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A COVARIANT APPROACH TO SYMMETRIZABLE AND CONSTRAINED HYPERBOLIC SYSTEMS

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A hyperbolic system with a convex extension is usually transformed in the symmetric form by taking the components of the main field as independent variables. However, the symmetric form can be obtained also in the original independent variables, which may have more physical meaning, by multiplying the system on the left by a suitable matrix P. Here the two methods are compared, showing also how to find the matrix P. The experience gained in this way, allows us to find also a new method to treat the systems with algebraic and differential constraints, without losing manifest covariance. The particular case of Lagrangian systems is also considered.

1. Introduction.

The problem of symmetrizing hyperbolic differential equations has been object of much interest in the literature; see, for example, [1]–[5], [10]–[13], [17], [18], [23], concerning some particular systems of equations of mathematical physics. The general case has also been considered, but at the cost of losing manifest covariance or of using independent variables which are not usually used in Physics; see [12], [13], [14], [15] for a little survey. To eliminate these drawbacks, we may proceed as follows. Let us consider the following system of M equations, in the N independent variables U^B,

$$\partial_\alpha \mathcal{A}_A^\alpha = f_A, \tag{1.1}$$

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which is equivalent to

$$\underline{\mathcal{A}}^{\alpha}\partial_{\alpha}\boldsymbol{U} = \boldsymbol{f}, \tag{1.2}$$

with

$$\mathcal{A}^{\alpha}_{AB} = \frac{\partial \mathcal{A}^{\alpha}_{A}}{\partial U^{B}}. \tag{1.3}$$

Let us suppose, firstly, that $M = N$. The case $M > N$ will be considered in the other sections.
System (1.2) is hyperbolic in the time-direction t^{α} (such that $t^{\alpha}t_{\alpha} = -1$) if and only if the following two conditions hold

1) $Det\ (\mathcal{A}^{\alpha}_{AB}t_{\alpha}) \neq 0$,
2) for any four-vector n^{α} such that $n^{\alpha}t_{\alpha} = 0$, $n^{\alpha}n_{\alpha} = 1$, the eigenvalue problem

$$\sum_{B=1}^{N} \mathcal{A}^{\alpha}_{AB}(n_{\alpha} - \lambda t_{\alpha})\delta U^{B} = 0 \tag{1.4}$$

has real eigenvalues λ and N linearly independent eigenvectors δU^{B}.
Obviously, these conditions are surely satisfied if the matrices $\underline{\mathcal{A}}^{\alpha}$ are symmetric and the matrix $\underline{\mathcal{A}}^{\alpha}t_{\alpha}$ is positive-definite. More generally, they are satisfied if two invertible matrices P and Q exist such that $P\underline{\mathcal{A}}^{\alpha}Q$ are symmetric and $P\underline{\mathcal{A}}^{\alpha}Qt_{\alpha}$ is positive-definite. In fact, in this case we have $Det\ (P\underline{\mathcal{A}}^{\alpha}Qt_{\alpha}) = (Det\ P)\ (Det\ \underline{\mathcal{A}}^{\alpha}t_{\alpha})\ (Det\ Q) \neq 0$ thus assuring condition 1); moreover system (1.4) can be multiplied on the left by the matrix P and transformed in

$$\sum_{B=1}^{N} (P\mathcal{A}^{\alpha}Q)_{AE}\ (n_{\alpha} - \lambda t_{\alpha})\delta W^{E} = 0\,, \tag{1.5}$$

with δW^{E} defined by $\delta U_{B} = Q_{BE}\delta W^{E}$. Obviously, problem (1.5) has real eigenvalues λ and N linearly independent eigenvectors δW^{E}, then assuring the same property also for system (1.4). These considerations suggest the following statement.

Statement 1. System (1.2) is equivalent to a symmetric one, if two invertible matrices P and Q exist, such that $P\underline{\mathcal{A}}^{\alpha}Q$ is a symmetric matrix.

The property of positive-definiteness of $\underline{\mathcal{A}}^{\alpha} t_{\alpha}$ will be considered in Theorem 3 below.

Now, if Statement 1 is satisfied, then also $\underline{\mathcal{A}}^{\alpha} S$ is symmetric with $S = Q(P^{-1})^T$; in fact, $\underline{\mathcal{A}}^{\alpha} S = P^{-1}\left(P\underline{\mathcal{A}}^{\alpha} Q\right)(P^{-1})^T$ holds. This yields the following theorem.

Theorem 1. *System (1.2) is equivalent to a symmetric one, if and only if an invertible matrix S exists, such that $\underline{\mathcal{A}}^{\alpha} S$ is a symmetric matrix.*

The necessary condition has been already proved above, the sufficient condition is obvious, because Statement 1 is a particular case of that in Theorem 1, with $Q = S$ and $P = I$, the identity matrix.
A more restrictive statement has been usually used in many applications in literature [11], [12], [3], [23], [25], [10], [17], [18], [1], that is,

Statement 2. System (1.2) is equivalent to a symmetric one, if it takes the symmetric form with a suitable change of independent variables.

In other words, it is required that the invertible functions $U^B = U^B(\boldsymbol{W})$ exist, such that the matrix

$$\mathcal{A}^{\alpha}_{AB} \frac{\partial U^B}{\partial W^E}$$

is symmetric; the new variables W^E are called "main field".
Obviously, this is a particular case of that in Theorem 1, when S is a Jacobian matrix; moreover, being more restrictive, it may guarantee more analitical properties. For example, it allows the study of weak solutions, since the original system of balance laws and the system symmetrized by premultiplication of some matrix (no longer containing balance laws) are equivalent only under differentiability conditions.
Starting from Statement 2, Friedrichs and Lax [12] conceived an idea which we adopt also using the less restrictive statement in Theorem 1 as starting point. More clearly, if $\underline{\mathcal{A}}^{\alpha} S$ is symmetric, then $(S^{-1})^T(\underline{\mathcal{A}}^{\alpha} S)S^{-1} = (S^{-1})^T\underline{\mathcal{A}}^{\alpha}$ is symmetric too; therefore, the statement in the following theorem is satisfied

Theorem 2. *System (1.2) is equivalent to a symmetric system if and only if an invertible matrix B exists, such that $B\underline{\mathcal{A}}^{\alpha}$ is symmetric.*

