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THE INVERSE ULTRAHYPERBOLIC MARCEL RIESZ KERNEL

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Let $R_\alpha^H(u)$ be the Marcel Riesz' ultrahyperbolic kernel defined by

$$= \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if} \quad x \in \Gamma_+ \\ 0 & \text{if} \quad x \notin \Gamma_+ \end{cases}$$

where $K_n(\alpha)$ is the constant defined by (2).

The distribution $R_{2k}^H(u)$ are elementary solutions of the n-dimensional ultrahyperbolic operator iterated k times:

$$\Box^k R_{2k}^H(u) = R_0^H(u) = \delta(x)$$

if p is odd ([10] page 10, formula IV, 1 and [2], pages 147-150) where

$$\Box^k = \left\{ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^k .$$

Let $M^\alpha(f)$ be the ultrahyperbolic Marcel Riesz operator defined by the formula

$$M^\alpha(f) = R_\alpha^H * f.$$

Our problem consists in to obtain the operator $N^\alpha = (M^\alpha)^{-1}$ such that if $M^\alpha(f) = \varphi$ then $N^\alpha \varphi = f$.

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In this the paper we prove that

$$N^{\alpha}\left(R_{\alpha}^{H}\right)^{-1}=\left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]^{-1}R_{-\alpha}^{H},$$

if p is odd and q even for all α,

$$N^{\alpha}=\left(R_{\alpha}^{H}\right)^{-1}=R_{-\alpha}^{H},$$

if p is odd and q odd for all α, and

$$N^{\alpha}=\left(R_{\alpha}^{H}\right)^{-1}=\left(\cos\frac{\alpha\pi}{2}\right)^{2}R_{-\alpha}^{H},$$

if p is even for all α such that $\frac{\alpha}{2}\neq 2s+1$, $s=0,1,2,\ldots$.

1. Introduction.

Let $x=(x_1,x_2,\ldots,x_n)$ be a point of R^n. We shall write $x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2=u$, $p+q=n$. By Γ^+ we designate the interior of forward cone : $\Gamma_+=\{x\in R^n : x_1>0, u>0\}$, and by $\overline{\Gamma}_+$ w edesignate its closure.
Similarly, Γ_- deisgnates the domain $\Gamma_-=\{x\in R^n : x_1<0, u>0\}$ and $\overline{\Gamma}_-$ designate its closure.
Let $F(\lambda)$ be a function of the scalar variable λ, and let $\phi(x)$ be a function endowed with the following properties:

a) $\phi(x)=F(u)$

b) supp$\phi(x)\subset\overline{\Gamma}_+$,

c) $e^{\langle x,y\rangle}\phi(x)\in L_1$ if $y\in V_-$,

where

$$V_-=\left\{y\in R^n : y_1>0, y_1^2+\cdots+y_p^2-y_{p+1}^2-\cdots-y_{p+q}^2>0\right\}.$$

We recall R^+ the family of functions $\phi(x)$ which satisfies conditions **a)**, **b)** and **c)**.

Similarly, we call A the family of functions which satisfies the conditions:

a′) $\phi(x)=F(x)$

b′) supp $\phi(x)\subset\overline{\Gamma}_-$,

c′) $e^{\langle x,y\rangle}\phi(x) \in L_1$ if $y \in V_+$,

where,

$$V_+ = \left\{y \in R^n : y_1 < 0,\, y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 > 0\right\}.$$

We shall consider the following functions of the family R introduced by Nozaki ([6], page 72):

$$R_\alpha(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if} \quad x \in \Gamma_+ \\ 0 & \text{if} \quad x \notin \Gamma_+ \end{cases} \tag{1}$$

Here α is a complex parameter, n the dimension of the space.

The constant $K_n(\alpha)$ is defined by:

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})} \tag{2}$$

p is the number of positive terms of:

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \tag{3}$$

$p + q = n$.

$R_\alpha(u)$, which is an ordinary function if $\Re(\alpha) \geq 0$, is a distribution of α.

We shall call $R_\alpha(u)$ the Marcel Riesz' ultra-hyperbolic kernel.

