LE MATEMATICHE
Vol. LIV (1999) – Fasc. I, pp. 91–97

A VARIANT ON MIRANDA-TALENTI ESTIMATE

GIUSEPPE DEVILLANOVA - FABRIZIO PUGLIESE

In this note we prove formula (1.1),which extends to functions in $W^{2,2}(\Omega)$ with zero normal derivative the analogous formula (1.2) by G. Talenti ([5]) on functions with zero trace. To prove (1.1) we use the technique introduced by C. Miranda in [3] and give a geometrical interpretation of his results (formula (2.17)).

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^n$ be a C^2-smooth, bounded domain. Let $u \in W^{2,2}(\Omega)$ be such that

$$u_0 = \frac{\partial u}{\partial n} = \sum_{i=1}^{n} p_i X_i = 0 \quad \text{on } \partial\Omega,$$

where $\boldsymbol{n} \equiv (X_1, ..., X_n)$ is the unit outward normal to $\partial\Omega$ and $p_i = \frac{\partial u}{\partial x_i}$, $i = 1, ..., n$. In this note we will show that for such functions u the following formula holds true:

$$\int_\Omega \sum_{i,k=1}^{n} \left(p_{ii} p_{kk} - p_{ik}^2\right) dx = -\int_{\partial\Omega} \sum_{i=1}^{n} p_i^2 k_n \, d\sigma, \tag{1.1}$$

where k_n is the normal curvature of $\partial\Omega$ along the direction of ∇u, i.e. the curvature of the intersection of $\partial\Omega$ with the plane determined by $\boldsymbol{n}$ and ∇u (which, under our assumption on u_0, is tangent to $\partial\Omega$).

Entrato in Redazione il 21 dicembre 1998.

Formula (1.1) extends to functions of $W^{2,2}(\Omega)$ with zero normal derivative the well-known formula by G. Talenti ([5]), concerning functions in $W^{2,2}(\Omega)$ with zero trace on $\partial\Omega$:

$$(1.2) \qquad \int_\Omega \sum_{i,k=1}^{n} \left(p_{ii}p_{kk} - p_{ik}^2\right) dx = -(n-1)\int_{\partial\Omega} \sum_{i=1}^{n} p_i^2 H \, d\sigma,$$

where H is the mean curvature of $\partial\Omega$ at x. We will derive (1.1) from a general formula due to Miranda (see (2.20) of [3]). Let us remark, however, that it remains unsolved the problem of finding the analogue of (1.1), (1.2) in the general case of a function $u \in W^{2,2}(\Omega)$ satisfying the condition $\frac{\partial u}{\partial l} = 0$ on $\partial\Omega$, where $\boldsymbol{l} \equiv (Y_1, ..., Y_n)$ is an oblique unit vector field.

From (1.1),(1.2), assuming that Ω is convex, one can obtain the inequality:

$$(1.3) \qquad \int_\Omega \sum_{i,k=1}^{n} p_{ik}^2 \, dx \le \int_\Omega (\triangle u)^2 \, dx,$$

valid for every $u \in W^{2,2}$ such that $u = 0$ or $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. (1.3) has been already proved by A. Maugeri ([2]) in the case $\frac{\partial u}{\partial n} = 0$. It plays a fundamental role in the theory of ”nearness” between operators, developed by S. Campanato ([1]) in order to study non-linear discontinuous elliptic and parabolic operators.

2. Proof of (1.1).

We can assume that $u \in C^2\left(\overline{\Omega}\right) \cap C^3(\Omega)$. In fact, once (1.1) has been proved in this special case, it can be extended to the case $u \in W^{2,2}$ by a well-known approximation method.

Keeping in mind that

$$\sum_{i,k=1}^{n} \left(p_{ii}p_{kk} - p_{ik}^2\right) = \sum_{i,k=1}^{n} \left[\frac{\partial}{\partial x_k}(p_{ii}p_k) - \frac{\partial}{\partial x_i}(p_k p_{ik})\right],$$

we obtain, by virtue of Gauss-Green formulas, the equality

$$\int_\Omega \sum_{i,k=1}^{n} \left(p_{ii}p_{kk} - p_{ik}^2\right) dx = -\int_{\partial\Omega} \sum_{i,k=1}^{n} p_i \left(p_{ik}X_k - p_{kk}X_i\right) d\sigma$$