This result follows from Theorem 1 with $B = (S^{-1})^T$. Vice versa, if it is satisfied, then Statement 1 follows too with $P = B$, $Q = I$. This property is very interesting, because it shows how system (1.2) can be transformed into symmetric form, no matter how the independent variables are chosen;

for example, we may choose independent variables which have more physical meaning than the main field. In fact, system (1.2) is equivalent to

$$B\underline{\mathcal{A}}^{\alpha}\partial_{\alpha}\boldsymbol{U} = B\boldsymbol{f},$$

where $B\underline{\mathcal{A}}^{\alpha}$ is a symmetric matrix. In Section 2 it is shown how the matrix B can be determined, when system (1.1) satisfies a supplementary conservation law. An apparent disadvantage of this technique is that the new system has not the divergence form; however, it is equivalent to the original system having the divergence form, because they can be obtained each from the other with a multiplication on the left by an invertible matrix. Therefore, all the analitical properties of the original system are preserved; for example, if the original system satisfies Statement 2, the above mentioned differentiability condition holds and, consequently, also the new system has the same weak solutions and schocks. In any case, the original system and the new one have the same eigenvalues and eigenvectors, which may be more easily found from the system in the symmetric form.
Another interesting aspect of this methodology is that it can be applied succesfully to systems with constraints; this subject is well studied in [6], [8] by Boillat, but only in a non-covariant form, in the presence only of differential constraints, and by using the main field instead of the variables which have more physical meaning. These drawbacks are eliminated in Section 3 of this paper; the Lagrangian systems are also considered as an example of physical application. Moreover, in Section 4, case will be considered which has both differential and algebraic constraints. From now on, I shall call "symmetrizable" a system, which is equivalent to a symmetric system.
Let us now consider system (1.2) where $\boldsymbol{U}$ are the components of the main field. The hyperbolicity of this system, in the time-like direction t_{α}, is easy to study; for example, it holds if the matrix $t_{\alpha}\underline{\mathcal{A}}^{\alpha}$ is positive-definite, which fact is equivalent to saying that the quadratic form $t_{\alpha}\mathcal{A}^{\alpha}_{AB}\,dU^{A}dU^{B}$ is positive-definite. Now, this quadratic form is equal to $t_{\alpha}(d\mathcal{A}^{\alpha}_{A})dU^{A}$, as can be seen by use of equation (1.3). Obviously, the positive-definiteness of this quadratic form, does not depend on the choice of the independent variables. Therefore the following theorem holds:

Theorem 3. *The System (1.1) is symmetrizable and hyperbolic in the time-like direction t_{α}, if it is symmetrizable and the quadratic form $t_{\alpha}(d\mathcal{A}^{\alpha}_{A})dU^{A}$ is positive-definite.*

This property has been already applied to un-constrained systems (see equation (1.9) of [24]); here I propose to extend it also for constrained systems, in a way that will be explained more clearly in Sections 3 and 4.

2. Symmetrization of systems satisfying a supplementary conservation law.

Usually, the symmetrizable and hyperbolic systems (1.1) considered in the literature are those that satisfy the supplementary condition

$$\partial_\alpha h^\alpha = g \tag{2.1}$$

for every solution of equations (1.1), where h^α, g are functions depending on the field variables.
By applying the results in [11], [12], [16], we know that this condition is equivalent to assuming the existence of the functions λ^A, called Lagrange multipliers, such that

$$\partial_\alpha h^\alpha - g - \lambda^A \left(\partial_\alpha \mathcal{A}_A^\alpha - f_A\right) = 0$$

holds for every value of the field variables; moreover, this last condition is equivalent to

$$dh^\alpha = \lambda^A d\mathcal{A}_A^\alpha;\ g = \lambda^A f_A. \tag{2.2}$$

If the quadratic form

$$Q = t_\alpha d\lambda^A d\mathcal{A}_A^\alpha, \tag{2.3}$$

is positive-definite, with t_α satisfying the condition $t_\alpha t^\alpha = -1$, we say that system (1.1) has a convex extension.
In this case, the functions λ^A are invertible and we may take them as independent variables; this idea can be found in [3], [23], [25]. By defining $h'^\alpha = -h^\alpha + \lambda^A \mathcal{A}_A^\alpha$, from equation $(2.2)_1$ it follows

$$dh'^\alpha = \mathcal{A}_A^\alpha d\lambda^A, \tag{2.4}$$

from which

$$\mathcal{A}_A^\alpha = \frac{\partial h'^\alpha}{\partial \lambda^A};$$

therefore, system (1.1) becomes

$$\frac{\partial^2 h'^\alpha}{\partial \lambda^A \partial \lambda^B} \partial_\alpha \lambda^B = f_A. \tag{2.5}$$

Consequently, the variables λ^B are the components of the main field and the hypothesis of Theorem 3 are satysfied with $U^A = \lambda^A$; therefore, a system with a convex extension is symmetrizable and hyperbolic.

Vice versa, if system (1.1) is symmetrizable and hyperbolic, from Statement 1 we have that it becomes symmetric when the components U^A of the main field are taken as independent variables; in other words, system (1.1) becomes

$$\frac{\partial \mathcal{A}_A^\alpha}{\partial U^B}\partial_\alpha U^B = f_A \qquad \text{with} \qquad \frac{\partial \mathcal{A}_A^\alpha}{\partial U^B} = \frac{\partial \mathcal{A}_B^\alpha}{\partial U^A}.$$

Now, this symmetry condition is the integrability condition of the problem

$$\frac{\partial h'^\alpha}{\partial U_A} = \mathcal{A}_A^\alpha, \tag{2.6}$$

in the unknown function h'^α.

Therefore, a function h'^α exists, such that (2.6) is satisfied, which is equivalent to

$$dh'^\alpha = \mathcal{A}_A^\alpha \, dU^A.$$

This relation becomes $dh^\alpha = U^A d\mathcal{A}_A^\alpha$, if we define $h^\alpha = -h'^\alpha + \mathcal{A}_A^\alpha U^A$.

In other words, condition (2.1) is satisfied with

$$g = U^A f_A \quad \text{and} \quad U^A = \lambda^A.$$

From the hyperbolicity of system (1.1), and from Theorem 3, it follows also that quadratic form (2.3) is positive-definite.

Therefore, it has been proved that a system is symmetrizable and hyperbolic, if and only if it has a convex extension.

