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THE CLASS OF HYPERGROUPS IN WHICH THE HEART IS THE SET OF IDENTITIES

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In this paper, the hypergroups, for which the heart is the set of identities, are studied.

Let us denote by ε the class of hypergroups H, for which the heart is the set of identities of H, that is $\omega_H = E_H$. We shall say that a hypergroup H is a ε-hypergroup, if $H \in \varepsilon$.

Theorem 1. *Let $H \in \varepsilon$. Then H is flat, that means $\omega_H \cap K = \omega_K$, for each K, subhypergroup of H.*

Proof. Since $\omega_H \cap K = E_H \cap K = \{e \in K \mid \forall x \in H : x \in xe \cap ex\} \subset \{e \in K \mid \forall x \in K : x \in xe \cap ex\} \subset \omega_K$, it remains to prove that $\omega_K \subset \omega_H \cap K$. Indeed, for each $x \in \omega_K$, and for each $e \in E_K$, we have $x\beta_K e$, then $\exists n \in N^*, \forall i \in \{1, 2, \dots, n\}, a_i \in K$, such that $\{x, e\} \subset \prod_{i=1}^{n} a_i$.
Since $e \in E_K$, it results $e \in I_p(H) = \{e' \in H \mid \exists x \in H : x \in xe' \cup e'x\} \subset \omega_H = E_H$. So, $E_H = \omega_H \supset \prod_{i=1}^{n} a_i$, and therefore $x \in \omega_H \cap K$.
Then, $\omega_H \cap K = \omega_K$.

Remark 2. *If $H \in \varepsilon$ and K is a subhypergroup of H, then $K \in \varepsilon$.*

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Indeed, we have $E_K \subset \omega_K = \omega_H \cap K = E_H \cap K$, and for each $a \in E_H \cap K$ and for each $x \in H$, we have $x \in xa \cap ax$, so $\forall x \in K$, $x \in ax \cap xa$, from which it results $a \in E_K$, so $E_H \cap K \subset E_K$.
Then $E_K = E_H \cap K = \omega_K$.

Theorem 3. *Let $f : H_1 \to H_2$ be a very good homomorphism between hypergroups, and let $H_2 \in \varepsilon$. Then*
a) $f(\omega_{H_1}) = f(I_p(H_1)) = E_{H_2}$; b) $f(H_1)$ is a complete part of H_2.

Proof. a) Let $i \in I_p(H_1)$, that is there is $x \in H_1$, such that $x \in xi$ or $x \in ix$. So, $f(x) \in f(x)f(i)$ or $f(x) \in f(i)f(x)$, from which $f(i) \in I_p(H_2) \subset \omega_{H_2} = E_{H_2}$.
Then $f(I_p(H_1)) \subset E_{H_2}$.
Conversely, let $\overline{e}$ be an arbitrary element of E_{H_2}. So $f(x) \in f(x)\overline{e}$, for all $x \in H_1$; we have $\overline{e} \in f(x)/f(x) = f(x/x) \subset f(I_p(H_1))$, whence $f(I_p(H_1)) \supset E_{H_2}$. Therefore, $f(I_p(H_1)) = E_{H_2}$, so $f(I_p(H_1)) = \omega_{H_2}$.
On the other hand, we have $f(\omega_{H_1}) \subset \omega_{H_2}$.
Indeed, $\forall x \in \omega_{H_1}$, $\forall i \in I_p(H_1)$, we have $x\beta_{H_1}i$, so $f(x)\beta_{H_2}f(i)$ and since $f(i) \in \omega_{H_2} = E_{H_2}$, it results $f(x) \in \omega_{H_2}$.
Hence, $f(\omega_{H_1}) \subset \omega_{H_2} = f(I_p(H_1)) \subset f(\omega_{H_1})$ and we have the equality.

b) From a), the subhypergroup $f(H_1)$ contains ω_{H_2}, so it a complete part of H_2.

