \quad \forall \epsilon \in [0, \epsilon(u)[. \tag{3.10}$$

Lemma 3.3. *Let f and φ be functions satisfying* (0.2), (0.8), *α be the constant given in* (0.8), *for p in $]n, +\infty]$ or $p = 'c1'$, let $f^p_{\hom}$ be the function defined in* (0.4) *and F''^p be the functional defined in* (1.13). *Assume $(dom f^p_{\hom})^\circ \neq \phi$. Let*

δ be the constant fixed in (2.1) *and let M be the constant given in Lemma 2.1. Then*

$$F''^{p}(\Omega, tu) \leq t \int_{\Omega} f^{p}_{\text{hom}}(Du)\, dx + (1-t)|\Omega| \int_{Y} f(y,0)\, dy \tag{3.11}$$

for every bounded open subset Ω of $\mathbb{R}^n$, every piecewise affine function u and for every $t \in \left(0, \frac{\alpha\ \delta}{\sigma^2(u)(\delta + M(2\|Du\|_{L^\infty(\Omega)}+1))}\right)$, where $\sigma(u)$ is given by (3.10) *(recall that we can assume $\alpha < 1$).*

Proof. The proof is performed only for p in $]n, +\infty[$. In the other cases the proof is similar. Let Ω and t be as above, let $u = \sum_{j=1}^{m} (u_{z_j} + s_j)\chi_{\overset{\circ}{P}_j}$ be a piecewise affine function and, for every j in $\{1, \dots, m\}$, set $\Omega_j = \Omega \cap \overset{\circ}{P}_j$. In order to prove (3.11) assume that

$$\sum_{j=1}^{m} f^{p}_{\text{hom}}(z_j)|\Omega \cap \overset{\circ}{P}_j| = \int_{\Omega} f^{p}_{\text{hom}}(Du)\, dx < +\infty. \tag{3.12}$$

Inequality (3.12) provides that z_j belongs to $\operatorname{dom} f^{p}_{\text{hom}}$ for every j in $\{1, \dots, m\}$. Hence, for every fixed ϑ in $(0, +\infty)$ and j in $\{1, \dots, m\}$, there exists a function v^j (depending on ϑ) in $W^{1,p}_{per}(Y)$ such that

$$|z_j + Dv^j| \leq \varphi \text{ a.e. in } Y \text{ and } f^{p}_{\text{hom}}(z_j) + \vartheta > \int_{Y} f(y, z_j + Dv^j)\, dy.$$

Consequently, by setting $v^j_h(x) = \frac{1}{h} v^j(hx)$ for every x in $\mathbb{R}^n$ and h in $\mathbb{N}$, from Lemma 3.1 it follows that

$$|z_j + Dv^j_h| \leq \varphi_h \quad \text{a.e. in } \mathbb{R}^n,\ \forall h \in \mathbb{N} \tag{3.13}$$

and, for every $\epsilon > 0$,

$$(f^{p}_{\text{hom}}(z) + \vartheta)|\Omega \cap \Omega^{+}_{j,\epsilon}| > \lim_{h} \int_{\Omega \cap \Omega^{+}_{j,\epsilon}} f(hx, z_j + Dv^j_h)\, dx. \tag{3.14}$$

For ϵ in $[0, \epsilon(u)[$ (see (3.10)) and for every j in $\{1, \dots, m\}$, let $\{\gamma^{\epsilon,j}_h\}_{h \in N}$ in $W^{1,p}_{\circ}(\Omega^{+}_{j,\epsilon})$ and γ^{ϵ}_j in $W^{1,p}_{\circ}(\Omega^{+}_{j,\epsilon})$ be given by Lemma 3.2 with $\gamma^{\epsilon,j}_h = 1$ in $\Omega^{-}_{j,\epsilon}$. For every ϵ in $[0, \epsilon(u)[$, let $\{w^{\epsilon}_h\}_{h \in N}$ be the sequence defined by

$$w^{\epsilon}_h = \sum_{j=1}^{m} (u_{z_j} + s_j + v^j_h)\gamma^{\epsilon,j}_h. \tag{3.15}$$

From Lemma 3.2, it results

$$w_h^\epsilon \to \sum_{j=1}^m \gamma_j^\epsilon(u_{z_j}+s_j)=w^\epsilon \ \text{ uniformly on } \overline{\Omega}, \text{ as } h\to+\infty. \tag{3.16}$$

Observe now that, to taking into account (3.7), $\sum\limits_{j=1}^m D\gamma_h^{\epsilon,j}=0$ a.e. in $\overline{\Omega}$ and that for a.e. x in $\mathbb{R}^n$ card $\{j\in\{1,\dots,m\}:\gamma_h^{\epsilon,j}(x)\neq 0\}\leq\sigma(u)$ and card$\{j\in\{1,\dots,m\}: D\gamma_h^{\epsilon,j}(x)\neq 0\}\leq\sigma(u)$. Moreover for every j in $\{1,\dots,m\}$, since $u=u_{z_j}+s_j$ on Ω_j, it results that

$$\sup_{x\in\Omega_{j,\epsilon}^+}|u_{z_j}(x)+s_j+v_h^j(x)-u(x)|\leq(2\|Du\|_{L^\infty(\Omega)}+1)\epsilon \tag{3.17}$$

for h sufficiently large. Hence by (3.9), (3.10), (3.13) and (3.17) it follows that

$$\begin{aligned} t|Dw_h^\epsilon| &\leq t\Big|\sum_{j=1}^m\big(D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j)+ \\ &+\gamma_h^{\epsilon,j}(z_j+Dv_h^j)\big)\Big| = t\Big|\sum_{j=1}^m\big(D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)+ \\ &+\gamma_h^{\epsilon,j}(z_j+Dv_h^j)\big)\Big| \leq t\sum_{j=1}^m\big(|D\gamma_h^{\epsilon,j}|\sup_{x\in\Omega_{j,\epsilon}^+}|u_{z_j}(x)+s_j+v_h^j(x)-u(x)|\big)+ \\ &+t\sum_{j=1}^m\big(\gamma_h^{\epsilon,j}|z_j+Dv_h^j|\big)\leq t\sigma^2(u)\frac{M}{\delta\epsilon}\varphi_h(x)(2\|Du\|_{L^\infty(\Omega)}+1)\epsilon+ \\ &+t\sigma(u)\varphi_h = t\sigma^2(u)\big(\frac{M}{\delta}(2\|Du\|_{L^\infty(\Omega)}+1)+1\big)\varphi_h\leq\varphi_h \end{aligned} \tag{3.18}$$

a.e. in Ω, for h sufficiently large. Taking into account the convexity of $f(x,\cdot)$, Lemma 3.2, (3.16), (3.18) and recalling that $\sum_{j=1}^m D\gamma_h^{\epsilon,j}=0$ a.e. in Ω, it results that

$$\begin{aligned} F''^p(\Omega,tw^\epsilon) &\leq \limsup_h\int_\Omega f(hx,tDw_h^\epsilon)\,dx\leq \\ &\leq t\limsup_h\int_\Omega f(hx,\sum_{j=1}^m\gamma_h^{\epsilon,j}(z_j+Dv_h^j))\,dx+ \end{aligned} \tag{3.19}$$

$$+ (1-t)\limsup_h \int_\Omega f\Big(hx, \frac{t}{1-t}\sum_{j=1}^m D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)\Big)\,dx \leq$$

$$\leq t\sum_{j=1}^m \limsup_h \int_\Omega \gamma_h^{\epsilon,j} f(hx, z_j + Dv_h^j)\,dx +$$

$$+ (1-t)\limsup_h \int_\Omega f\Big(hx, \frac{t}{1-t}\sum_{j=1}^m D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)\Big)\,dx \leq$$

$$\leq t\sum_{j=1}^m \limsup_h \int_{\Omega\cap\Omega_{j,\epsilon}^+} f(hx, z_j + Dv_h^j)\,dx +$$

$$+ (1-t)\limsup_h \sum_{j=1}^m \int_{\Omega_{j,\epsilon}^-} f(hx, 0)\,dx +$$

$$+ (1-t)\limsup_h \int_{\Omega\setminus\cup_{i=1}^m\Omega_{i,\epsilon}^-} f\Big(hx, \frac{t}{1-t}\sum_{j=1}^m D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)\Big)\,dx.$$