If system (1.1) satisfies this condition, it becomes symmetric when the components of the main field are taken as independent variables; from Theorem 2, it becomes symmetric also in the original independent variables, if we multiply it on the left by a suitable invertible matrix B. Now we can prove that this matrix B is

$$\frac{\partial \lambda^A}{\partial U^C};$$

in other words, system (1.1) can be substituted by

$$\frac{\partial \lambda^A}{\partial U^C}\frac{\partial \mathcal{A}_A^\alpha}{\partial U^B}\partial_\alpha U^B = \frac{\partial \lambda^A}{\partial U^C} f_A, \tag{2.7}$$

and we can prove that the matrix

$$\frac{\partial \lambda^A}{\partial U^C}\frac{\partial \mathcal{A}_A^\alpha}{\partial U^B}$$

is symmetric. In fact, from equation (2.4) we have

$$\frac{\partial h'^\alpha}{\partial U^C} = \mathcal{A}^\alpha_A \frac{\partial \lambda^A}{\partial U^C},$$

from which follows

$$\frac{\partial \lambda^A}{\partial U^C} \frac{\partial \mathcal{A}^\alpha_A}{\partial U^B} = \frac{\partial^2 h'^\alpha}{\partial U^C \partial U^B} - \mathcal{A}^\alpha_A \frac{\partial^2 \lambda^A}{\partial U^C \partial U^B}.$$

It remains to prove the hyperbolicity of system (2.7), i.e. that the quadratic form

$$t_\alpha \frac{\partial \lambda^A}{\partial U^C} \frac{\partial \mathcal{A}^\alpha_A}{\partial U^B} dU^C \, dU^B$$

is positive-definite. Now this quadratic form is exactly that in equation (2.3); therefore, it is positive-definite.
To have a better understanding of these arguments, let us apply them to an easy example of physical application, i.e. the equation of Eulerian fluid dynamics. In this case, system (1.1) is

$$\begin{aligned} &\partial_\alpha V^\alpha = 0, \\ &\partial_\alpha \left[(E+P) u^\alpha u^\beta + P \, g^{\alpha\beta} \right] = 0, \end{aligned} \tag{2.8}$$

with $n = (-V_\alpha V^\alpha)^{1/2}$, $u^\alpha = n^{-1} V^\alpha$.
Here E is the total energy density, P the pressure, n the number particle density, u^α the 4-velocity of the fluid; these variables are such that the Gibbs relation holds

$$\frac{1}{n} dE + (E+P) \, d\left(\frac{1}{n}\right) = T \, dS,$$

where S is the entropy density and T the temperature.
System (2.8) has a convex extension with

$$h^\alpha = -nSu^\alpha \quad , \quad \lambda^A \equiv \left(\frac{E+P}{nT} - S \quad , \quad \frac{u^\alpha}{T} \right).$$

If we take T and V^β as independent variables, from the Gibbs relation we obtain

$$E_T = \frac{\partial E}{\partial T} = nT \frac{\partial S}{\partial T}, \quad E_n = \frac{\partial E}{\partial n} = \frac{E+P}{n} + nT \frac{\partial S}{\partial n},$$

whose integrability condition is

$$E_n = \frac{E+P}{n} - \frac{TP_T}{n}.$$

By use of this relation, we see that the system (2.8) in the form (2.7) reads

$$\text{(2.9)}\quad \begin{aligned} &\frac{E_T}{T^2} u^\alpha \partial_\alpha T + \frac{P_T}{nT} h^\alpha_\beta \partial_\alpha V^\beta = 0, \\ &\frac{P_T}{nT} h^\alpha_\gamma \partial_\alpha T + \frac{1}{n^2 T}\left[(E+P)u^\alpha h_{\gamma\beta} - nP_n\left(2g^\alpha_{(\gamma} u_{\beta)} + u^\alpha u_\gamma u_\beta\right)\right]\partial_\alpha V^\beta = 0, \end{aligned}$$

where $h^\alpha_\beta = g^\alpha_\beta + u^\alpha u_\beta$.
System (2.9) is manifestly symmetric and hyperbolic.
Similarly, if we take P and V^β as independent variables, the Gibbs relation implies that

$$S_P = \frac{1}{nT} E_P; \quad S_n = \frac{1}{nT} E_n - \frac{E+P}{n^2 T},$$

from which it follows

$$-\frac{1}{nT^2} E_P T_n = -\frac{1}{nT^2} E_n T_P - \frac{1}{n^2 T} + \frac{E+P}{n^2 T^2} T_P;$$

consequently, system (2.8) in the form (2.7) reads

$$\text{(2.10)}\quad \begin{aligned} &\frac{1}{T^2} T_P E_P u^\alpha \partial_\alpha P + \left(\frac{1}{nT} h^\alpha_\beta - \frac{1}{T^2} T_n E_P u^\alpha u_\beta\right)\partial_\alpha V^\beta = 0, \\ &\left(\frac{1}{nT} h^\alpha_\gamma - \frac{1}{T^2} T_n E_P u^\alpha u_\gamma\right)\partial_\alpha P + \\ &+\left[\frac{E+P}{n^2 T} h_{\beta\gamma} + \frac{T_n}{T^2}\left(E_n - \frac{E+P}{n}\right) u_\beta u_\gamma\right] u^\alpha \partial_\alpha V^\beta = 0, \end{aligned}$$

which is symmetric hyperbolic.
At last, if we take E and V^β as independent variables, we have

$$S_E = \frac{1}{nT}, \quad S_n = -\frac{E+P}{n^2 T},$$

from which

$$T_n = \frac{T}{n} P_E - \frac{E+P}{n} T_E.$$

System (2.8) in the form (2.7) reads

$$\frac{T_E}{T^2}u^\alpha \partial_\alpha P + \left(\frac{P_E}{nT}g^\alpha_\beta + T_E\frac{E+P}{nT^2}u^\alpha u_\beta\right)\partial_\alpha V^\beta = 0, \tag{2.11}$$

$$\left(\frac{P_E}{nT}g^\alpha_\gamma + T_E\frac{E+P}{nT^2}u^\alpha u_\gamma\right)\partial_\alpha P +$$
$$+\left[-2\frac{P_n}{nT}u_{(\gamma}h^\alpha_{\beta)} + \frac{E+P}{n^2T}h_{\gamma\beta}u^\alpha + \left(\frac{P_n}{nT} - T_n\frac{E+P}{nT^2}\right)u_\gamma u_\beta u^\alpha\right]\partial_\alpha V^\beta = 0;$$

It is again symmetric hyperbolic.
In the next section, this method will be applied succesfully to systems with differential constraints, with an application to the Lagrangian systems.
Before concluding this section, I want to observe that the condition of positive-definiteness on the quadratic form (2.3) is particularly significant, when t_α is field-independent and system (1.1) has no differential constraints. In fact, in this case, this condition implies that $t_\alpha \mathcal{A}^\alpha_A = V_A$ are invertible functions of U_A; taking them as independent variables, from equation $(2.2)_1$ it follows

$$\lambda^A = \frac{\partial(t_\alpha h^\alpha)}{\partial V_A}.$$

Consequently, the quadratic form (2.3) becomes

$$\frac{\partial^2(h^\alpha t_\alpha)}{\partial V_A \partial V_B} dV_B\, dV_A.$$

From this fact, we see that the quadratic form (2.3) is positive-definite if and only if $h^\alpha t_\alpha$ is a convex function of the variables V_A. Now, in many physical problem, $h^\alpha t_\alpha$ is the entropy density; therefore, the condition on the quadratic form (2.3) to be positive-definite may be called "convexity of the entropy density with respect to the variables $t_\alpha \mathcal{A}^\alpha_A$". Obviously, this is equivalent to saying that the function $h'^\alpha t_\alpha$ is a convex function of the variables λ^A.