Remark 4. *The complete hypergroups form a subclass of ε. The are ε-hypergroups, which are not complete (see* [1]*, Th. 266). 1-hypergroups are also ε-hypergroups.*

Proposition 5. *A K_H-hypergroup is a ε-hypergroup if and only if H is a ε-hypergroup.*

Proof. "$\Leftarrow$" By Th.384, [1], we have $\omega_{K_H} = K(\omega_H) = K(E_H)$ and by Th. 377, [1], we have $K(E_H) = E(K_H)$, so $\omega_{K_H} = E(K_H)$, whence $K_H \in \varepsilon$.
"$\Rightarrow$" We have $K(\omega_H) = \omega_{K_H} = E(K_H) = K(E_H)$, whence $K(\omega_H) = \bigcup_{x\in\omega_H} A(x) = \bigcup_{x\in E_H} A(x) = K(E_H)$.
If we suppose that there is $x_0 \in \omega_H - E_H$, then $A(x_0) \subset \bigcup_{x\in\omega_H} A(x) = \bigcup_{x\in E_H} A(x)$, so there is $x_1 \in E_H$, such that $A(x_0) = A(x_1)$, because $\forall (x, y) \in H^2$, $x \neq y \Rightarrow A(x) \cap A(y) = \emptyset$.
Hence, $x_0 = x_1 \in E_H$, that is false. Therefore, $\omega_H = E_H$, that is $H \in \varepsilon$.

Proposition 6. *The direct product of ε-hypergroups is a ε-hypergroup.*

Proof. Let $H = \prod_{i \in I} H_i$. Let us notice that if $x = (x_i)_{i \in I}$ and $e = (e_i)_{i \in I}$, where $\{x_i, e_i\} \subset H_i$ and $e_i \in E_{H_i}$, then $x \in xe = (x_i e_i)_{i \in I}$ and $x \in ex = (e_i x_i)_{i \in I}$, that means $e \in E_H$.
We have $x \beta_n^H e$ if and only if $a^1 = (a_i^1)_{i \in I}, a^2 = (a_i^2)_{i \in I}, \dots, a^n = (a_i^n)_{i \in I}$ exist in H, such that $x \in \prod_{k=1}^{n} a^k \ni e$, that is if and only if $\forall i \in I$, $x_i \in \prod_{k=1}^{n} a_i^k \ni e_i$. So, $x_i \beta_n^H e_i$, that is $x_i \in \omega_{H_i} = E_{H_i}$, so $x \in E_H$. Hence $\omega_H = E_H$.

Definition 7. (see [6]). Let H be a hypergroup, G a group and $\{A_i\}_{i \in G}$ a family of nonempty sets, such that:
1) $A_1 = H$ (1 is the identity of G);
2) $\forall (i, j) \in G^2, i \neq j \Rightarrow A_i \cap A_j = \emptyset$.

Let us define on $K = \underset{i \in G}{\cup} A_i$ the following hyperoperation:
$\forall (x, y) \in H^2, x \otimes y = xy$;
$\forall (x, y) \in A_i \times A_j, x \otimes y = A_k$, if $(i, j) \neq (1, 1)$ and $ij = k$.
The hypergroup $\langle K, \otimes \rangle$ is called *(H, G)-hypergroup.*

Proposition 8. *Let H be a hypergroup.*
$H = E_H$ if and only if any (H, G)-hypergroup is a ε-hypergroup.

Proof. Let us notice that $\omega_K = H$ and $\forall y \in A_1 = H$, $\forall x \in A_j$, $j \neq 1$, $x \in x \otimes y = y \otimes x$. So, $y \in E_K \Leftrightarrow y \in E_H$. Hence, $\omega_K = E_K \Leftrightarrow H = E_H$.

Remark 9. *If $H \in \varepsilon$, then $\forall x \in H$, $i_l(x) = i_r(x)$, where*

$$i_l(x) = \{x' \in H \mid E_H \cap x'x \neq \emptyset\} \text{ and } i_r(x) = \{x'' \in H \mid E_H \cap xx'' \neq \emptyset\}.$$

Proof. Indeed, if $x' \in i_l(x)$, then $\exists e \in E_H$, such that $e \in x'x$, so $x'x \subset \omega_H = E_H$. Hence $x' \in ex' \subset (x'x)x' = x'(xx')$, whence $xx' \cap I_p(H) \neq \emptyset$, so $xx' \subset \omega_H = E_H$, that means $x' \in i_r(x)$. Then $i_l(x) \subset i_r(x)$ and similarly, we have $i_l(x) \supset i_r(x)$. Therefore, $\forall x \in H$, $i_l(x) = i_r(x)$.