On the other hand Lemma 3.2 and (3.17) provide that

$$\frac{t}{1-t}\Big|\sum_{j=1}^m D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)\Big| \leq \tag{3.20}$$

$$\leq \frac{t}{1-t}\sum_{j=1}^m |D\gamma_h^{\epsilon,j}\,|\,|u_{z_j}+s_j+v_h^j-u| \leq$$

$$\leq \frac{t}{1-t}\sum_{j=1}^m |D\gamma_h^{\epsilon,j}| \sup_{\Omega\cap\Omega_{j,\epsilon}^+} |u_{z_j}+s_j+v_h^j-u| \leq$$

$$\leq \frac{t}{1-t}\sigma^2(u)\frac{M}{\delta\epsilon}\varphi_h(x)(2\|Du\|_{L^\infty(\Omega)}+1)\epsilon =$$

$$= \frac{t}{1-t}\sigma^2(u)\frac{M}{\delta}(2\|Du\|_{L^\infty(\Omega)}+1)\varphi_h \leq \alpha\varphi_h$$

$$\text{a.e. in}\quad \Omega\setminus\bigcup_{i=1}^m \Omega_{i,\epsilon}^- \quad \text{for } h \text{ large enough.}$$

From Lemma 1.10 and (3.20) it follows that

$$\limsup_h \int_{\Omega\setminus\cup_{j=1}^m\Omega_{j,\epsilon}^-} f\Big(hx, \frac{t}{1-t}\sum_{j=1}^m D\gamma_h^{\epsilon,j}(u_{z_j}+s_j+v_h^j-u)\Big)\,dx \leq \tag{3.21}$$

$$\leq c\Big|\Omega \setminus \bigcup_{i=1}^{m} \Omega^{-}_{i,\epsilon}\Big|$$

where c is the constant defined in (1.31). Then, by combining (3.19) with (3.14) and (3.21) and by making use of the periodicity of $f(\cdot, 0)$, it results

$$F''^{p}(\Omega, tw^{\epsilon}) \leq t \sum_{j=1}^{m} |\Omega \cap \Omega^{+}_{j,\epsilon}|(f^{p}_{\mathrm{hom}}(z_j) + \vartheta) + + (1-t)|\Omega| \int_{Y} f(y,0)\, dy + (1-t)c\Big|\Omega \setminus \bigcup_{i=1}^{m} \Omega^{-}_{i,\epsilon}\Big|. \tag{3.22}$$

As in (3.17), it results

$$\sup_{\Omega} \Big|w^{\epsilon} - u| = \sup_{\Omega} |\sum_{j=1}^{m} \gamma^{\epsilon}_{j}(u_{z_j} + s_j - u)\Big| \leq$$

$$\leq \sum_{j=1}^{m} \sup_{\Omega \cap \Omega^{+}_{j,\epsilon}} |\gamma^{\epsilon}_{j}|\, |u_{z_j} + s_j - u| \leq \sigma(u)(2\|Du\|_{L^{\infty}(\Omega)} + 1)\epsilon.$$

Consequently

$$w^{\epsilon} \to u \quad \text{in} \quad C^{\circ}(\Omega) \text{ as } \epsilon \to 0. \tag{3.23}$$

Then, from (3.22), (3.23) and the $C^{\circ}(\Omega)-$lower semicontinuity of $F''^{p}(\Omega, \cdot)$ it results

$$F''^{p}(\Omega, tu) \leq \liminf_{\epsilon \to 0^{+}} F''^{p}(\Omega, tw^{\epsilon}) \leq \leq t \sum_{j=1}^{m} |\overline{\Omega}_j|(f^{p}_{\mathrm{hom}}(z_j) + \vartheta) + + (1-t)|\Omega| \int_{Y} f(y,0)\, dy + (1-t)c\Big|\Omega - \bigcap_{j=1}^{m} \partial P_j\Big|. \tag{3.24}$$

By passing to the limit in (3.24), as $\vartheta \to 0^{+}$, and by recalling that

$$\sum_{j=1}^{m} |\overline{\Omega}_j|\, f^{p}_{\mathrm{hom}}(z_j) = \int_{\Omega} f^{p}_{\mathrm{hom}}(Du)\, dx \text{ and that } |\Omega \cap \cup_{j=i}^{m} \partial P_j| = 0,$$

inequality (3.11) is proved. □

Arguing as in Lemma 3.4 in [18], the Lemma 3.3 and the Proposition 1.6 provide the following result:

Lemma 3.4. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8)*, for p in $]n, +\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4) *and F''^p be the functional defined in* (1.13)*. Assume* $(\text{dom } f^p_{\text{hom}})^\circ \neq \phi$*. Let δ be the constant fixed in* (2.1) *and M be the constant given in Lemma 2.1. Then there exists a costant c, such that*

$$F''^p(\Omega, tu) \leq t \int_\Omega f^p_{\text{hom}}(Du)\, dx + (1-t)|\Omega| \int_Y f(y, 0)\, dy \tag{3.25}$$

for every bounded open subset Ω of $\mathbb{R}^n$, every u in $C^1(\mathbb{R}^n)$ and every t in $\left(0, \frac{\delta}{c(\delta + M(2\|Du\|_{L^\infty(\Omega)}+1))}\right)$.

Arguing as in Lemma 3.5 in [18], Lemma 3.4 and Proposition 1.6 provide the following result:

Proposition 3.5. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8)*, for p in $]n, +\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4) *and F''^p be the functional defined in* (1.13)*. Assume* $(\text{dom } f^p_{\text{hom}})^\circ \neq \phi$*. Let δ be the constant fixed in* (2.1) *and M be the constant given in Lemma 2.1.*

Then there exist two positive costants r in $[0, \delta[$ *and c in* $[0, +\infty[$ *such that*

$$u \in Lip_{\text{loc}}, \quad \|Du\|_{L^\infty(\Omega)} \leq r \Rightarrow F''^p_-(\Omega, u) \leq c|\Omega| < +\infty \tag{3.26}$$

for every bounded open subset Ω of $\mathbb{R}^n$.

4. A representation result on the class of the affine functions.

In this section we give a representation result of $F^p(\Omega, \cdot)$ on the class of the affine functions.

Recall that, for a given z in $\mathbb{R}^n$, u_z denotes the function defined by

$$u_z : x \in \mathbb{R}^n \to xz \in \mathbb{R}.$$

Lemma 4.1. *Let f and φ be functions satisfying* (0.2)*, for p in $]n, +\infty]$ or $p =' c1'$, let $\overline{f}^p_{\text{hom}}$ be the function defined in* (1.29) *and F''^p be the functional defined in* (1.13)*. Then*

$$F''^p(\Omega, u_z) \leq |\Omega| \overline{f}^p_{\text{hom}}(z) \tag{4.1}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every z in $\mathbb{R}^n$.

Proof. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $z \in \mathbb{R}^n$.

In order to prove (4.1), assume that $\overline{f}^p_{\hom}(z) < +\infty$. Then, for every $\epsilon > 0$ there exists $0 < t_\epsilon < 1$ such that for every t in $]t_\epsilon, 1[$ there exists v in $W^{1,p}_{per}(Y)$ ($v \in C^1_{per}(Y)$if $p = 'c1'$) such that $|tz + Dv| \leq \varphi$ a.e. in Y and

$$\overline{f}^p_{\hom}(z) + \epsilon \geq \int_Y f(y, tz + Dv)\, dy. \tag{4.2}$$

Set $v_h(x) = \frac{1}{h} v(hx), x \in \mathbb{R}^n,\ h \in \mathbb{N}$. Then $v_h \to 0$ in $C^\circ(\Omega)$ and $|tz + Dv_h| \leq \varphi_h$ a.e. in Ω. Consequently, by virtue of (4.2), it results

$$\begin{aligned} F''^p(\Omega, tu_z) &\leq \limsup_h \int_\Omega f(hx, tz + Dv_h) dx = \\ &= |\Omega| \int_Y f(y, tz + Dv)\, dy \leq |\Omega| (\overline{f}^p_{\hom}(z) + \epsilon). \end{aligned} \tag{4.3}$$

As $t \to 1^-$ and $\epsilon \to 0$, the thesis follows from (4.3). □

In order to prove the reverse inequality in (4.1) recall the following result proved in Lemma 3.2 in [20].