3. Symmetrization of systems with differential constraints, endowed with a convex extension.

Let us consider system (1.1), but in the case $M > N$; obviously, $M - N$ rows of the matrix $t_\alpha \mathcal{A}^\alpha_{AB}$ are linear combinations of the others; therefore, the corresponding $M - N$ equations of system (1.1) can be substituted by $M - N$

different equations, where no derivative with respect to time appears, in the time-like direction t_α. Then we say that system (1.1) has $M - N$ differential constraints. We say that system (1.1), with differential constraints, is endowed with a convex extension, if the following conditions are satisfied
1) the functions h^α, λ^A exist, such that equation $(2.2)_1$ is satisfied;
2) the quadratic form $t_\alpha d\lambda^A d\mathcal{A}^\alpha_A$ is positive-definite.
Obviously, these conditions are equivalent to assuming that the supplementary equation (2.1) holds for every solution of system (1.1), with g given by equation $(2.2)_2$.
But, unlike the case considered in Section 2, now the functions λ^A are not invertible. In any case, however, the rectangular matrix $\frac{\partial\lambda^A}{\partial U^C}$ has rank N, because $t_\alpha d\lambda^A d\mathcal{A}^\alpha_A$ is positive-definite.
Consequently, we can multiply system (1.1) by this matrix, obtaining again system (2.7) of N equations, in the N unknowns U^B. As before, we can prove that the matrix

$$\frac{\partial\lambda^A}{\partial U^C}\frac{\partial\mathcal{A}^\alpha_A}{\partial U^B}$$

is symmetric, and that the matrix

$$t_\alpha\frac{\partial\lambda^A}{\partial U^C}\frac{\partial\mathcal{A}^\alpha_A}{\partial U^B}$$

is positive-definite; therefore system (2.7) is symmetric and hyperbolic.
Now system (1.1) is equivalent to system (2.7) and to $N - M$ non evolutive equations; these last ones may be imposed on the initial manifold and, after that, omitted because they will be automatically satisfied as consequence of system (2.7) and of such initial conditions.
Let us consider now an example of physical application, the Lagrangian systems. They are described by equations such as

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial q^h_\alpha} = \frac{\partial\mathcal{L}}{\partial q^h}, \tag{3.1}$$

in the p unknowns q^h, and with $q^h_\alpha = \partial_\alpha q^h$.
The problem of studying system (3.1) as a system compatible with a supplementary conservation law, was developed by Boillat [4]–[8], in non-covariant formulation and, later, by Strumia [27] in covariant form and with constrained equations. This last author writes the Euler-Lagrange equations (3.1) as

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial q^h_\alpha} = \frac{\partial\mathcal{L}}{\partial q^h}, \quad \partial_\alpha q^h = q^h_\alpha, \quad \partial_\alpha q^h_\beta - \partial_\beta q^h_\alpha = 0, \tag{3.2}$$

in the $5p$ independent variables q^h, q^h_α.
System (3.2) is compatible with a supplementary conservation law, with

$$h^\alpha = \frac{\partial \mathcal{L}}{\partial q^h_\alpha} q^\beta_h t_\beta - \mathcal{L} t^\alpha, \quad g = 0,$$

while the Lagrange multipliers are

$$\lambda^A \equiv \left(q^h_\beta t^\beta, -\frac{\partial \mathcal{L}}{\partial q^h} t^\alpha, \frac{\partial \mathcal{L}}{\partial q^h_{[\beta}} t^{\alpha]} \right).$$

Strumia [27] transforms system (3.2) into a symmetric one, by taking λ^A as variables; these Lagrange multipliers are also constrained variables. The method of the present paper allows us to transform system (3.2) into a symmetric one also with q^h, q^h_α as independent variables; these variables are not constrained. The transformed system (2.7), for this case, is given by

$$\left(\partial_\alpha \frac{\partial \mathcal{L}}{\partial q^h_\alpha} - \frac{\partial \mathcal{L}}{\partial q^h} \right) \frac{\partial \left(q^h_\beta t^\beta \right)}{\partial U^C} - \left(\partial_\alpha q^h - q^h_\alpha \right) \frac{\partial}{\partial U^C} \left(\frac{\partial \mathcal{L}}{\partial q^h} t^\alpha \right) +$$
$$+ \left(\partial_\alpha q^h_\beta - \partial_\beta q^h_\alpha \right) \frac{\partial}{\partial U^C} \left(\frac{\partial \mathcal{L}}{\partial q^h_{[\beta}} t^{\alpha]} \right) = 0,$$

that is,

$$-t^\alpha \left(\partial_\alpha q^h - q^h_\alpha \right) \frac{\partial^2 \mathcal{L}}{\partial q^h \partial q^k} + t^\beta \left(\partial_\alpha q^h_\beta - \partial_\beta q^h_\alpha \right) \frac{\partial^2 \mathcal{L}}{\partial q^h_\alpha \partial q^k} = 0,$$
$$\left(\partial_\alpha \frac{\partial \mathcal{L}}{\partial q^k_\alpha} - \frac{\partial \mathcal{L}}{\partial q^k} \right) t^\delta - t^\alpha \left(\partial_\alpha q^h - q^h_\alpha \right) \frac{\partial^2 \mathcal{L}}{\partial q^h \partial q^k_\delta} +$$
$$+ t^\beta \left(\partial_\alpha q^h_\beta - \partial_\beta q^h_\alpha \right) \frac{\partial^2 \mathcal{L}}{\partial q^h_\alpha \partial q^k_\delta} = 0,$$

which can be expressed as

(3.3)
$$A^\alpha_{kl} \partial_\alpha q^l + B^\alpha_{kl\gamma} \partial_\alpha q^{l\gamma} = -t^\alpha q^h_\alpha \frac{\partial^2 \mathcal{L}}{\partial q^h \partial q^k},$$
$$B^\alpha_{lk\delta} \partial_\alpha q^l + C^\alpha_{k\delta l\gamma} \partial_\alpha q^{l\gamma} = \frac{\partial \mathcal{L}}{\partial q^k} t^\delta - t^\alpha q^h_\alpha \frac{\partial^2 \mathcal{L}}{\partial q^h \partial q^k_\delta},$$

where

$$A^\alpha_{kl} = A^\alpha_{lk} = -t^\alpha \frac{\partial^2 \mathcal{L}}{\partial q^l \partial q^k}, \quad B^\alpha_{kl\gamma} = t_\gamma \frac{\partial^2 \mathcal{L}}{\partial q^l_\alpha \partial q^k} - t^\alpha \frac{\partial^2 \mathcal{L}}{\partial q^{l\gamma} \partial q^k},$$

$$C^\alpha_{k\delta l\gamma} = C^\alpha_{l\gamma k\delta} = t_\delta \frac{\partial^2 \mathcal{L}}{\partial q^k_\alpha \partial q^{l\gamma}} + t_\gamma \frac{\partial^2 \mathcal{L}}{\partial q^l_\alpha \partial q^{k\delta}} - t^\alpha \frac{\partial^2 \mathcal{L}}{\partial q^{l\gamma} \partial q^{k\delta}}.$$