By putting $p = 1$ in (1) and (2) and remembering the Legrende's duplication formula of $\Gamma(z)$:

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{4}$$

([4], Vol. I, page 5, formula 15) the formula (1) reduces to:

$$M_\alpha = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if} \quad x \in \Gamma_+ \\ 0 & \text{if} \quad x \notin \Gamma_+. \end{cases} \tag{5}$$

Here

$$u = x_1^2 - x_2^2 - \cdots - x_n^2, \tag{6}$$

and

$$H_n(\alpha) = 2^{\alpha-1}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n+2}{2}\right). \tag{7}$$

$M_\alpha(u)$ is precisely, the hyperbolic kernel of Marcel Riesz ([7], page 31).
S.E. Trione in [9], page 8, formula (III.6) defined the convolution $R^H_\alpha(u) * f(x)$ by the following formula:

$$R^H_\alpha(u) * f(x) = \int_{R^n} f(x)\frac{(u(x-t))^{\frac{\alpha-n}{2}}}{K_n(\alpha)}\,dx \quad \text{([9], page 8)} \tag{8}$$

Making several change of variables and proceeding as Gelfand-Shilov ([5], pages 253–254) the formula (8) can be rewrite

$$R^H_\alpha(u) * f(x) = \frac{1}{K_n(\alpha)}\int_{R^n} u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1} G_{\frac{\alpha-n}{2}}(u)\,du \tag{9}$$

where

$$G_{\frac{\alpha-n}{2}}(u) = \frac{1}{4}\int_0^1 (1-t)^{\frac{\alpha-n}{2}} t^{\frac{q-2}{2}} \Psi_1(u,tu)\,dt \tag{10}$$

and

$$\Psi_1(u,tu) = \Psi_1(u,\nu) = \Psi(r,s) = \int f(x)\,d\Omega_p d\Omega_q \quad \text{([5], page 253).}$$

2. The properties of $R^H_\alpha(u)$.

The following formula are valid for all $\alpha, \beta \in C$ (where C are complex numbers).

(11) $R^H_\alpha * R^H_\beta = R_{\alpha-\beta} + T_{\alpha,\beta}$ if p is odd ([1], page 121, formula (I,2,17))

$$R^H_\alpha * R^H_\beta = \frac{\cos\frac{\alpha\pi}{2}\cdot\cos\frac{\beta\pi}{2}}{\cos(\frac{\alpha+\beta}{2})} R^H_{\alpha+\beta} \tag{12}$$

if p is even ([1], page 123, formula (I,2,25))

$$R^H_\alpha * R^H_{-2k} = R^H_{\alpha-2k} \quad \text{([1], page 123, formula (I,2,26))} \tag{13}$$

$$R^H_{-2k} = \Box^k \delta \quad \text{([1], page 123, formula (I,2,27))} \tag{14}$$

$$\Box^k R^H_\alpha = R_{\alpha-2k} \quad \text{([1], page 123, formula (I,2,29))} \tag{15}$$

$$R_\alpha * R^H_{2k} = R_{\alpha+2k} \quad \text{([1], page 123, formula (I,2,30))} \tag{16}$$

$$\Box^k R^H_{2k} = R_0 = \delta \quad \text{([1], page 123, formula (I,2,33))} \tag{17}$$

where

$$\Box^k = \left\{ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^k . \tag{18}$$

$$T_{\alpha,\beta} = T_{\alpha,\beta}(P\pm, i0) = \frac{2\pi i}{4} \frac{C(\frac{-\alpha-\beta}{2})}{C(\frac{-\alpha}{2})C(\frac{-\beta}{2})}[H^+_{\alpha+\beta} - H^-_{\alpha+\beta}] \tag{19}$$

$$C(\gamma) = \Gamma(\gamma)\Gamma(1-\gamma) \tag{20}$$

$$H^\pm_\gamma = H_\gamma(P\pm, i0) = e^{\pm\frac{q\pi i}{2}} a\Big(\frac{\gamma}{2}\Big)(u \pm i0)^{\frac{\gamma-n}{2}} \tag{21}$$

$$a(\frac{\gamma}{2}) = \Gamma(\frac{n-\gamma}{2})[2^\gamma \pi^{\frac{n}{2}} \Gamma(\frac{\gamma}{2})]^{-1} \tag{22}$$

$$(u \pm i0)^\lambda = \lim_{\varepsilon\to 0}(u \pm i\varepsilon|x|^2)^\lambda \quad \text{([5], page 275)} \tag{23}$$

$$|x|^2 = x_1^2 + \cdots + x_n^2$$

and

$$u = u(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 .$$

3. The convolution $R^H_\alpha * R^H_\beta$ for these case $\beta = -\alpha$.

Now we will go to study property $R^H_\alpha * R^H_\beta$ when $\beta = -\alpha$.
From (11) and (12) we have

$$R^H_\alpha * R^H_\beta = R^H_{\alpha+\beta} + T_{\alpha+\beta} \quad \text{if } p \text{ is odd and } q \text{ is even} \tag{24}$$