Lemma 4.2. [20]. *Let f and φ be functions satisfying* (0.2) *and for p in $]n, +\infty]$ or $p = 'c1'$, let F'^p be the functional defined in* (1.13). *Then*

$$\begin{aligned} r_1^{-n} F'^p(x_1 + r_1 Y, u_z) &= r_2^{-n} F'^p(x_2 + r_2 Y, u_z), \\ &\forall x_1, x_2 \in \mathbb{R}^n,\ \forall r_1, r_2 \in]0, +\infty[,\ \forall z \in \mathbb{R}^n. \end{aligned} \tag{4.4}$$

Proposition 4.3. *Let f and φ be functions satisfying* (0.2), (0.8), *for p in $]n, +\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\hom}$ be the function defined in* (1.29) *and F'^p be the functional defined in* (1.13). *Assume that* $(\operatorname{dom} f^p_{\hom})^\circ \neq \phi$. *Then*

$$|\Omega| \overline{f}^p_{\hom}(z) \leq F'^p(\Omega, u_z) \tag{4.5}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every $z \in \mathbb{R}^n$.

Proof. First assume $\Omega = Y$ and fix z in $\mathbb{R}^n$.

In order to prove (4.5) assume that $F'^p(Y, u_z) < +\infty$. Then, by virtue of Lemma 1.7, z belongs to dom $f^p_{\hom}$. Consequently, Proposition 1.6 provide that tz belongs to dom $f^p_{\hom}$ for every t in [0,1]. Hence for every t in [0,1] there exists $v \in W^{1,p}_{per}(Y)(C^1_{per}(Y)$ if $p = 'c1')$ such that $|tz + Dv| \leq \varphi$ a.e. in Y.

For every h in $\mathbb{N}$, set $v_h(x) = \frac{1}{h}v(hx)$, x in $\mathbb{R}^n$. Obviously

$$v_h \to 0 \text{ in } C^\circ(Y) \text{ as } h \to +\infty \text{ and } |tz + Dv_h| \leq \varphi_h \text{ a.e. in } Y. \tag{4.6}$$

Since $F'^p(Y, u_z) < +\infty$, by virtue of Lemma 1.11, $F'^p(Y, tu_z) < +\infty$ too. Hence there exist $\{u_h\}_{h\in N}$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ $(C^1(\mathbb{R}^n)$ if $p = 'c1')$ and a subsequence $\{h_k\}_{k\in N}$ of $\mathbb{N}$ such that $u_h \to tu_z$ in $C^\circ(Y)$, $|Du_{h_k}| \leq \varphi_{h_k}$ a.e. in Y and

$$F'^p(Y, tu_z) \geq \lim_k \int_Y f(h_k x, Du_{h_k})\, dx. \tag{4.7}$$

Let Ω' and Ω'' be open subsets of $\mathbb{R}^n$ such that $\Omega' \subset\subset \Omega'' \subset\subset Y$, let $\{\psi_h\}_{h\in N}$ in $W^{1,p}_\circ(\Omega'')(Lip_\circ(\Omega'')$ if $p = +\infty$, $C^1_\circ(\Omega'')$ if $p = 'c1')$ be given by Lemma 2.1 with $\psi_h = 1$ in $\overline{\Omega}'$ and set

$$w_k = \psi_{h_k} u_{h_k} + (1 - \Psi_{h_k})(v_{h_k} + tu_z). \tag{4.8}$$

Obviously $w_k - tu_z$ is in $W^{1,p}_{per}(Y)(C^1_{per}(Y)$ if $p = 'c1')$, $w_k \to tu_z$ in $C^\circ(Y)$ and

$$\begin{aligned} t|Dw_k| &\leq t\big(\psi_{h_k}\varphi_{h_k} + (1 - \psi_{h_k})\varphi_{h_k} + \|u_{h_k} - v_{h_k} - tu_z\|_{C^\circ(Y)}|D\psi_{h_k}|\big) \leq \\ &\leq t\Big(\varphi_{h_k} + \|u_{h_k} - v_{h_k} - tu_z\|_{C^\circ(Y)} \frac{M}{\delta \operatorname{dist}(\Omega', \partial\Omega'')}\varphi_{h_k}\Big) \quad \forall t \in [0, 1]. \end{aligned} \tag{4.9}$$

From (4.6) and (4.9) it follows that

$$t|Dw_k| \leq \varphi_{h_k}, \quad \forall t \in [0, 1[\text{ and } k \text{ large enough depending on } t. \tag{4.10}$$

Consequently, vii) of Proposition 1.6 provides that

$$\begin{aligned} &\int_Y f(h_k x, tDw_k)\, dx \geq \\ &\geq \inf\Big\{ \int_Y f(h_k x, t^2 z + Dv)\, dx : v \in W^{1,p}_{per}(Y)\ (C^1_{per}(Y) \text{ if } p = 'c1'), \\ &\qquad |t^2 z + Dv| \leq \varphi_{h_k} \text{ a.e. in } Y\Big\} = f^p_{\text{hom}}(t^2 z). \end{aligned} \tag{4.11}$$

By (4.11) and the convexity of $f(x, \cdot)$ it follows that

$$f^p_{\text{hom}}(t^2 z) \leq t \int_Y f\big(h_k x, \psi_{h_k} Du_{h_k} + (1 - \psi_{h_k})(Dv_{h_k} + tz)\big)\, dx + \tag{4.12}$$

$$
\begin{aligned}
&+(1-t)\int_Y f\big(h_k x, \frac{t}{1-t}(u_{h_k}-v_{h_k}-tu_z)D\psi_{h_k}\big)\,dx \le \\
&\le t\int_Y \psi_{h_k} f(h_k x, Du_{h_k})\,dx + t\int_Y (1-\psi_{h_k}) f(hx, Dv_{h_k}+tz)\,dx + \\
&+(1-t)\int_Y f\big(h_k x, \frac{t}{1-t}(u_{h_k}-v_{h_k}-tu_z)D\psi_{h_k}\big)\,dx \le \\
&\le \int_Y f(h_k x, Du_{h_k})\,dx + \int_{Y\setminus\Omega'} f(hx, Dv(h_k x)+tz)\,dx + \\
&+(1-t)\int_Y f\big(h_k x, \frac{t}{1-t}(u_{h_k}-v_{h_k}-tu_z)D\psi_{h_k}\big)\,dx \quad \forall t\in[0,1[.
\end{aligned}
$$

Since

$$u_{h_k}-v_{h_k}-tu_z \to 0 \text{ in } C^\circ(Y) \text{ as } k\to+\infty,$$

by virtue of (2.5) of Lemma 2.1

$$
\begin{aligned}
&\forall t\in[0,1[\quad \exists k_t\in\mathbb{N} : \forall k(\in\mathbb{N}) > k_t \\
&\left|\frac{t}{1-t}(u_{h_k}-v_{h_k}-tu_z)D\psi_{h_k}\right| \le \frac{t}{1-t}\|u_{h_k}-v_{h_k}-tu_z\|_{C^\circ(Y)}\cdot \\
&\cdot\frac{M}{\delta\operatorname{dist}(\Omega',\partial\Omega'')}\varphi_{h_k} \le \alpha\varphi_{h_k} \quad \text{a.e. in } Y,
\end{aligned}
$$

where α is the constant given in (0.8). Consequently Lemma 1.10 provides that

$$
\begin{aligned}
(4.13)\qquad &\limsup_{k\to+\infty}\int_Y f\Big(h_k x, \frac{t}{1-t}(u_{h_k}-v_{h_k}-tu_z)D\psi_{h_k}\Big)\,dx \le \\
&\le c|\Omega|,\ \forall t\in[0,1[,
\end{aligned}
$$

where c is the constant defined in (1.31). Hence, by passing to the limit in (4.12) as $k\to+\infty$, (0.2), (4.7) and (4.13) provide that

$$f^p_{\text{hom}}(t^2 z) \le F'^p(Y, tu_z)+|Y\setminus\Omega'|\int_Y f(y, tz+Dv)\,dy+(1-t)c|\Omega|,\ \ \forall t\in[0,1[,$$

from which, by virtue of Lemma 1.11,

$$
\begin{aligned}
(4.14)\qquad &f^p_{\text{hom}}(t^2 z) \le tF'^p(Y,u_z) + (1-t)\int_Y f(y,0)\,dy + \\
&+|Y\setminus\Omega'|\int_Y f(y, tz+Dv)\,dy + (1-t)c|\Omega|,\ \ \forall t\in[0,1[.
\end{aligned}
$$