System (3.3) is manifestly symmetric.
It is also hyperbolic, if quadratic form (2.3) is positive-definite, a requirement corresponding to the convexity of the entropy density, which is assumed by Boillat and Strumia. In our case, quadratic form (2.3) becomes

$$\begin{aligned}
Q &= t_\alpha dq^k \left(A^\alpha_{kl} dq^l + B^\alpha_{kl\gamma} dq^{l\gamma}\right) + t_\alpha dq^{k\delta} \left(B^\alpha_{lk\delta} dq^l + C^\alpha_{k\delta l\gamma} dq^{l\gamma}\right) = \\
&= \frac{\partial^2 \mathcal{L}}{\partial q^l \partial q^k} dq^l dq^k + 2 \left(t_\gamma t^\alpha \frac{\partial^2 \mathcal{L}}{\partial q^{l\alpha} \partial q^k} + \frac{\partial^2 \mathcal{L}}{\partial q^{l\gamma} \partial q^k}\right) dq^k dq^{l\gamma} + \\
&+ \left(\frac{\partial^2 \mathcal{L}}{\partial q^{l\gamma} \partial q^{k\delta}} + t^\alpha t_\delta \frac{\partial^2 \mathcal{L}}{\partial q^{k\alpha} \partial q^{l\gamma}} + t^\alpha t_\gamma \frac{\partial^2 \mathcal{L}}{\partial q^{l\alpha} \partial q^{k\delta}}\right) dq^{k\delta} dq^{l\gamma} = \\
&= -\frac{\partial^2 \mathcal{L}}{\partial q^{l0} \partial q^{k0}} dq^{l0} dq^{k0} + \frac{\partial^2 \mathcal{L}}{\partial q^{ki} \partial q^{lj}} dq^{ki} dq^{lj} + \\
&+ 2 \frac{\partial^2 \mathcal{L}}{\partial q^k \partial q^{lj}} dq^k dq^{lj} + \frac{\partial^2 \mathcal{L}}{\partial q^l \partial q^k} dq^l dq^k = \\
&= \frac{\partial^2 \mathcal{L}}{\partial q^{l0} \partial q^{k0}} dq^{l0} dq^{k0} + dq^{lj} d\left(\frac{\partial \mathcal{L}}{\partial q^{lj}}\right) + dq^k d\left(\frac{\partial \mathcal{L}}{\partial q^k}\right).
\end{aligned}$$

In the next section, the case of systems with algebraic constraints will be considered.

4. Symmetrization of systems with differential and algebraic constraints.

Let us consider system (1.1) with the field variables constrained by n functionally independent algebraic relation

$$\Phi_I(U^B) = 0, \qquad \text{with} \quad I = 1, \cdots, n. \tag{4.1}$$

Moreover, we consider the case $n < N$, $M \geq N - n$. From equation (4.1) we can obtain $N - n$ variables, as functions of the remaining ones; more generally, there are different possibilities of choosing $N - n$ independent parameters q^h, such that

$$U^B = U^B(q^h) \tag{4.2}$$

is a solution of equation (4.1), identical with respect to q^h.
We consider the case where the functions $h^\alpha(U^B), \lambda^A(U^B)$ exist, such that equation $(2.7)_1$ holds, and the quadratic form $t_\alpha d\lambda^A d\mathcal{A}^\alpha_A$ is positive-definite, for every choice of the parametrical representation (4.2).
Also in this case, Strumia [26] says that the constrained system (1.1) is endowed with a convex extension. Strumia proves also that this system assumes the symmetric form, if the Lagrange multipliers λ^A are taken as new variables, which are also algebraically constrained. But the matrix of the coefficients of $\partial_\alpha\lambda^A$ is singular; consequently, so as to obtain the eigenvectors, we have to consider n different equations, which are the differentials of the algebraic constraints on λ^A. In this way we obtain a set of $N + n$ equations (Strumia considers the case $M = N$) in the N unknowns $d\lambda^A$; consequently, the symmetric form is lost. This obstacle can be overcome, as will be shown below.
But the main interest of this paper is to obtain the symmetric hyperbolic form also in the original variables U^B; there are two possible approaches to obtain this result.
The first one is that of choosing one of the parametrical representations (4.2) and to proceed with the unconstrained variables q^h as in Section 3. Unfortunately, in this way, the originally covariant form may be lost.
The second approach is the following one.
Let us consider the n linearly independent equations

$$\frac{\partial \Phi_I}{\partial U^B} X^B = 0, \tag{4.3}$$

in the N unknowns X^B. System (4.3) is expressed in covariant form, so that its general solution is a covariant linear function of p parameters Y^C, i.e. $X^B = X^B(Y^C)$. Obviously $p \geq N - n$; if I request $p = N - n$, the covariant form may be lost. The matrix

$$X^B_C = \frac{\partial X^B}{\partial Y^C}$$

has rank $N - n$ and is such that

$$\frac{\partial \Phi_I}{\partial U^B} X^B_C = 0, \tag{4.4}$$

for every $I = 1, \cdots, n$; $C = 1, \cdots, p$.
Let us multiply system (1.1) by $X_C^B \frac{\partial \lambda^A}{\partial U^B}$, obtaining the new system

$$X_C^B \frac{\partial \lambda^A}{\partial U^B} \partial_\alpha \mathcal{A}_A^\alpha = X_C^B \frac{\partial \lambda^A}{\partial U^B} f_A = S_C, \tag{4.5}$$

in the N variables U^B constrained by (4.1).
We can now prove that

i) system (4.5) is symmetrizable and hyperbolic;
ii) the eigenvalues λ associated to system (4.5) can be expressed in covariant form; in particular, they are the solutions of the equation

$$S_{N-n} \left[(n_\alpha - \lambda t_\alpha) X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} X_{C'}^{B'} \right] = 0, \tag{4.6}$$

where $S_i(\mathcal{M})$ denotes the orthogonal invariant of order i, of the matrix $\mathcal{M}$;
iii) a set of $N-n$ linearly independent eigenvectors, associated to system (4.5), can be obtained and expressed in covariant form.