Proposition 10. *If $H \in \varepsilon$, then $\forall x \in H, i_l(x) = C(x')$, where $x' \in i_l(x) = i_r(x)$, and $C(x')$ is the complete closure of x' in H.*

Proof. Let $y \in i_l(x)$. So, $\exists e \in E_H : e \in yx$, whence $x' \in (yx)x' = y(xx') \subset yE_H = C(y)$. Hence, $C(x') = C(y) \ni y$; then $i_l(x) \subset C(x')$.
Conversely, if $z \in C(x')$, then $z \in x'E_H = E_H x'$, whence $zx \subset E_H(x'x) \subset E_H$, so $z \in i_l(x)$; then $i_l(x) \supset C(x')$. Therefore, $\forall x \in H$, $i_l(x) = C(x')$.

Let H be a hypergroup with identities and let us define on H the following relation:

$$xRy \Leftrightarrow \exists z \in H : xz \supset E_H \subset yz.$$

Remark 11. *If H is a hypergroup with identities, then $\beta \supset R$.*

Indeed, we have $xRy \Leftrightarrow \exists z \in H : xz \subset E_H \supset yz$. Let $z' \in i_r(z)$. Then $x \in xzz' \subset E_H z'$, whence $C(x) = C(z')$ and similarly, we have $C(y) = C(z')$, so $C(x) = C(y)$, that is $x\beta y$.

Theorem 12. *Let H be a hypergroup with identities. $H \in \varepsilon$ if and only if $\beta = R$.*

Proof. "$\Rightarrow$" We have to verify only the inclusion $\beta \subset R$. Let $\{x, y\} \subset H$, such that $x\beta y$. Then $C(x) = C(y) = i_l(y')$, where $y' \in i_l(y)$, because $H \in \varepsilon$ (see Prop. 10). So, $\{x, y\} \subset i_l(y')$, whence $\exists e_1 \in E_H$, such that $e_1 \in xy'$. Hence, $xy' \subset \omega_H = E_H$ and similarly, we have $yy' \subset E_H$. Therefore, xRy.
"$\Leftarrow$" Now, let us suppose $\beta = R$. We shall check that $\omega_H = E_H$. Let $x \in \omega_H$ and $e \in E_H$. We have $e\beta x$, whence eRx. So, $\exists z \in H : xz \subset E_H \subset ez \ni z$ and then $x \in xz \subset E_H$, that means $x \in E_H$. Therefore, $\omega_H = E_H$, that is $H \in \varepsilon$.

Definition 13. (see [1], 372). Let $\langle H, \cdot\rangle$ be a hypergroup. We define $\forall (x, y) \in H^2, x \otimes y = C_H(xy)$. Then $\langle H, \otimes\rangle$ is a hypergroup, which we call the *completion* of $\langle H, \cdot\rangle$, and we shall denote $\Delta(H)$.

Theorem 14. *Let $\langle H, \cdot\rangle$ be an arbitrary hypergroup. Then $\Delta(H) \in \varepsilon$.*

Proof. Let us denote the group $\langle H/\beta, \cdot\rangle$ by H' and let us consider the following $K_{H'}$-hypergroup: $K_{H'} = \bigcup\limits_{\bar{x}\in H'} A(\bar{x})$, where $A(\bar{x}) = \beta(x)$ and $\forall (a, b) \in K_{H'} \otimes K_{H'}, a \bullet b = \bigcup\limits_{\bar{z}\in g(a)g(b)} A(\bar{z})$, where $\forall a \in K_{H'}$, $g(a) = \bar{x} \Leftrightarrow a \in A(\bar{x})$.
We shall verify that $K_{H'} = H$ and $a \bullet b = C_H(ab)$, that means $\langle K_{H'}, \bullet\rangle$ is the completion of $\langle H, \cdot\rangle$.
Since $g(a) = \bar{x} \Leftrightarrow a \in \beta(x) \Leftrightarrow \bar{a} = \bar{x}$, it results $a \bullet b = \bigcup\limits_{\bar{z}\in\bar{a}\bar{b}} A(\bar{z})$.
We have $K_{H'} = \bigcup\limits_{\bar{x}\in H'} A(\bar{x}) = \bigcup\limits_{x\in H} \beta(x) = H$ and $a \bullet b = \bigcup\limits_{\bar{z}\in\bar{a}\bar{b}} A(\bar{z}) = \bigcup\limits_{\bar{z}\in\bar{a}\bar{b}} \beta(z)$.
Let $\varphi_H : H \to H/\beta = H'$ be the canonical projection.
By [1], Theorems 66, 67, we have:

$$\bar{z} \in \bar{a}\bar{b} \Leftrightarrow \varphi_H(z) \in \varphi_H(a)\varphi_H(b) = \varphi_H(ab) \Leftrightarrow z \in \varphi_H^{-1}\varphi_H(ab) = C_H(ab).$$