As Ω' increases to Y and $t \to 1^-$, from (4.14) it follows that

$$\overline{f}^p_{\text{hom}}(z) \le F'(Y, u_z). \tag{4.15}$$

Consider now the general case with Ω a bounded open subset of $\mathbb{R}^n$. For every k in $\mathbb{N}$, let Q^k_j, B^k_j, $j = 1, \dots, m_k$, be cubes of type $x + rY$ such that

$$\begin{cases} Q^k_i \cap Q^k_j = \phi \quad \text{if } i \neq j, \ \bigcup\limits_{j=1}^{m_k} Q^k_j \subseteq \Omega, \ |\Omega \setminus \bigcup\limits_{j=1}^{m_k} Q^k_j| < \frac{1}{k}, \\ B^k_j \subset\subset Q^k_j, \quad |Q^k_j \setminus B^k_j| < \dfrac{1}{km_k} \quad \forall\ j \in \{1, \dots, m_k\}. \end{cases} \tag{4.16}$$

From (4.16) and (2.30) of Proposition 2.3 it follows that

$$\begin{aligned} F'^p(\Omega, u_z) \ge F'^p_-(\Omega, u_z) \ge F'^p_-\Big(\bigcup_{j=1}^{m_k} Q^k_j, u_z\Big) \ge \\ \ge \sum_{j=1}^{m_k} F'^p_-(Q^k_j, u_z) \ge \sum_{j=1}^{m_k} F'^p(B^k_j, u_z). \end{aligned} \tag{4.17}$$

On the other hand, Lemma 4.2 and (4.15) provide that

$$F'^p(B^k_j, u_z) = |B^k_j| F'^P(Y, u_z) \ge |B^k_j| \overline{f}^p_{\text{hom}}(z), \ j = 1, \dots m_k, \ \forall k \in \mathbb{N}. \tag{4.18}$$

Combining (4.17) with (4.18) and (4.16) it results

$$\begin{aligned} F'^p(\Omega, u_z) \ge \sum_{j=1}^{m_k} |B^k_j| \ \overline{f}^p_{\text{hom}}(z) = \\ = \Big|\bigcup_{j=1}^{m_k} B^k_j\Big| \overline{f}^p_{\text{hom}}(z) \ge (|\Omega| - \frac{2}{k}) \ \overline{f}^p_{\text{hom}}(z). \end{aligned} \tag{4.19}$$

As $k \to \infty$ in (4.19), inequality (4.5) is proved. □

Combining Lemma 4.1 with Proposition 4.3, we obtain the following result:

Corollary 4.4. *Let f and φ be functions satisfying* (0.2)*,* (0.8)*, for p in $]n, +\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\text{hom}}$ be the function defined in* (1.29) *and F'^p, F''^p be the functionals defined in* (1.13)*. Assume that* $(\text{dom}\, f^p_{\text{hom}})^\circ \neq \phi$*. Then*

$$F'^p(\Omega, u_z) = F''^p(\Omega, u_z) = |\Omega| \ \overline{f}^p_{\text{hom}}(z) \tag{4.20}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every z in $\mathbb{R}^n$.

5. Some abstract results.

For every u in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for every y in $\mathbb{R}^n$ denote by u^y the function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ defined by

$$u^y(x) = u(x - y) \quad \forall x \in \mathbb{R}^n . \tag{5.1}$$

Let U be a vector subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u^y \in U$ for every u in U and for every y in $\mathbb{R}^n$, and, for every bounded open subset Ω of $\mathbb{R}^n$, let τ_Ω be a topology on U.

Moreover let V be a subspace of U τ_Ω-dense for every bounded open subset Ω of $\mathbb{R}^n$.

For every bounded open subset Ω of $\mathbb{R}^n$, consider a functional $G(\Omega, \cdot)$ satisfying

$$\begin{gathered} G(\Omega, \cdot) : U \to [0, +\infty], \\ G(\Omega, \cdot) \text{ convex and } \tau_\Omega\text{-lower semicontinuous,} \end{gathered} \tag{5.2}$$

and define the following relaxed functional of $G(\Omega, \cdot)$ on U

$$\begin{aligned} (sc^-(\tau_\Omega)G)(\Omega, u) = \inf\{&\liminf_h G(\Omega, u_h) : \\ &\{u_h\}_{h\in\mathbb{N}} \subseteq V, u_h \overset{\tau_\Omega}{\to} u\}, \quad u \in U. \end{aligned} \tag{5.3}$$

Moreover for every bounded open subset Ω of $\mathbb{R}^n$ define

$$(sc^-(\tau_\Omega)G)_-(\Omega, u) = \sup_{A\subset\subset\Omega} (sc^-(\tau_A)G)(A, u), \quad u \in U. \tag{5.4}$$

In [26], by mean of a Jensen type inequality (see Proposition 4.1, in [40]), the following result is proved, in order to give sufficient conditions to deduce the identity of $G_-(\Omega, u)$ and $(sc^-(\tau_\Omega)G)_-(\Omega, u)$.

Proposition 5.1. *Let G be as in* (5.2). *Assume that $G(\cdot, u)$ is increasing for every u in U and*

(5.5) $G(\Omega_r^-, u^y) \le G(\Omega, u)$
for every bounded open subset Ω of $\mathbb{R}^n$, $u \in U$, $r > 0$ and for every $y \in \mathbb{R}^n$ with $|y| < r$

and that

(5.6) *for every $u \in U$ if u_ϵ is the regularized function of u as in* (1.10)*, then $u_\epsilon \in V$ for every $\epsilon > 0$ and $u_\epsilon \overset{\tau_\Omega}{\to} u$ for every bounded open subset Ω of $\mathbb{R}^n$.*

Then

(5.7) $(sc^-(\tau_\Omega)G)_-(\Omega, u) = G_-(\Omega, u)$
for every bounded open subset Ω of $\mathbb{R}^n$, $u \in U$.

6. A representation result on $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$.

In this section, following the same outlines of cap. 5 in [20], we give first a representation result of F^p_- on C^1, where F^p is defined in (1.14). Then, we obtain a representation result of F^p_- on $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$.

Lemma 6.1. [20]. *Let f and φ be functions satisfying* (0.2), (0.8), *for p in $]n, +\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\mathrm{hom}}$ be the function defined in* (1.29) *and F''^p be the functional defined in* (1.13). *Assume that* $(\mathrm{dom}\, f^p_{\mathrm{hom}})^\circ \neq \phi$.

Then

$$F'^p_-(\Omega, u) \geq \int_\Omega \overline{f}^p_{\mathrm{hom}}(Du)\, dx \tag{6.1}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every piecewise affine function u.

Proof. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u = \sum_{j=1}^m (u_{z_j} + s_j)\chi_{\overset{\circ}{P_j}}$ be a piecewise affine function. For every j in $\{1, \ldots, m\}$ set $\Omega_j = \Omega \cap P_j^\circ$ and, for ϵ sufficiently small, let $\Omega_1^\epsilon, \ldots, \Omega_m^\epsilon$ be open subsets of $\mathbb{R}^n$ with $\Omega_j^\epsilon \subset\subset \Omega_j$, $|\Omega_j \setminus \Omega_j^\epsilon| < \epsilon$ for every j in $\{1, \ldots, m\}$.

Since the functional $F'^p_-(\cdot, u)$ is increasing, from Proposition 2.3, (1.18), Corollary 4.4 and (1.19) it follows that

$$F'^p_-(\Omega, u) \geq F'^p_-(\bigcup_{j=1}^m \Omega_j, u) \geq \sum_{j=1}^m F'^p_-(\Omega_j, u) \geq \tag{6.2}$$

$$\geq \sum_{j=1}^m F'^p(\Omega_j^\epsilon, u_{z_j}) = \sum_{j=1}^m |\Omega_j^\epsilon|\, \overline{f}^p_{\mathrm{hom}}(z_j) = \int_{\bigcup_{j=1}^m \Omega_j^\epsilon} \overline{f}^p_{\mathrm{hom}}(Du)\, dx.$$

Inequality (6.1) is obtained by passing to the limit, as $\epsilon \to 0$, in (6.2).

Lemma 6.2. [20]. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8), *for p in $]n, +\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\mathrm{hom}}$ be the function defined in* (1.29) *and F''^p be the functional defined in* (1.13). *Assume that* $(\mathrm{dom}\, f^p_{\mathrm{hom}})^\circ \neq \phi$.