On i).
To prove this first property, it is not necessary to worry about the covariant form. Then, let $U^B(q^h)$ be a parametrical representation of the solutions of equation (4.1); consequently, we have

$$\frac{\partial \Phi_I}{\partial U^B} \frac{\partial U^B}{\partial q^h} = 0, \quad \text{i.e.} \quad \frac{\partial U^B}{\partial q^h}$$

is a solution of system (4.3) whose set of solutions has the generators X_C^B. Therefore L_h^C exist such that

$$\frac{\partial U^B}{\partial q^h} = L_h^C X_C^B.$$

By multiplying system (4.5) by L_h^C we obtain

$$\frac{\partial U^B}{\partial q^h} \frac{\partial \lambda^A}{\partial U^B} \partial_\alpha \mathcal{A}_A^\alpha = \frac{\partial U^B}{\partial q^h} \frac{\partial \lambda^A}{\partial U^B} f_A. \tag{4.7}$$

This new system is equivalent to (4.5); in fact, $\frac{\partial U^B}{\partial q^h}$ is also a set of generators of the solutions of equation (4.3), even if they are not expressed in covariant form.

Therefore $\mathcal{L}_C^h$ exists such that $X_C^B = \mathcal{L}_C^h \frac{\partial U^B}{\partial q^h}$; by multiplying system (4.7) by $\mathcal{L}_C^h$, we obtain system (4.5).
Now system (4.7) is

$$\frac{\partial \lambda^A}{\partial q^h} \partial_\alpha \mathcal{A}_A^\alpha = \frac{\partial \lambda^A}{\partial q^h} f_A,$$

which is symmetric and hyperbolic, because it is the same system obtained, with the first approach, in the unconstrained variables q^h.
On ii).
The eigenvalues λ and eigenvectors dU^B, associated to system (4.5) are the solutions of

$$(n_\alpha - \lambda t_\alpha)\, X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} dU^{B'} = 0, \tag{4.8}$$
$$\frac{\partial \Phi_I}{\partial U^{B'}} dU^{B'} = 0.$$

From equation $(4.8)_2$ we obtain

$$dU^{B'} = X_{C'}^{B'} dY^{C'}, \tag{4.9}$$

and $(4.8)_1$ becomes

$$(n_\alpha - \lambda t_\alpha)\, X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} X_{C'}^{B'} dY^{C'} = 0, \tag{4.10}$$

i.e.

$$(\mathcal{B}_{CC'} - \lambda \mathcal{A}_{CC'})\, dY^{C'} = 0, \tag{4.11}$$

with

$$\mathcal{B}_{CC'} = n_\alpha X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} X_{C'}^{B'}; \quad \mathcal{A}_{CC'} = t_\alpha X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} X_{C'}^{B'}. \tag{4.12}$$

We notice that $\mathcal{B}_{CC'}$ and $\mathcal{A}_{CC'}$ are symmetric matrices, as will be proved in the appendix.
It is true that the following passages will not be expressed in covariant form, but the important thing is that the final result will satisfy this requirement.
Let $\boldsymbol{W}_{N-n+1}, \cdots, \boldsymbol{W}_N$ be n orthonormal solutions of

$$X_{C'}^{B'} W^{C'} = 0. \tag{4.13}$$

We have that $\mathcal{B}_{CC'} W^{C'}_{N-n+i} = 0$, $\mathcal{A}_{CC'} W^{C'}_{N-n+i} = 0$. Moreover $\mathcal{A}_{CC'}$ has rank $N - n$, because condition i) is satisfied. Let us consider a set of N orthonormal eigenvectors of the matrix $\mathcal{A}$, with respect to the identical matrix I, with $\boldsymbol{W}_{N-n+1}, \cdots, \boldsymbol{W}_N$ as last elements; let us consider the matrix $\mathcal{P}$ which has these eigenvectors as columns. Let us change the variables by means of

$$dY^{C'} = \mathcal{P}^{C'}_{D'} dZ^{D'}, \tag{4.14}$$

and multiply equation (4.11) by $\mathcal{P}^C_D$; it becomes

$$\left(\mathcal{P}^C_D \mathcal{B}_{CC'} \mathcal{P}^{C'}_{D'} - \lambda \quad \mathcal{P}^C_D \mathcal{A}_{CC'} \mathcal{P}^{C'}_{D'}\right) dZ^{C'} = 0. \tag{4.15}$$

Now,

$$\mathcal{P}^C_D \mathcal{B}_{CC'} \mathcal{P}^{C'}_{D'} = \left\| \begin{matrix} B^* & \mathbf{0}_{N-n,n} \\ \mathbf{0}_{n,N-n} & \mathbf{0}_{n,n} \end{matrix} \right\|$$

$$\mathcal{P}^C_D \mathcal{A}_{CC'} \mathcal{P}^{C'}_{D'} = \left\| \begin{matrix} A^* & \mathbf{0}_{N-n,n} \\ \mathbf{0}_{n,N-n} & \mathbf{0}_{n,n} \end{matrix} \right\|$$

with B^*, A^* symmetric $(N-n) \times (N-n)$ matrices.
If the first $N - n$ components of $dZ^{C'}$ are zero, then equation (4.14) yields that $dY^{C'}$ is a linear combination of $\boldsymbol{W}_{N-n+1}, \cdots, \boldsymbol{W}_N$; after that, by (4.9) and (4.11) we obtain $dY^{B'} = 0$, which is not acceptable as eigenvector. Therefore we look for solutions of the system (4.15), with $dZ^{C'}$ having at least a non-zero component between the first $N - n$ ones.
This fact shows that the eigenvalues λ are the solutions of $|B^* - \lambda A^*| = 0$, which can be expressed as

$$S_{N-n}\left(\mathcal{P}^C_D \mathcal{B}_{CC'} \mathcal{P}^{C'}_{D'} - \lambda \quad \mathcal{P}^C_D \mathcal{A}_{CC'} \mathcal{P}^{C'}_{D'}\right) = 0.$$

But $S_i(\mathcal{M}) = S_i\left(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right)$, because the characteristic equations of $\mathcal{M}$ and of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}$ are the same. Therefore (4.16) is equal to $S_{N-n}\left(\mathcal{B}_{CC'} - \lambda \quad \mathcal{A}_{CC'}\right) = 0$, from which equation (4.6) follows. We notice also, that equation (4.6) is expressed in covariant form, because its first member is an orthogonal invariant of a covariant matrix.
On iii).
The existence of $N - n$ linearly independent eigenvectors, is a consequence of property i); they can be expressed in covariant form because they are the solutions of the system (4.8), which satisfies this property.

NOTE: System (4.5) is not symmetric, even if it is symmetrizable. However, the symmetry property is useful to study the eigenvalues and eigenvectors, i.e. to

solve the system (4.8), which is equivalent to (4.9) and (4.11); this last system is expressed in symmetric form, so that we can consider this result as very satisfactory.
To see better how this method works, let us consider two simple examples of physical applications.