According to the elegant technique used by Miranda ([3]) to evaluate the surface integral, let us introduce the operators:

$$\delta_i : u \in C^1\left(\overline{\Omega}\right) \mapsto u_i \in C^1(\partial\Omega),$$

where

$$u_i \stackrel{\text{def}}{=} p_i - u_0 X_i \quad i = 1, \dots, n. \tag{2.1}$$

These scalar expressions are equivalent to the vectorial one:

$$\delta u = \nabla u - u_0 \boldsymbol{n}, \tag{2.2}$$

where $\delta u \equiv (u_1, \dots, u_n)$ is the projection of ∇u on the hyperplane $T_x(\partial\Omega)$, tangent to $\partial\Omega$ at x. Let us fix on $\partial\Omega$ a system of local, C^2−smooth curvilinear coordinates $t_1, \dots, t_{n-1}$:

$$\begin{cases} x_1 = x_1(t_1, \dots, t_{n-1}) \\ \dots\dots\dots\dots\dots\dots\dots \\ x_n = x_n(t_1, \dots, t_{n-1}) \end{cases} \tag{2.3}$$

with $(t_1, ..., t_{n-1})$ varying in the domain $T \subseteq \mathbb{R}^{n-1}$. Let us also assume that such coordinates are orthogonal, i.e.:

$$\frac{\partial \boldsymbol{x}}{\partial t_i} \cdot \frac{\partial \boldsymbol{x}}{\partial t_j} = \begin{cases} 0 & i \neq j \\ E_i = \left\| \frac{\partial \boldsymbol{x}}{\partial t_i} \right\|^2 & i = j \end{cases}, \tag{2.4}$$

for $i, j = 1, \dots, n-1$. From (2.2), (2.4) we obtain:

$$\delta u = \sum_{k=1}^{n-1} \frac{1}{E_k} \left(\delta u \cdot \frac{\partial \boldsymbol{x}}{\partial t_k} \right) \frac{\partial x}{\partial t_k} = \sum_{k=1}^{n-1} \frac{1}{E_k} \left(\nabla u \cdot \frac{\partial \boldsymbol{x}}{\partial t_k} \right) \frac{\partial \boldsymbol{x}}{\partial t_k} = \tag{2.5}$$

$$= \sum_{k=1}^{n-1} \frac{1}{E_k} \frac{\partial u}{\partial t_k} \frac{\partial \boldsymbol{x}}{\partial t_k}$$

or, in cartesian coordinates:

$$u_i = \sum_{k=1}^{n-1} \frac{1}{E_k} \frac{\partial u}{\partial t_k} \frac{\partial x_i}{\partial t_k} \qquad i = 1, \dots, n. \tag{2.6}$$

Let us remark that (2.5), (2.6) *are still valid for functions defined only on* $\partial\Omega$ (in fact $\delta u = \operatorname{grad}(u|_{\partial\Omega})$, where grad is the gradient operator on the riemannian manifold $\partial\Omega$, see [6]). Let us furtherly remark that δ_i has the following properties:

$$\begin{aligned} (u+v)_i &= u_i + v_i \\ (uv)_i &= uv_i + u_i v, \end{aligned} \tag{2.7}$$

i.e. it is a derivation of the algebra $C^2(\partial\Omega)$. Following [3], we will express "spatial" derivatives p_{ij} in terms of "superficial" ones $u_{rs} = (u_r)_s$. First of all, let us evaluate $u_{0r} = (u_0)_r$:

$$(2.8) \qquad u_{0r} = \left(\frac{\partial u}{\partial n}\right)_r = \left(\sum_{i=1}^n p_i X_i\right)_r = \sum_{i=1}^n (p_i)_r X_i + \sum_{i=1}^n p_i X_{ir} =$$

$$= \sum_{i=1}^n \left(p_{ir} - \frac{\partial p_i}{\partial n} X_r\right) X_i + \sum_{i=1}^n (u_i + u_0 X_i) X_{ir} =$$

$$= \frac{\partial p_r}{\partial n} - \theta X_r + \sum_{i=1}^n u_i X_{ir},$$

where $\theta = \sum_{i=1}^n \frac{\partial p_i}{\partial n} X_i = \frac{\partial \nabla u}{n} \cdot \boldsymbol{n}$ and $u_0 \sum_{i=1}^n X_{ir} X_i = 0$ in virtue of the successive formula (2.13).