Then

$$F'^p_-(\Omega, u) \leq \int_\Omega \overline{f}^p_{\mathrm{hom}}(Du)\, dx \tag{6.3}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every u in $C^1(\mathbb{R}^n)$.

Proof. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $u \in C^1(\mathbb{R}^n)$. Let δ be the constant fixed in (2.1), let r be in $]0, \delta[$ and let c be the constant given by Proposition 3.5.

First assume that $Du(x)$ belongs to $(\mathrm{dom}\overline{f}^p_{\mathrm{hom}})^\circ$ for every x in $\overline{\Omega}$.

For every ν in $\mathbb{N}$ let $\{Q^\nu_j\}_{j\in N}$ be a sequence of open cubes of $\mathbb{R}^n$ with sidelenght $\frac{1}{\nu}$ such that $|\mathbb{R}^n \setminus \bigcup\limits_{j=1}^{\infty} Q^\nu_j| = 0$ and $Q^\nu_i \cap Q^\nu_j = \phi$ if $i \neq j$. Denote x^ν_j the center of Q^ν_j and set $z^\nu_j = Du(x^\nu_j)$. Moreover, for every $\epsilon > 0$, let $Q^{\nu,\epsilon}_j$ be the cube with center in x^ν_j and faces parallel to the ones Q^ν_j and sidelenght $\frac{1}{\nu} + \epsilon$.

Since u is in $C^1(\mathbb{R}^n)$, it turns out that $\mathrm{dist}(\bigcup\limits_{x\in\overline{\Omega}} Du(x), \partial(\mathrm{dom}\overline{f}^p_{\mathrm{hom}})) > 0$. Hence there exists $t_\circ$ in $]0, 1[$ such that $\frac{1}{t_\circ} Du(x)$ belongs to $(\mathrm{dom}\overline{f}^p_{\mathrm{hom}})^\circ$ for every x in Ω.

Let $\Omega' \subset\subset \Omega$ and let us choose $\nu \in \mathbb{N}$ and the cubes Q^ν_j so that $Q^\nu_j \subset\subset \Omega$ for every $j = \{1, \dots, m_\nu\}$, $\Omega' \subset\subset \left(\bigcup_{j=1}^{m_\nu} \overline{Q}^\nu_j\right)^\circ$ and

$$\frac{1}{1-t_\circ} \sup_{x\in Q^\nu_j} |Du(x) - z^\nu_j| \leq \frac{1}{2} r \quad \forall j \in \{1, \dots, m_\nu\}. \tag{6.4}$$

Let $\epsilon_\nu > 0$ be such that $Q^{\nu,\epsilon_\nu}_j \subset\subset \Omega$ for every j in $\{1, \dots, m_\nu\}$ and, by (6.4),

$$\frac{1}{1-t_\circ} \sup_{x\in Q^{\nu,\epsilon}_j} |Du(x) - z^\nu_j| \leq r \quad \forall j \in \{1, \dots, m_\nu\},\ \forall \epsilon \in]0, \epsilon_\nu[. \tag{6.5}$$

From (6.5) and Proposition 3.5 it follows that

$$F''^p_-\big(Q^{\nu,\epsilon}_j, \frac{1}{1-t_\circ}(u - u_{z^\nu_j})\big) \leq c|Q^{\nu,\epsilon}_j|, \forall j \in \{1, \dots, m_\nu\}, \forall \epsilon \in]0, \epsilon_\nu[. \tag{6.6}$$

Proposition 2.3, the convexity of F''^p_- and (6.6) provide that

$$\begin{aligned} F''^p_-(\Omega', u) &\leq F''^p_-\big(\bigcup_{j=1}^{m_\nu} Q^{\nu,\epsilon}_j, u\big) \leq \sum_{j=1}^{m_\nu} F''^p_-(Q^{\nu,\epsilon}_j, u) = \\ &= \sum_{j=1}^{m_\nu} F''^p_-\big(Q^{\nu,\epsilon}_j, \frac{t_\circ u_{z^\nu_j}}{t_\circ} + (1-t_\circ)\frac{u - u_{z^\nu_j}}{1-t_\circ}\big) \leq \end{aligned} \tag{6.7}$$

$$\leq \sum_{j=1}^{m_\nu} \big(t_\circ F''^{p}_{-}\big(Q_j^{\nu,\epsilon}, \frac{u_{z_j^\nu}}{t_\circ}\big) + (1-t_\circ) F''^{p}_{-}\big(Q_j^{\nu,\epsilon}, \frac{u-u_{z_j^\nu}}{1-t_\circ}\big)\big) \leq$$

$$\leq \sum_{j=1}^{m_\nu} \big(t_\circ F''^{p}_{-}\big(Q_j^{\nu,\epsilon}, \frac{u_{z_j^\nu}}{t_\circ}\big) + (1-t_\circ)c|Q_j^{\nu,\epsilon}|\big).$$

Combining (6.7) with Corollary 4.4, it results

$$F''^{p}_{-}(\Omega', u) \leq \sum_{j=1}^{m_\nu} \big(t_\circ |Q_j^{\nu,\epsilon}| \overline{f}^{p}_{\infty}\big(\frac{z_j^\nu}{t_\circ}\big) + (1-t_\circ)c|Q_j^{\nu,\epsilon}|\big). \tag{6.8}$$

As $\epsilon \to 0$ in (6.8), since for $j = 1, \dots, m_\nu$, $\frac{z_j^\nu}{t}$ belongs to $\mathrm{dom}\overline{f}^{p}_{\mathrm{hom}}$,

$$F''^{p}_{-}(\Omega', u) \leq t_\circ \sum_{j=1}^{m_\nu} |Q_j^\nu| \overline{f}^{p}_{\infty}(\frac{z_j^\nu}{t_\circ}) + (1-t_\circ)c|\Omega|. \tag{6.9}$$

Now, since u belongs to $C^1(\mathbb{R}^n)$ and $\big(\bigcup_{x\in\overline{\Omega}} \big\{\frac{1}{t_\circ} Du(x)\big\}\big) \subseteq (\mathrm{dom}\overline{f}^{p}_{\infty})^\circ$, and $\overline{f}^{p}_{\mathrm{hom}}$ is continuous on $(\mathrm{dom}\overline{f}^{p}_{\mathrm{hom}})^\circ$, as $\nu \to \infty$ it results

$$\lim_{\nu} \sum_{j=1}^{m_\nu} |Q_j^\nu| \overline{f}^{p}_{\mathrm{hom}}\big(\frac{z_j^\nu}{t_\circ}\big) = \int_\Omega \overline{f}^{p}_{\mathrm{hom}}\big(\frac{1}{t_\circ} Du(x)\big)\, dx < +\infty. \tag{6.10}$$

Therefore, if $Du(x)$ belongs to $(\mathrm{dom}\overline{f}^{p}_{\infty})^\circ$ for every x in Ω, inequality (6.3) is obtained from (6.9), (6.10), taking into account the continuity of $\overline{f}^{p}_{\infty}$ on $(\mathrm{dom}\overline{f}^{p}_{\infty})^\circ$ and passing to the limit for Ω' increasing to Ω and $t_\circ \to 1^-$.

On the other hand, if there exists $E \subseteq \Omega$ such that $|E| > 0$ and

$$Du(x) \notin cl(\mathrm{dom}\,\overline{f}^{p}_{\mathrm{hom}}), \quad \forall x \in E,$$

then

$$\int_\Omega \overline{f}^{p}_{\mathrm{hom}}(Du)\, dx = +\infty$$

and (6.3) holds. Therefore assume that

$$Du(x) \in cl(\mathrm{dom}\overline{f}^{p}_{\mathrm{hom}}) \qquad \forall x \in \Omega. \tag{6.11}$$

(6.11) and Lemma 1.8 provide that $tDu(x)$ belongs to $(\mathrm{dom}\overline{f}^p_{\mathrm{hom}})^\circ$, for every t in $]0,1[$ and x in $\overline{\Omega}$. Consequently, by applying (6.3) with the function tu, it results

$$F''^p_-(\Omega, tu) \leq \int_\Omega \overline{f}^p_{\mathrm{hom}}(tDu)\,dx \qquad \forall t \in]0,1[. \tag{6.12}$$

Hence, by virtue of the convexity of $\overline{f}^p_{\mathrm{hom}}$, it follows

$$F''^p_-(\Omega, tu) \leq t \int_\Omega \overline{f}^p_{\mathrm{hom}}(Du)\,dx + (1-t)|\Omega| f^p_{\mathrm{hom}}(0) \quad \forall t \in]0,1[. \tag{6.13}$$

Finally, by passing to liminf for $t \to 1^-$ in (6.13), the semicontinuity of $F''^p_-(\Omega,\cdot)$ provides (6.3). $\square$

We prove, now the representation result of F^p_- on $C^1(\mathbb{R}^n)$.