$$R^H_\alpha * R^H_\beta = R^H_{\alpha+\beta} + T_{\alpha+\beta} \quad \text{if } p \text{ is odd and } q \text{ is odd} \tag{25}$$

$$R^H_\alpha * R^H_\beta = \frac{\cos\frac{\alpha\pi}{2}\cdot\cos\frac{\beta\pi}{2}}{\cos(\frac{\alpha+\beta}{2})} R^H_{\alpha+\beta} \quad \text{if } p \text{ is even and } q \text{ is odd} \tag{26}$$

and

$$R^H_\alpha * R^H_\beta = \frac{\cos\frac{\alpha\pi}{2}\cdot\cos\frac{\beta\pi}{2}}{\cos(\frac{\alpha+\beta}{2})} R^H_{\alpha+\beta} \quad \text{if } p \text{ is even and } q \text{ is even} \tag{27}$$

where $T_{\alpha,\beta}$ is defined by (19).
From (19) we have.

$$T_{\alpha-\alpha} = \lim_{\beta\to\alpha} T_{\alpha,\beta} = \frac{\pi}{2} i \lim_{\gamma\to 0} C(-\frac{\gamma}{2}) \frac{1}{C(-\frac{\alpha}{2})C(\frac{\alpha-\gamma}{2})} \tag{28}$$

$$[H^+\gamma - H^-_\gamma] = \frac{\pi i}{2} \lim_{\gamma\to 0} \frac{C(\frac{-\gamma}{2})}{C(-\frac{\alpha}{2})C(\frac{\alpha-\gamma}{2})} \cdot \lim_{\gamma\to 0}[H^+\gamma - H^-_\gamma]$$

where $\gamma = \alpha + \beta$.
On the other hand, using (21) and (22) we have,

$$\begin{aligned}\lim_{\gamma\to 0}[H^+\gamma - H^-_\gamma] &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\{\lim_{\gamma\to 0} e^{-\frac{\gamma\pi i}{2}} \lim_{\gamma\to 0} e^{\pm\frac{q\pi i}{2}} \frac{(P+i0)^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})} \\ &\quad - \lim_{\gamma\to 0} e^{\frac{\gamma\pi i}{2}} \lim_{\gamma\to 0} e^{\pm\frac{q\pi i}{2}} \frac{(P-i0)^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})}\} \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\{\lim_{\gamma\to 0} e^{\frac{-\gamma\pi i}{2}} e^{\pm\frac{q\pi i}{2}} \frac{\operatorname*{Res}_{\beta-\frac{n}{2}}(P+i0)^\beta}{\operatorname*{Res}_{\beta-\frac{n}{2}}\Gamma(\beta+\frac{n}{2})} \\ &\quad - \lim_{\gamma\to 0} e^{\frac{\gamma\pi i}{2}} e^{\pm\frac{q\pi i}{2}} \frac{\operatorname*{Res}_{\beta-\frac{n}{2}}(P-i0)^\beta}{\operatorname*{Res}_{\beta-\frac{n}{2}}\Gamma(\beta+\frac{n}{2})}\end{aligned} \tag{29}$$

Now taking into account that

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k}(P\pm i0)^{\lambda}=\frac{e^{\pm\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}\Box^{k}\delta, \tag{30}$$

if n is odd ([3], page 16).

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k}(P\pm i0)^{\lambda}=\frac{e^{\pm\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}\Box^{k}\delta, \tag{31}$$

if p and q are even (n even)([3], page 116).

and

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k}(P\pm i0)^{\lambda}=0 \quad \text{if } p \text{ and } q \text{ are odd } (n \text{ even}) \text{ ([3], page 116).} \tag{32}$$

where $\Box^k$ is defined by (18). We have.

$$\begin{aligned}\lim_{\gamma\to 0}[H^{+}\gamma-H_{\gamma}^{-}]&=\frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\cdot\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\Big\{\lim_{\gamma\to 0}e^{-\frac{\gamma\pi i}{2}}-\lim_{\gamma\to 0}e^{\frac{\gamma\pi i}{2}}\}\delta(x)\\&=\lim_{\gamma\to 0}[-2i\sin\frac{\gamma\pi}{2}]\delta\end{aligned} \tag{33}$$

Now, using the formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin z\pi} \quad \text{([4], Vol. I, page 3)} \tag{34}$$

and (30), from (29), we have

$$\begin{aligned}\lim_{\gamma\to 0}[H^{+}\gamma-H_{\gamma}^{-}]&=\lim_{\gamma\to 0}\frac{1}{\Gamma(-\frac{\gamma}{2})\Gamma(1+\frac{\gamma}{2})}(2\pi i)\delta(x_1,x_2,\dots,x_n)\\&=\lim_{\gamma\to 0}\frac{(-2\pi i)}{\Gamma(\frac{\gamma}{2})\Gamma(1-\frac{\gamma}{2})}\delta(x_1,x_2,\dots,x_n)\end{aligned} \tag{35}$$

if n is odd.