We can now evaluate, using (2.8), the "surface" second derivatives:

$$u_{rs} = (p_r - u_0 X_r)_s = p_{rs} - \frac{\partial p_r}{\partial n} X_s - u_{0s} X_r - u_0 X_{rs} =$$

$$= p_{rs} - u_{0r} X_s - \theta X_r X_s + \sum_{i=1}^n u_i X_{ir} X_s - u_{0s} X_r - u_0 X_{rs}$$

Therefore:

$$(2.9) \qquad p_{rs} = u_{rs} + u_{0r} X_s + \theta X_r X_s - \sum_{i=1}^n u_i X_s X_{ir} + u_{0s} X_r + u_0 X_{rs}$$

The X_{rs} satisfy two remarkable relations. Firstly

$$(2.10) \qquad X_{rs} = X_{sr}$$

In fact, recalling Weingarten formulas:

$$(2.11) \qquad \frac{\partial \boldsymbol{n}}{\partial t_k} = -\sum_{i=1}^{n-1} \frac{1}{E_i} D_{ki} \frac{\partial \boldsymbol{x}}{\partial t_i}, \quad k = 1, \ldots, n-1$$

(where $D_{ki} = \frac{\partial^2 \boldsymbol{x}}{\partial t_i \partial t_k} \cdot \boldsymbol{n} = -\frac{\partial \boldsymbol{x}}{\partial t_k} \cdot \frac{\partial \boldsymbol{n}}{\partial t_i}$ are the coefficients of the second quadratic form B on $\partial\Omega$), we get:

$$(2.12) \qquad X_{rs} = (X_r)_s = \sum_{i=1}^{n-1} \frac{1}{E_k} \frac{\partial X_r}{\partial t_k} \frac{\partial x_s}{\partial t_k} = -\sum_{i,k=1}^{n-1} \frac{D_{ki}}{E_i E_k} \frac{\partial x_s}{\partial t_k} \frac{\partial x_r}{\partial t_i},$$

which proves (2.10).
The second useful relation involving the X_{rs} is:

$$\sum_{r=1}^{n} X_{rs} X_r = \sum_{r=1}^{n} X_{sr} X_r = 0, \tag{2.13}$$

which is obtained by applying the operator δ_s to the right and left hand of the identity $\|\boldsymbol{n}\|^2 = \sum_{r=1}^{n} X_r^2 = 1$.

Setting, for the sake of brevity:

$$\Sigma = \sum_{i,k=1}^{n} p_i \left(p_{ik} X_k - p_{kk} X_i\right)|_{\partial\Omega},$$

we evaluate Σ by using relations (2.9), (2.10), (2.13). Firstly, from (2.9) we get:

$$\Sigma = \sum_{i,k=1}^{n} p_i \Big(u_{ik} X_k + u_{0i} X_k^2 - \sum_{r=1}^{n} u_r X_k^2 X_{ri} + u_0 X_k X_{ik} - u_{kk} X_i - \\ - u_{0k} X_k X_i + \sum_{r=1}^{n} u_r X_k X_{rk} X_i - u_0 X_{kk} X_i \Big),$$

i.e., by (2.13) and the relation $\sum_{i=1}^{n} X_i^2 = 1$:

$$\Sigma = \sum_{i=1}^{n} p_i \left(\delta u_i \cdot \boldsymbol{n}\right) + \nabla u \cdot \delta u_0 - \sum_{r,i=1}^{n} p_i u_r X_{ri} - \\ - u_0 \sum_{k=1}^{n} u_{kk} - u_0 \delta u_0 \cdot \boldsymbol{n} - u_0^2 \sum_{k=1}^{n} X_{kk}$$

But δu_i, δu_0 are tangent to $\partial\Omega$, so $\delta u_i \cdot \boldsymbol{n} = \delta u_0 \cdot \boldsymbol{n} = 0$. Hence, reminding (2.2), we obtain at last:

$$\Sigma = \delta u_0 \cdot \delta u - \sum_{i,r=1}^{n} p_i u_r X_{ri} - u_0 \sum_{r=1}^{n} u_{rr} - u_0^2 \sum_{k=1}^{n} X_{kk} \tag{2.14}$$

This expresson can be furtherly simplified. Indeed, from (2.12), (2.13), (2.1) it follows that:

$$\sum_{i,r=1}^{n} p_i u_r X_{ri} = - \sum_{j,k=1}^{n-1} D_{kj} \left(\sum_{r=1}^{n} \frac{1}{E_j} \frac{\partial x_r}{\partial t_j} u_r \right) \left(\sum_{i=1}^{n} \frac{1}{E_k} \frac{\partial x_i}{\partial t_k} u_i \right) = \tag{2.15}$$

$$= -\sum_{j,k=1}^{n-1} D_{kj} \left(\frac{1}{E_j} \frac{\partial \boldsymbol{x}}{\partial t_j} \cdot \delta u \right) \left(\frac{1}{E_k} \frac{\partial \boldsymbol{x}}{\partial t_k} \cdot \delta u \right) =$$