Proposition 6.3. [20]. *Let f and φ be the functions satisfying* (0.2), (0.7), (0.8). *For p in $]n,+\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\mathrm{hom}}$ be the function defined in* (1.29) *and F'^p, F''^p be the functionals defined in* (1.13). *Assume that* $(\mathrm{dom} f^p_{\mathrm{hom}})^\circ \neq \phi$.
Then

$$F'^p_-(\Omega, u) = F''^p_-(\Omega, u) = \int_\Omega \overline{f}^p_{\mathrm{hom}}(Du)\,dx \tag{6.14}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every u in $C^1(\mathbb{R}^n)$.

Proof. Let Ω be a bounded open subset of $\mathbb{R}^n$ and u in $C^1(\mathbb{R}^n)$.
Prove that

$$F'^p_-(\Omega, u) \geq \int_\Omega \overline{f}^p_{\mathrm{hom}}(Du)\,dx. \tag{6.15}$$

Assume $F'^p_-(\Omega, u) < +\infty$. Denote G_Ω the restriction of $F'^p_-(\Omega,\cdot)$ to $W^{1,\infty}(\Omega)$. Obviously u is in $\mathrm{dom}G_\Omega$.

If $u_\circ$ denotes the function defined by $u_\circ(x) = 0$ for every x in $\mathbb{R}^n$, Proposition 3.5 provides that $u_\circ$ belongs to $(\mathrm{dom}G_\Omega)^\circ$, where the interior is taken in the $W^{1,\infty}(\Omega)$-topology. Consequently tu belongs to $(\mathrm{dom}G_\Omega)^\circ$ for every t in $]0,1[$ (see, for example [31], pag. 413) and therefore

(6.16) G_Ω is $W^{1,\infty}(\Omega)$-continuous at tu for every $t \in]0,1[$.

For every t in $]0,1[$ let $\{u^t_h\}_{h\in N}$ be a sequence of piecewise affine functions such that $u^t_h \to tu$ in $W^{1,\infty}(\Omega)$ as $h \to +\infty$ (see for example [32], pag. 309). Then from (6.16), Lemma 6.1, Fatou's Lemma and the l.s.c. of $\overline{f}^p_{\text{hom}}$ (see Lemma 1.8), it follows that

$$G_\Omega(tu) = \lim_h G_\Omega(u^t_h) = \lim_h F'^p_-(\Omega, u^t_h) \geq \tag{6.17}$$

$$\geq \liminf_h \int_\Omega \overline{f}^p_{\text{hom}}(Du^t_h)\,dx \geq \int_\Omega \overline{f}^p_{\text{hom}}(tDu)\,dx, \quad \forall t \in]0,1[.$$

On the other hand, by virtue of Lemma 1.11, it results that

$$G_\Omega(tu) \leq tG_\Omega(u) + (1-t)|\Omega| \int_\Omega f(y,0)\,dy \qquad \forall t \in]0,1[. \tag{6.18}$$

Hence, combining (6.17) with (6.18) it follows that

$$\int_\Omega \overline{f}^p_{\text{hom}}(tDu)\,dx \leq tF'^p_-(\Omega,u) + (1-t)|\Omega| \int_\Omega f(y,0)\,dy, \ \forall t \in]0,1[. \tag{6.19}$$

As $t \to 1^-$ in (6.19), Fatou's lemma and Lemma 1.8 provide (6.15).

The conclusion follows from (6.15) and Lemma 6.2. $\square$

To extend (6.14) on $W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$, we recall two lemmas. For the proof of these lemmas compare the proof of the Lemma 5.5 and Lemma 5.6 in [20].

For every u in $C^\circ(\mathbb{R}^n)$ and y in $\mathbb{R}^n$, define the function u^y as in (5.1).

Lemma 6.4. [20]. *Let f and φ be functions satisfying* (0.2) *and, for p in $]n,+\infty]$ or $p = {}'c1'$, let F''^p, F'^p be the functionals defined in* (1.13). *Then*

$$F'^p_-(\Omega^-_r, u^y) \leq F'^p_-(\Omega, u) \qquad F''^p_-(\Omega^-_r, u^y) \leq F''^p_-(\Omega, u) \tag{6.20}$$

for every bounded open subset Ω of $\mathbb{R}^n$, for every $u \in C^\circ$, $r > 0$, $y \in \mathbb{R}^n$ such that $|y| < r$.

Let f and φ be functions satisfying (0.2), (0.8) and, for p in $]n,+\infty]$ or $p' = c1'$, let f^p_{hom} be the function defined in (0.4) and $\overline{f}^p_{\text{hom}}$ be the function defined in (1.29). For every bounded open subset Ω of $\mathbb{R}^n$, define the functionals

$$F^p_{\text{hom}}(\Omega,\cdot) : u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \to \int_\Omega f^p_{\text{hom}}(Du)\,dx, \tag{6.21}$$

$$\overline{F}^p_{\text{hom}}(\Omega,\cdot) : u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \to \int_\Omega \overline{f}^p_{\text{hom}}(Du)\,dx. \tag{6.22}$$

Moreover let $(sc^-(C^\circ(\Omega))F^p_{\text{hom}})(\Omega,\cdot)$ be defined by (5.3) with $G = F^p_{\text{hom}}$, $U = W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$, $V = C^1(\mathbb{R}^n)$, let $(sc^-(C^\circ_\circ(\Omega))F^p_{\text{hom}})(\Omega,\cdot)$ be defined by (5.3) with $G = F^p_{\text{hom}}$, $U = W^{1,1}_\circ(\Omega) \cap C^\circ(\mathbb{R}^n)$, $V = C^1_\circ(\Omega)$ and let $(sc^-(C^\circ(\Omega))F^p_{\text{hom}})_-(\Omega,\cdot)$ be defined by (5.4). Analogously for $\overline{F}^p_{\text{hom}}$.

Lemma 6.5. [20]. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8) *and for p in $]n, +\infty]$ or $p = 'c1'$, let F^p_{hom}, $\overline{F}^p_{\text{hom}}$ be the functionals defined in* (6.21), (6.22). *Assume that* $(\text{dom}\, f^p_{\text{hom}})^\circ \neq \phi$.

Then

$$\begin{aligned}(6.23)\qquad \overline{F}^p_{\text{hom}}(\Omega, u) &= (sc^-(C^\circ(\Omega))\overline{F}^p_{\text{hom}})_-(\Omega, u) = \\ &= (sc^-(C^\circ(\Omega))F^p_{\text{hom}})_-(\Omega, u)\end{aligned}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every u in $W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$,

$$\begin{aligned}(6.24)\qquad \overline{F}^p_{\text{hom}}(\Omega, u) &= (sc^-(C^\circ_\circ(\Omega))\overline{F}^p_{\text{hom}})(\Omega, u) = \\ &= (sc^-(C^\circ_\circ(\Omega))F^p_{\text{hom}})(\Omega, u)\end{aligned}$$

for every bounded open subset Ω of $\mathbb{R}^n$ with Lipschitz boundary and for every u in $W^{1,1}_\circ(\Omega) \cap C^\circ(\mathbb{R}^n)$.

Prove now the representation result.

Proposition 6.6. [20]. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8), *for $p \in]n, +\infty]$ or $p = 'c1'$, let $\overline{f}^p_{\text{hom}}$ be the function defined in* (1.29) *and F'^p, F''^p be the functionals defined in* (1.13). *Assume that* $(\text{dom} f^p_{\text{hom}})^\circ \neq \phi$.

Then

$$(6.25)\qquad (F'^p)_-(\Omega, u) = (F''^P)_-(\Omega, u) = \int_\Omega \overline{f}^p_{\text{hom}}(Du)dx$$

for every bounded open subset Ω of $\mathbb{R}^n$ and for every u in $W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$.