From (29) and (32) we have,

$$\lim_{\gamma\to 0}[H^{+}\gamma-H_{\gamma}^{-}]=\frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\cdot\lim_{\gamma\to 0}\frac{(-2\pi i)}{\Gamma(\frac{\gamma}{2})\Gamma(1-\frac{\gamma}{2})}\cdot 0. \tag{36}$$

if p and q are odd (n even).

From (28) and considering (20), (34), (35) and (36) we have,

$$T_{\alpha,-\alpha} = \frac{\pi i}{2} \lim_{\gamma \to 0} \frac{\Gamma(-\frac{\gamma}{2})\Gamma(1+\frac{\gamma}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{\alpha-\gamma}{2})\Gamma(1+\frac{\gamma-\alpha}{2})} \cdot \left[\frac{-2\pi i}{\Gamma(\frac{\gamma}{2})\Gamma(1-\frac{\gamma}{2})}\delta(x_1,\ldots,x_n)\right] = \left(\frac{\pi}{\Gamma(1-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}\right)^2 \delta(x_1,\ldots,x_n) = (\sin\frac{\alpha\pi}{2})^2\delta(x_1,\ldots,x_n), \tag{37}$$

if p is odd and q is even, and

$$T_{\alpha,-\alpha} = \frac{\pi i}{2} \lim_{\gamma \to 0} \frac{\Gamma(-\frac{\gamma}{2})\Gamma(1+\frac{\gamma}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{\alpha-\gamma}{2})\Gamma(1+\frac{\gamma-\alpha}{2})} \cdot \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\frac{(-2\pi i)}{\Gamma(\frac{\gamma}{2})\Gamma(1-\frac{\gamma}{2})} \cdot 0 = \left(\frac{\pi}{\Gamma(1-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}\right)^2 \cdot 0 = 0, \tag{38}$$

if p and q are odd.

From (24), (25), (26) and (27) and using the property (17) and the formulae (37) and (38) we obtain

$$R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha} = \delta(x_1,\ldots,x_n) + (\pi \sin\frac{\alpha\pi}{2})^2\delta(x_1,\ldots,x_n) = \left[1+(\sin\frac{\alpha\pi}{2})^2\right]\delta(x_1,\ldots,x_n), \tag{39}$$

if p is odd and q is even.

$$R^H_\alpha * R^H_{-\alpha} = R^H_0 + T_{\alpha,-\alpha} = R^H_0 = \delta(x_1,\ldots,x_n), \tag{40}$$

if p and q are odd.

$$R_\alpha^H * R_{-\alpha}^H = \left(\cos\frac{\alpha\pi}{2}\right)^2 R_0^H = \left(\cos\frac{\alpha\pi}{2})^2\delta(x_1,\ldots,x_n).\right. \tag{41}$$

if p is even and q is odd, and

$$R_\alpha^H * R_{-\alpha}^H = \left(\cos\frac{\alpha\pi}{2}\right)^2 R_0^H = \left(\cos\frac{\alpha\pi}{2})^2\delta(x_1,\ldots,x_n).\right. \tag{42}$$

if p is even and q is even.

4. The inverse ultrahyperbolic Marcel Riesz kernel.

Let $M^\alpha(f)$ be the ultrahyperbolic Marcel Riesz operator defined by the formula

$$M^\alpha(f) = R_\alpha^H * f \tag{43}$$

where $f \in S$ and S is the Schwartz space of functions ([8], page 233).

Our objective is to obtain the operator $N^\alpha = (M^\alpha)^{-1}$ such that if

$$\varphi = M^\alpha f \quad \text{then} \quad N^\alpha\varphi = f. \tag{44}$$

The following theorem express that if we put, by definition $M^\alpha = R_\alpha^H$ then $(M^\alpha)^{-1} = (R_\alpha^H)^{-1} = \left[1 + (\sin\frac{\alpha\pi}{2})^2\right]^{-1} R_\alpha^H$ if p is odd and q is even for all complex α, $(M^\alpha)^{-1} = (R_\alpha^H)^{-1}$ if p is odd and q odd for all complex α, and $(M^\alpha)^{-1} = (R_\alpha^H)^{-1}$ if p is odd and q odd for all complex α, and $(M^\alpha)^{-1} = (R_\alpha^H)^{-1} = \left[(\cos\frac{\alpha\pi}{2})^2\right]^{-1} R_{-\alpha}^H$ if p is even for all complex α such that $\frac{\alpha}{2} \neq 2s+1, s = 0, 1, 2, \cdots$

Now we shall state our main theorem.