$$= -B\,(\delta u, \delta u) = -\,\|\delta u\|^2\, k_n\,(\delta u)\,,$$

where B denotes the second fundamental quadratic form on $\partial\Omega$ and by $k_n\,(\delta u)$ we mean the normal curvature of $\partial\Omega$ along the direction of δu (i.e. the curvature of the curve obtained intersecting $\partial\Omega$ with the plane containing vectors $\mathbf{n}$ and δu). Recall that $k_n\,(\delta u)$ is related to the principal curvatures $\lambda_1, \ldots, \lambda_{n-1}$ of $\partial\Omega$ at x by Euler's formula:

$$k_n\,(\delta u) = \sum_{i=1}^{n} \lambda_i \cos^2 \phi_i\,,$$

where the ϕ_i's are the angles between δu and the principal directions.

Principal curvatures and principal directions are the eigenvalues and the eigenvectors, respectively, of the *shape operator* $\mathcal{L}$ on $\partial\Omega$, i.e. the linear symmetric operator on $T_x\,(\partial\Omega)$ defined by:

$$\mathcal{L}\,(\boldsymbol{v}) \cdot \boldsymbol{w} = B\,(\boldsymbol{v}, \boldsymbol{w}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in T_x\,(\partial\Omega)$$

Let us recall that the matrix of $\mathcal{L}$ with respect to the base $\left\{ \frac{\partial \boldsymbol{x}}{\partial t_1}, \ldots, \frac{\partial \boldsymbol{x}}{\partial t_{n-1}} \right\}$ is, reminding (2.4), $\left\| D_{ij}/E_i \right\|_{i,j=1,\ldots,n-1}$ (see [4]). Therefore, using once more (2.12) and (2.4), we get:

$$(2.16) \quad \sum_{k=1}^{n} X_{kk} = -\sum_{i,j=1}^{n-1} \frac{D_{ij}}{E_i E_j} \left(\sum_{k=1}^{n} \frac{\partial x_k}{\partial t_i} \frac{\partial x_k}{\partial t_j} \right) = -\sum_{i,j=1}^{n-1} \frac{D_{ij}}{E_i E_j} \delta_{ij} E_j =$$

$$= -\sum_{i=1}^{n-1} \frac{D_{ii}}{E_i} = -tr\,\mathcal{L} = -\,(n-1)\,H,$$

where H is the mean curvature of $\partial\Omega$ at x ([4]).

So we have found at last the following formula for Σ :

$$(2.17) \qquad \Sigma = \delta u \cdot \delta u_0 + \|\delta u\|^2\, k_n\,(\delta u) - u_0 \sum_{r=1}^{n} u_{rr} + (n-1)\, u_0^2 H$$

Let us apply (2.17) to Dirichlet's and Neumann's boundary problems. In Dirichlet's case ($u|_{\partial\Omega} = 0$) functions u_i, u_{ij} identically vanish on $\partial\Omega$, so (2.17) becomes:

$$(2.18) \qquad \Sigma_{Dir} = (n-1)\, u_0^2 H,$$

a result already found by Talenti ([5]). If Ω is a convex domain, then $H \leq 0$ on $\partial\Omega$, so in this case Σ_{Dir} is negative on $\partial\Omega$, and hence (1.3) holds true.

In the case of Neumann's boundary condition ($u_0 = 0$), (2.17) becomes:

$$\Sigma_{Neum.} = \|\delta u\|^2 \, k_n \, (\delta u) \tag{2.19}$$

In this case, too, convexity assumption for $\partial\Omega$ implies that $\Sigma_{Neum} \leq 0$ on the whole boundary, and therefore (1.3) holds true.

REFERENCES

[1] S. Campanato, *On the condition of nearness between operators,* Ann. Mat. Pura e Appl., 167 (1994), pp. 243–256.

[2] A. Maugeri, *Il problema di derivata normale per equazioni paraboliche lineari,* Le Matematiche, 27 (1972), pp. 87–93.

[3] C. Miranda, *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui,* Ann. Mat. Pura e Appl., 63 (1963), pp. 353–386.

[4] E. Sernesi, *Geometria 2,* Bollati-Boringhieri, Torino, 1997.

[5] G. Talenti, *Sopra una classe di equazioni ellittiche a coefficienti discontinui,* Ann. Mat. Pura e Appl., 69 (1965), pp. 285–304.

[6] E. Vesentini - G. Gentili - F. Podestà, *Lezioni di Geometria Differenziale,* Bollati-Boringhieri, Torino,1997.

Dipartimento di Matematica e Informatica,
D.M.I.
Università degli Studi di Salerno,
Via S. Allende,
84081 Baronissi, (SA) (ITALY)
e-mail: pugliese@matna2.dma.unina.it