4.1. *The equations of fluiddynamics.*

Let us consider again the equations

$$\begin{aligned} &\partial_\alpha (n u^\alpha) = 0, \\ &\partial_\alpha \left[(E+P) u^\alpha u^\beta + P g^{\alpha\beta} \right] = 0, \end{aligned} \tag{4.17}$$

but in the six variables n, T, u^α constrained by $u^\alpha u_\alpha = -1$. System (4.3) becomes

$$0 \cdot X' = 0, \quad 0 \cdot X'' = 0, \quad 2u_\alpha X^\alpha = 0,$$

which has the solution $X^\alpha = h^\alpha_\gamma Y^\gamma$; consequently, the matrix X^B_C is

$$\begin{Vmatrix} 1 & 0 & 0_\beta \\ 0 & 1 & \mathbf{0}_\beta \\ \mathbf{0}_\gamma & \mathbf{0}_\gamma & h^\alpha_\gamma, \end{Vmatrix}$$

and system (4.5) reads

$$\begin{array}{lllllll} \dfrac{P_n}{nT} u^\alpha \partial_\alpha n & + & & & \dfrac{P_n}{T} h^\alpha_\beta \partial_\alpha u^\beta & = 0, \\ & & \dfrac{E_T}{T^2} u^\alpha \partial_\alpha T & + & \dfrac{P_T}{T} h^\alpha_\beta \partial_\alpha u^\beta & = 0, \\ \dfrac{P_n}{T} h^\alpha_\gamma \partial_\alpha n & + & \dfrac{P_T}{T} h^\alpha_\gamma \partial_\alpha T & + & \dfrac{E+P}{T} h_{\gamma\beta} u^\alpha \partial_\alpha u^\beta & = 0. \end{array} \tag{4.18}$$

The corresponding equation (4.6) is

$$\begin{aligned} \frac{(E+P)^2}{nT^6} P_n \, (\phi_\alpha u^\alpha)^3 \Big\{ &(E+P) E_T \, (\phi_\alpha u^\alpha)^2 - \\ &- h^{\alpha\beta} \phi_\alpha \phi_\beta \left[n P_n E_T + T \, (P_T)^2 \right] \Big\} = 0, \end{aligned} \tag{4.19}$$

where $\phi_\alpha = n_\alpha - \lambda \, t_\alpha$.

4.2. *The equations of magnetofluiddynamics*.

In this case system (1.1) is given by the nine equations

$$
\begin{aligned}
&\partial_\alpha (n u^\alpha) = 0, \\
&\partial_\alpha \left[(E + P + b^2) u^\alpha u^\beta + (P + b^2) g^{\alpha\beta} - b^\alpha b^\beta \right] = 0, \\
&\partial_\alpha (u^\alpha b^\beta - b^\alpha u^\beta) = 0.
\end{aligned}
\tag{4.20}
$$

for the determination of the ten variables n, T, u^α, b^α (related to the magnetic field), constrained by

$$u_\alpha u^\alpha = -1, \quad u_\alpha b^\alpha = 0.$$

The Lagrange multipliers are

$$\lambda^A \equiv \left(\frac{E + P}{nT} - S \quad , \quad \frac{u^\beta}{T} \quad , \quad \frac{b^\beta}{T} \right).$$

System (4.3) becomes

$$2 u_\delta X^\delta = 0, \quad b_\delta X^\delta + u_\delta x^\delta = 0,$$

which has the solution

$$X^\delta = h^\delta_\gamma Y^\gamma, \quad x^\delta = b_\gamma Y^\gamma u^\delta + h^\delta_\gamma y^\gamma;$$

consequently, the matrix X^B_C is

$$\left\| \begin{array}{llll} 1 & 0 & \mathbf{0}_\beta & \mathbf{0}_\beta \\ 0 & 1 & \mathbf{0}_\beta & \mathbf{0}_\beta \\ \mathbf{0}_\gamma & \mathbf{0}_\gamma & h_{\delta\gamma} & u_\delta b_\gamma \\ \mathbf{0}_\gamma & \mathbf{0}_\gamma & u_\gamma b_\delta & h_{\delta\gamma} \end{array} \right\| ;$$

System (4.5) is given by

$$\frac{\partial \lambda^A}{\partial U^B} \partial_\alpha \mathcal{A}^\alpha_A = \frac{\partial \lambda^A}{\partial U^B} f_A,$$

premultiplied by the transposite of this matrix X^B_C, i.e.

$$
\begin{array}{lllll}
\frac{P_n}{nT} u^\alpha \partial_\alpha n + & & \frac{P_n}{T} g^\alpha_\beta \partial_\alpha u^\beta & & = 0, \\
& \frac{E_T}{T^2} u^\alpha \partial_\alpha T + & \frac{P_T}{T} g^\alpha_\beta \partial_\alpha u^\beta & & = 0, \\
\frac{P_n}{T} h^\alpha_\gamma \partial_\alpha n + & \frac{P_T}{T} h^\alpha_\gamma \partial_\alpha T + & \frac{1}{T} E^\alpha_{\gamma\beta} \partial_\alpha u^\beta + & \frac{1}{T} C^\alpha_{\gamma\beta} \partial_\alpha b^\beta & = 0, \\
& & \frac{1}{T} D^\alpha_{\gamma\beta} \partial_\alpha u^\beta + & \frac{1}{T} h_{\gamma\beta} u^\alpha \partial_\alpha b^\beta & = 0,
\end{array}
\tag{4.21}
$$

with

$$E^{\alpha}_{\gamma\beta} = (E + P + b^2)h_{\gamma\beta}u^{\alpha} - b_{\gamma}b^{\alpha}u_{\beta},$$
$$C^{\alpha}_{\gamma\beta} = h^{\alpha}_{\gamma}b_{\beta} - b^{\alpha}h_{\gamma\beta} + u^{\alpha}b_{\gamma}u_{\beta},\ D^{\alpha}_{\gamma\beta} = b_{\gamma}g^{\alpha}_{\beta} - h_{\gamma\beta}b^{\alpha}.$$

System (4.21) is not symmetric, but the corresponding eigenvectors are the solutions of

$$\begin{array}{lllll}
\frac{P_n}{nT}u^{\alpha}\phi_{\alpha}dn + & & \frac{P_n}{T}g^{\alpha}_{\beta}\phi_{\alpha}du^{\beta} & & = 0,\\
& \frac{E_T}{T^2}u^{\alpha}\phi_{\alpha}dT + & \frac{P_T}{T}g^{\alpha}_{\beta}\phi_{\alpha}du^{\beta} & & = 0,\\
\frac{P_n}{T}h^{\alpha}_{\gamma}\phi_{\alpha}dn + & \frac{P_T}{T}h^{\alpha}_{\gamma}\phi_{\alpha}dT + & \frac{1}{T}E^{\alpha}_{\gamma\beta}\phi_{\alpha}du^{\beta} + & \frac{1}{T}C^{\alpha}_{\gamma\beta}\phi_{\alpha}db^{\beta} & = 0,\\
& & \frac{1}{T}D^{\alpha}_{\gamma\beta}\phi_{\alpha}du^{\beta} + & \frac{1}{T}h_{\gamma\beta}u^{\alpha}\phi_{\alpha}db^{\beta} & = 0,\\
& u_{\beta}du^{\beta} = 0, & & u_{\beta}db^{\beta} + b_{\beta}du^{\beta} & = 0.
\end{array}$$