Then $a \bullet b = \bigcup\limits_{z\in C_H(ab)} \beta(z) = \bigcup\limits_{z\in C_H(ab)} \varphi_H^{-1}\varphi_H(z) = \bigcup\limits_{z\in C_H(ab)} C_H(z) = C_H(ab)$.
Therefore, $\langle K_{H'}, \bullet\rangle = \Delta(H)$.
On the other hand, since H' is a group, it results it is a ε-hypergroup and by a preceding Proposition, every $K_{H'}$-hypergroup is also a ε-hypergroup. Hence, $\Delta(H) \in \varepsilon$.

We can define a functor between the hypergroups category and the ε-hypergroups category.

Remark 15. *There are ε-hypergroups H, such that $H \neq \Delta(H)$ and there are nontotal ε-hypergroups H, such that $H = \Delta(H)$.*

Indeed, for proving the first part, we can consider the following example (see Examples 13, [1]): let $\langle G, \cdot\rangle$ be a group, and $\forall(x, y) \in G^2$, $x \circ y = \langle x, y\rangle$ be the subgroup generated by $\{x, y\}$. Then $\langle G, \circ\rangle$ is a ε-hypergroup, for which $\omega_G = G$, and so $\Delta(G)$ is the total hypergroup on G.
For proving the second part of the above remark, we can consider T a total hypergroup, and A a nonempty set, such that $A \cap T = \emptyset$. Let $H = A \cup T$ and

$$\forall(x, y) \in H^2, x \circ y = \begin{cases} A, & \text{if } (x, y) \in A \times T \cup T \times A \\ T, & \text{if } (x, y) \in A^2 \\ T, & \text{if } (x, y) \in T^2 \end{cases}$$

Then $\langle H, \circ\rangle$ is a ε-hypergroup, for which $H = \Delta(H)$.

Proposition 16. *If H is a hypergroup, then $\omega_{\Delta H} = \omega_H$.*

Proof. By the above Theorem, we have $\omega_{\Delta(H)} = E_{\Delta(H)}$. We have $e \in E_{\Delta(H)} \Leftrightarrow \forall a \in H, a \in e \otimes a \cap a \otimes e = C_H(ea) \cap C_H(ae)$. Let a be a partial identity of H, and we shall denote a by i. From $i \in C_H(ei) \cap C_H(ie) = C_H(e)$, it results $C_H(e) = C_H(i) = \omega_H$, that means $e \in \omega_H$. So, $E_{\Delta(H)} \subset \omega_H$.
Conversely, if $x \in \omega_H$, then $\forall a \in H$, $C_H(ax) = ax\omega_H = a\omega_H = C_H(a) = \omega_H a = \omega_H xa = C_H(xa)$, whence $\forall a \in H$, $a \in C_H(a) = C_H(ax) = C_H(xa) = a \otimes x \cap x \otimes a$, that is $x \in E_{\Delta(H)}$.
Then, $\omega_{\Delta(H)} = E_{\Delta(H)} = \omega_H$.

Proposition 17. *Let H_1 and H_2 be two hypergroups, and $f : \Delta(H_1) \to \Delta(H_2)$ a good homomorphism. Then $f(\omega_{H_1}) = \omega_{H_2}$.*

Proof. $\forall(a, b) \in H_1^2$, we have $f(a \otimes b) = f(a) \otimes f(b)$, that is $f(C_{H_1}(ab)) = C_{H_2}(f(a)f(b))$. Let a and b be in ω_{H_1}. Then $C_{H_1}(ab) = \omega_{H_1}$ and from $a\beta_{H_1}b$, it results $f(a)\beta_{H_2}f(b)$, so $\{f(a), f(b)\} \subset \omega_{H_2}$, whence $C_{H_2}(f(a)f(b)) = \omega_{H_2}$. Then $f(\omega_{H_1}) = \omega_{H_2}$.

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