Proof. By virtue of Lemma 6.4, the functionals F'^p and F''^p satisfy the assumptions of Proposition 5.1 with $U = W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$, $V = C^1(\mathbb{R}^n)$, $\tau_\Omega = C^\circ(\Omega)$. Then, by virtue of Proposition 5.1 it results

(6.26)

$$\begin{cases} (F'^p)_-(\Omega, u) = ((F'^p)_-)_-(\Omega, u) = (sc^-(C^\circ(\Omega))(F'^p)_-)_-(\Omega, u), \\ (F''^p)_-(\Omega, u) = ((F''^p)_-)_-(\Omega, u) = (sc^-(C^\circ(\Omega))(F''^p)_-)_-(\Omega, u), \end{cases}$$

for every bounded open subset Ω of $\mathbb{R}^n$ and u in $W^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$.

Finally, combining (6.26) with Proposition 6.3 and the first equality of (6.23), the representation result (6.25) holds. $\square$

7. The convergence of minimun points.

Let q in $]n,+\infty[$ and $p \geq q$ or $p = 'c1'$. For every bounded open subset Ω of $\mathbb{R}^n$, denote $sc^-(C^\circ_\circ(\Omega))^{p,q}$ the operator defined by (5.3) with $U = W^{1,q}_\circ(\Omega)$ and $V = W^{1,p}_\circ(\Omega)$ ($V = C^1(\mathbb{R}^n)$ if $p = 'c1'$).

Moreover denote $u_\circ$ the function defined by $u_\circ(x) = 0$ for every x in $\mathbb{R}^n$.

Theorem 7.1. *Let f and φ be functions satisfying* (0.2), (0.7), (0.8). *For p in $]n,+\infty]$ or $p = 'c1'$, let f^p_{hom} be the function defined in* (0.4), *$\overline{f}^p_{\text{hom}}$ be the function defined in* (1.29), *F^p_h be the functionals defined in* (1.11) *and F^p_{hom} be the functional defined in* (6.21). *For every p in $]n,+\infty]$ or $p = 'c1'$, for every bounded open subset Ω of $\mathbb{R}^n$ with Lipschitz boundary and β in $L^1(\Omega)$ define*

$$(7.1)\qquad m^p_h(\Omega,\beta) = \inf\Big\{\int_\Omega f(hx, Du)\,dx + \int_\Omega \beta u\,dx :$$
$$u \in W^{1,p}_\circ(\Omega)(u \in C^1_\circ(\Omega) \text{ if } p = 'c1'), |Du(x)| \leq \varphi(hx) \quad \text{for a.e. } x \text{ in } \Omega\Big\}$$

and, by denoting $Argmin(G)$ the set of minimum points of a functional G, for every $p \geq q$ or $p = 'c1'$ (where q is given in (0.7)*) define*

$$(7.2)\qquad M^p_h(\Omega,\beta) = \operatorname{Argmin}\Big\{sc^-(C^0_0(\Omega))^{p,q}F^p_h(\Omega,u) +$$
$$+\int_\Omega \beta u\,dx : \ u \in W^{1,q}_\circ(\Omega)\Big\}.$$

Assume $(\operatorname{dom} f^p_{\text{hom}})^\circ \neq \phi$.

Then, for every $n < p \leq q$, $m^p_h(\Omega,\beta) = m^q_h(\Omega,\beta)$ and the sequence $\{m^q_h\}$ converges, as $h \to +\infty$, to

$$(7.3)\quad m^q(\Omega,\beta) = \inf\Big\{\int_\Omega f^q_{\text{hom}}(Du)\,dx + \int_\Omega \beta u\,dx : u \in W^{1,q}_\circ(\Omega)\Big\} =$$
$$= \min\Big\{\int_\Omega \overline{f}^q_{\text{hom}}(Du)\,dx + \int_\Omega \beta u\,dx : u \in W^{1,q}_\circ(\Omega)\Big\} =$$
$$= \min\Big\{sc^-(C^\circ_\circ(\Omega))F^q_{\text{hom}}(\Omega,u) + \int_\Omega \beta u\,dx : u \in W^{1,q}_\circ(\Omega)\Big\}$$

and every sequence $\{u_h\}_{h\in N}$ such that $u_h \in M^q_h$ for every h in $\mathbb{N}$ is compact in $C^\circ(\Omega)$ and the subsequences that converge in $C^\circ(\Omega)$ converge to solutions of problems in (7.3).

Moreover, for every p *in* $]q, +\infty]$ *or* $p = 'c1'$, *the sequence* $\{m_h^p\}_{h\in N}$ *converges, as* $h \to +\infty$, *to*

$$(7.4)\quad m^p(\Omega,\beta) = \inf\left\{\int_\Omega f_{\text{hom}}^p(Du)\,dx + \int_\Omega \beta u\,dx : u \in W_\circ^{1,q}(\Omega)\right\} =$$
$$= \min\left\{\int_\Omega \overline{f}_{\text{hom}}^p(Du)\,dx + \int_\Omega \beta u\,dx : u \in W_\circ^{1,q}(\Omega)\right\} =$$
$$= \min\left\{sc^-(C_\circ^\circ(\Omega))F_{\text{hom}}^p(\Omega,u) + \int_\Omega \beta u\,dx : u \in W_\circ^{1,q}(\Omega)\right\}$$

where $W_\circ^{1,p}(\Omega)$ *in the first equality of* (7.4) *has to be replaced by* $C_\circ^1(\Omega)$ *when* $p = 'c1'$. *Every sequence* $\{u_h\}_{h\in N}$ *such that* $u_h \in M_h^p$ *for every* h *in* $\mathbb{N}$ *is compact in* $C^\circ(\Omega)$ *and the subsequences that converge in* $C^\circ(\Omega)$ *converge to minimum points solutions of minimum problems in* (7.3).

Assume $(\text{dom} f_{\text{hom}}^p)^\circ = \phi$.

Then $u_\circ$ *is the only solution of* $m^p(\Omega,\beta)$ *and the sequence* $\{u_h\}_{h\in N}$ *converges in* $C^\circ(\Omega)$ *to* $u_\circ$. *Moreover, if in addition we assume that*

$$(7.5)\qquad f(y,0) = \min_{z\in R^n} f(y,z) \quad \textit{for a.e. } y \textit{ in } Y,$$

it turns out that the sequence $\{m_h^p(\Omega,\beta)\}_{h\in N}$ *converges to* $m^p(\Omega,\beta)$ *and that*

$$m^p(\Omega,\beta) = |\Omega| \int_Y f(y,0)\,dy.$$

Proof. Assume $(\text{dom} f_{\text{hom}}^p)^\circ \neq \phi$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and β in $L^1(\Omega)$. The functional $u \in W_{\text{loc}}^{1,q}(\mathbb{R}^n) \to \int_\Omega \beta u$ is $C^\circ(\Omega)$-continuous on $W_{\text{loc}}^{1,q}(\mathbb{R}^n)$. Moreover, by virtue of (0.7), the functionals $F_h^p(\Omega,\cdot) + \int_\Omega \beta(\cdot)$ are equicoercive on $W_\circ^{1,q}(\Omega)$ in the topology of $C^\circ(\Omega)$. Therefore Theorem 1.5, Proposition 2.2 and Proposition 6.6 provide that

$$(7.6)\qquad \lim_h m_h^p(\Omega,\beta) = m^p(\Omega,\beta) \text{ for every } p \geq q \text{ or } p = 'c1'.$$

The last equalities in the right hand sides of (7.3) and (7.4) follow by (6.24) of Lemma 6.5. The convergence of solutions follows from the equicoerciveness of functionals F_h^p in the same way. If $(\text{dom } f_{\text{hom}}^p)^\circ = \phi$, the thesis follows by arguing as in Theorem 5.2 in [18]. □

Concerning to the Lavrentieff phenomenon, in Theorem 7.1 the minimum values m_h^p effectively depend on p. Recall the following examples given in [20]:

Example 7.2. *Let* $n = 1$, f *be function satisfying* (0.2) *and* K *be closed set such that* $K \subseteq Y$, $|K| = \frac{1}{2}$, $\overset{\circ}{K} = \phi$ *and* $\varphi = \chi_k$ *and* K_1 *be a subset of* K *such that* $|K_1| = \frac{1}{4}$. *Let* $q > n$ *and let* $\beta = 0$, $\Omega = Y$ *and* $f(x, z) = \chi_{k_1}(x)|z - 1|^q$. *For* p *in* $]n, +\infty]$ *or* $p = 'c1'$, *let* m_h^p, $h \in \mathbb{N}$, m^p *be as in Theorem 7.1. Then* f *and* φ *satisfy assumption of Theorem 7.1 and for every* p *in* $]n, +\infty]$

$$m^p = m_h^p = 0 < \frac{1}{4} = m_h^{c1} = m^{c1} \quad \textit{for every } \; h \in \mathbb{N}.$$

Acknowledgements. Work performed as part of the research project "Relaxation and Homogenization Methods in the Study of Composite Materials" of the Progetto Strategico C.N.R. 1996 "Applicazione della Matematica per la Tecnologia e la Società".