Theorem 1. *If $\varphi = M^\alpha(f)$ where $M^\alpha(f)$ is defined by* (43)*, $f \in S$, then*

1. *$N^\alpha\varphi = f$ where*

$$N^\alpha = (M^\alpha)^1 = (R_\alpha^H)^{-1} = \left[1 + (\sin\frac{\alpha\pi}{2})^2\right]^{-1} R_{-\alpha}^H \tag{45}$$

if p is odd and q is even for all complex α.

2. $N^{\alpha}\varphi = f$ *where*

$$N^{\alpha} = (M^{\alpha})^{1} = (R^{H}_{\alpha})^{-1} = R^{H}_{-\alpha} \tag{46}$$

if p is odd and q is odd for all α.

3. $N^{\alpha}\varphi = f$ *where*

$$N^{\alpha} = (M^{\alpha})^{1} = (R^{H}_{-\alpha})^{-1} = \left[\left(\cos\frac{\alpha\pi}{2}\right)^{2}\right]^{-1} R^{H}_{-\alpha} \tag{47}$$

if p is even for all complex α such that $\frac{\alpha}{2} \neq 2s+1$, $s = 0, 1, 2, \ldots$.

Proof. From the definition formulae (43) and (8) we have.

$$M^{\alpha}(f) = R^{H}_{\alpha} * f = \varphi$$

where R^{H}_{α} is defined by the formula (2) for $\alpha \in C$ and $f \in S$ (Schwartz space of functions [8], page 233). Then, in view of (39) we obtain

$$\begin{aligned}
&\left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]^{-1} R^{H}_{-\alpha} * (R^{H}_{\alpha} * f) \\
&= \left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]^{-1} (R^{H}_{-\alpha} * (R^{H}_{\alpha}) * f \\
&= \left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]^{-1} \left\{\left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]\delta\right\} * f \\
&= \delta * f = \delta
\end{aligned} \tag{48}$$

if p is odd and q is even for all complex α.

Therefore

$$\left[1+(\sin\frac{\alpha\pi}{2})^{2}\right]^{-1} R^{H}_{-\alpha} = (M^{\alpha})^{-1} = (R^{H}_{\alpha})^{-1} \tag{49}$$

if p is odd and q even for all complex α.

Similarly, using (40) we obtain

$$R^{H}_{-\alpha} * (R^{H}_{\alpha} * f) = (R^{H}_{-\alpha} * R^{H}_{\alpha} * f) = \delta * f = f \tag{50}$$

if p and q are odd for all complex α.

Therefore

$$R^H_{-\alpha} = (M^\alpha)^{-1} = (R^H_\alpha)^{-1} \tag{51}$$

if p and q are odd for all complex α.

On the othere hand, using (41) and (42) we have

$$\begin{aligned} &\left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} R^H_\alpha * (R^H_\alpha * f) \\ &= \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} (R^H_\alpha * R^H_\alpha) * f \\ &= \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} \left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right] \delta * f \\ &= \delta * f = f \end{aligned} \tag{52}$$

if p is even for all complex α such that $\frac{\alpha}{2} \neq 2s+1$, $s = 0, 1, 2, \ldots$.

Therefore

$$\left[\left(\cos\frac{\alpha\pi}{2}\right)^2\right]^{-1} R^H_{-\alpha} = (M^\alpha)^{-1} = (R^H_\alpha)^{-1} \tag{53}$$

if p is even for all complex α such that $\frac{\alpha}{2} \neq 2s+1$, $s = 0, 1, 2, \ldots$.

Formulae (49), (51), and (53) are the desired result and this finished the proof of theorem. □

In particular putting $p = 1$ in (45) and (46) and taking into account (2) and (5) we obtain the inverse hyperbolic Marcel Riesz kernel. In fact putting $p = 1$ in (45) an d(46) and considering that $R^H_\alpha(u) = R_\alpha(u)$ if $p = 1$ we have

$$N^\alpha = (R_\alpha)^{-1} = \left[1 + (\sin\frac{\alpha\pi}{2})^2\right]^{-1} R_{-\alpha} \tag{54}$$

if q is even, and

$$N^\alpha = (R_\alpha)^{-1} = R_{-\alpha} \tag{55}$$

if q is odd.

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