REFERENCES

[1] H. Attouch, *Varational Convergence for Functions and Operators,* Applicable Mathematics Series, Pitman, London, 1984.

[2] J.M., Ball - V.J. Mizel, *One-dimensional Variational Problems whose Minimizers do not satisfy the Euler-Lagrange Equation,* Arch. Rat. Mech. Anal., 90 (1985), pp. 325–388.

[3] A. Bensoussan - J.L. Lions - G. Papanicolaou, *Asymptotic Analysis for Periodic Structures,* North Holland, Amsterdam, 1978.

[4] H. Brezis - M. Sibony, *Équivalence de deux inéquations variationnelles et applications,* Arch. Rational Mech. Anal., 41 (1971), pp. 254–265.

[5] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations,* Longman Scientific & Technical, 1989.

[6] G. Buttazzo - G. Dal Maso, *Γ-limits of integral functionals,* J. Analyse Math., 37 (1980), pp. 145–185.

[7] A.L. Caffarelli - N.M. Riviere, *On the Lipschitz character of the stress tensor when twisting an elastic- plastic bar,* Arch. Rat. Mech Anal., 69 (1979), pp. 31–36.

[8] L. Carbone, *Sur la convergence des intégrales du type de l'énergie sur des functions a gradient borné,* J. Math. Pures Appl., 56 (1977), pp. 79–84.

[9] L. Carbone, *Γ-convergence des intégrales sur des fonctions avec des contraintes sur le gradient,* Comm. Part. Diff. Eq., 2 (1977), pp. 627–651.

[10] L. Carbone, *Sull'omogeneizzazione di un problema variazionale con vincoli sul gradiente,* Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., 63 (1977), pp. 10–14.

[11] L. Carbone, *Sur un probléme d'homogénéisation avec des contraintes sur le gradient,* J. Math. Pures Appl., 58 (1979), pp. 275–297.

[12] L. Carbone - A. Esposito Corbo - R. De Arcangelis, *Homogenization of Neumann Problems for Unbounded Integral Functionals,* Boll. UMI, (8) 2-B (1999), pp. 463–491.

[13] L. Carbone - R. De Arcangelis, *Unbounded functionals: Applications to the homogenization of gradient constrained problems,* Ric. di Mat., 48 (1999), pp. 139–182.

[14] L. Carbone - S. Salerno, *On a problem of homogenization with quickly oscillating constraints on the gradient,* J. Math. Anal. Appl., 90 (1982), pp. 219–250.

[15] L. Carbone - S. Salerno, *Further results on a problem of homogenization with constraints on the gradient,* J. Anal. Math., 44 (1984-85), pp. 1–20.

[16] L. Carbone - S. Salerno, *Homogenization with unbounded constraints on the gradient,* Nonlinear Anal., 9 (1985), pp. 431–444.

[17] L. Carbone - C. Sbordone, *Some properties of Γ-limits of integral functionals,* Ann. Mat. Pura Appl., 122 (1979), pp. 1–60.

[18] A. Corbo Esposito - R. De Arcangelis, *Homogenization of Dirichlet problems with nonnegative bounded constraints on the gradient,* J. Analyse Math., 64 (1994), pp. 53–96.

[19] A. Corbo Esposito - R. De Arcangelis, *The Lavrentieff phenomenon and different processes of homogenization,* Comm. Part. Diff. Eq., 17 (1992), pp. 1503–1538.

[20] A. Corbo Esposito - F. Serra Cassano, *A Lavrentieff phenomenon for problems of homogenization with constraints on the gradient,* Ric. di Mat., 46 (1997), pp. 127–159.

[21] G. Dal Maso, *An Introduction to Γ-Convergence,* Birkhäuser-Verlarg, 1993.

[22] C. D'Apice - U. De Maio, *A homogenization result of unbounded variational functionals,* Rend. Accad. Naz. XL, 20-1 (1996), pp. 65–93.

[23] R. De Arcangelis, *A General homogenization result for almost periodic functionals,* J. Math. Anal. Appl., 156 (1991), pp. 358–380.

[24] R. De Arcangelis - G. Gargiulo, *Homogenization of integral functionals with linear growth defined on vector-valued functions,* Nodea, 2 (1995), pp. 371–416.

[25] R. De Arcangelis - A. Gaudiello - G. Paderni, *Some cases of homogenization of linearly coercive gradient constrained variational problems,* M^3AS, 6 (1996), pp. 901–940.

[26] R. De Arcangelis - A. Vitolo, *Some cases of homogenization with unbonded oscillating constraints on the gradient,* Asymp. Anal., 5 (1992), pp. 397–428.

[27] E. De Giorgi, *G-operators and Γ-convergence,* Proceedings of International Congress of Mathematicians, Warszawa, August 16-24, 1983, PWN, Warszawa (1984), pp. 1175–1191.

[28] E. De Giorgi - T. Franzoni, *Su un tipo di convergenza variazionale,* Rend. Sem. Mat. Brescia, 3 (1979), pp. 63–101.

[29] E. De Giorgi - G. Letta, *Une notion générale de convergence faible pour des fonctions croissantes d'ensemble,* Ann. Sc. Norm. Sup. Pisa, 4 (1977), pp. 61–99.

[30] E. De Giorgi - S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine,* Boll. Un. Mat. Ital., 8 (1973), pp. 391–411.

[31] N. Dunford - J.T. Schwartz, *Linear Operators, Part I,* Wiley Interscience Publiations, 1957.

[32] I. Ekeland - R. Temam, *Convex Analysis and Variational Problems,* North-Holland American Elsevier, 1976.

[33] N. Fusco, *Γ-convergenza unidimensionale,* Boll. Un. Mat. Ital., (5) 16-B (1979), pp. 74–86.

[34] H. Lanchon, *Solution du probléme de torsion élastoplastique d'une barre cylindrique de section quelconque,* C.R. Acad. Sci. Paris, 269 (1969), pp. 791–794.

[35] H. Lanchon, *Sur la solution du probléme de torsion élastoplastique d'une barre cylindrique de section multiconnexe,* C.R. Acad. Sci. Paris, 271 (1970), pp. 1137–1140.

[36] H. Lanchon, *Torsion élastoplastique d'une barre cylindrique de section simplement ou multiplement connexe,* J. Mécanique, 13 (1974), pp. 267–320.

[37] M. Lavrentieff, *Sur quelques problémes du calcul des variations,* Ann. Mat. Pura Appl., 4 (1926), pp. 7–29.

[38] B. Mania', *Sopra un esempio di Lavrentieff,* Boll. Un. Mat. Ital., 13 (1934), pp. 147–153.

[39] P. Marcellini, *Periodic Solution and homogenization of non linear variational problems,* Ann. Mat. Pura Appl., 117 (1978), pp. 139–152.

[40] P. Marcellini - C. Sbordone, *Homogenization of non-uniformly elliptic operators,* Applicable Anal., 8 (1978), pp. 101–113.

[41] T.W. Ting, *Elastic-plastic torsion problem III,* Arch. Rat. Mech. Anal., 34 (1969), pp. 228–244.

[42] T.W. Ting, *Elastic-plastic torsion of convex cylindrical bars,* J. Math. Mech., 19 (1969), pp. 531–551.

[43] T.W. Ting, *Elastic-plastic torsion of simply connected cylindrical bars,* Indiana Univ. Math. J., 20 (1971), pp. 1074–1076.

C. D'Apice,
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata,
Università di Salerno,
Via Ponte don Melillo,
Fisciano, SA (ITALY),

T. Durante,
Dipartimento di Ingegneria Industriale,
Università di Cassino,
Via G. Di Biasio 43,
Cassino, FR (ITALY),

A. Gaudiello,
Dipartimento di Ingegneria Agraria ed Agronomia del Territorio,
Università di Napoli "Federico II",
Via Università 100,
Portici, NA (ITALY).