These last two equations give

$$du^{\beta} = h^{\beta}_{\gamma}Y^{\gamma}, \quad db^{\beta} = b_{\gamma}Y^{\gamma}u^{\beta} + h^{\beta}_{\gamma}y^{\gamma}$$

in the new unknowns Y^{γ}, y^{γ}; by substituting these relations in the other equations, they become

$$\begin{array}{lllll}
\frac{P_n}{nT}u^{\alpha}\phi_{\alpha}dn + & & \frac{P_n}{T}h^{\alpha}_{\delta}\phi_{\alpha}Y^{\delta} & & = 0,\\
& \frac{E_T}{T^2}u^{\alpha}\phi_{\alpha}dT + & \frac{P_T}{T}h^{\alpha}_{\delta}\phi_{\alpha}Y^{\delta} & & = 0,\\
\frac{P_n}{T}h^{\alpha}_{\gamma}\phi_{\alpha}dn + & \frac{P_T}{T}h^{\alpha}_{\gamma}\phi_{\alpha}dT + & \frac{1}{T}F_{\gamma\delta}\phi_{\alpha}u^{\alpha}Y^{\delta} + & \frac{1}{T}G_{\gamma\delta}y^{\delta} & = 0,\\
& & \frac{1}{T}G_{\delta\gamma}Y^{\delta} + & \frac{1}{T}h_{\gamma\delta}u^{\alpha}\phi_{\alpha}y^{\delta} & = 0,
\end{array}$$

with

$$F_{\gamma\delta} = (E + P + b^2)h_{\gamma\delta} - b_{\gamma}b_{\delta}, \quad G_{\gamma\delta} = \phi_{\alpha}\left(h^{\alpha}_{\gamma}b_{\delta} - b^{\alpha}h_{\gamma\delta}\right),$$

so that the above system is a symmetric one. The matrix of the coefficients has rank 8; by calculating its S_8, we obtain that equation (4.6) for this case reads,

$$\frac{P_n}{n}T^{-9}E_T a^2 A N_4 = 0, \tag{4.22}$$

with

$$a = \phi_\alpha u^\alpha, \quad A = (E + P + b^2)(\phi_\alpha u^\alpha)^2 - (\phi_\alpha b^\alpha)^2,$$

$$N_4 = (E + P)\left(E + P - nP_n - \frac{T(P_T)^2}{E_T}\right)(\phi_\alpha u^\alpha)^4 -$$

$$-(E + P)\left(nP_n + \frac{T(P_T)^2}{E_T} + b^2\right)(\phi_\alpha u^\alpha)^2 \phi_\beta \phi^\beta +$$

$$+\left(nP_n + \frac{T(P_T)^2}{E_T}\right)(\phi_\alpha b^\alpha)^2 \phi_\beta \phi^\beta .$$

Obviously, this method can be applied also to the symmetric system obtained by Strumia [27], for Lagrangian systems; in fact, he obtains this system by taking the Lagrange multipliers as constrained variables. However, I do not exploit this problem further in detail, because the results will not be more significative than those in Section 3.

For the sake of completeness, I recall that a different approach to constrained systems is that of considering an extended set of field equations and of independent variables; I conceived this method applying it in some physical problems [19], [20]. Boillat [9] has shown how this approach can be used for all systems, which have only differential constraints that are linear functions of the independent variables; the systems considered by Boillat are not expressed in covariant form. Later, I have eliminated the hypothesis of linearity [21]. The problem with algebraic constraints and expressed in covariant form is exhausted in the present paper. The case where a convex extension is not present, has already been treated in [22].

A. Appendix. On the symmetry of the matrices $\mathcal{A}_{CC'}$ and $\mathcal{B}_{CC'}$.

In Section 4, systems (1.1) have been considered, which are endowed with a convex extension $(2.2)_1$ and present both differential and algebraic constraints (4.1). Now, from equation (4.1) we can obtain $N - n$ of the variables U^B as functions of the remaining ones; by changing the names of these variables, we can obtain $U^{n+1}, \cdots, U^N$ as functions of $U^1, \cdots, U^n$. After that equation $(2.2)_1$ gives

$$(A.1) \qquad \frac{\partial h^\alpha}{\partial U^i} - \lambda^A \frac{\partial \mathcal{A}_A^\alpha}{\partial U^i} + \sum_{j=n+1}^{N} \left(\frac{\partial h^\alpha}{\partial U^j} - \lambda^A \frac{\partial \mathcal{A}_A^\alpha}{\partial U^j}\right)\frac{\partial U^j}{\partial U^i} = 0,$$

with $i = 1, \cdots, n$. From equation (4.1) we have also

$$\frac{\partial \Phi_I}{\partial U^i} + \sum_{j=n+1}^{N} \frac{\partial \Phi_I}{\partial U^j} \frac{\partial U^j}{\partial U^i} = 0. \tag{$A.2$}$$

Let $\| P^{kI} \|$ be the inverse matrix of $\| \frac{\partial \Phi_I}{\partial U^j} \|$, so that

$$P^{kI} \frac{\partial \Phi_I}{\partial U^j} = \delta^k_j. \tag{$A.3$}$$

From equation (A.2) we obtain

$$\frac{\partial U^j}{\partial U^i} = -P^{jI} \frac{\partial \Phi_I}{\partial U^i}. \tag{$A.4$}$$

Let us define $\omega^{I\alpha}$ as

$$\omega^{I\alpha} = \sum_{j=n+1}^{N} \left(\frac{\partial h^\alpha}{\partial U^j} - \lambda^A \frac{\partial \mathcal{A}^\alpha_A}{\partial U^j} \right) P^{jI}. \tag{$A.5$}$$

After that, equation (A.1) becomes

$$\frac{\partial h^\alpha}{\partial U^i} - \lambda^A \frac{\partial \mathcal{A}^\alpha_A}{\partial U^i} - \omega^{I\alpha} \frac{\partial \Phi_I}{\partial U^i} = 0 \quad \text{for}\, i = 1, \cdots, n. \tag{$A.6$}$$

Now, if $i > n$, from (A.3) we have

$$\omega^{I\alpha} \frac{\partial \Phi_I}{\partial U^i} = \frac{\partial h^\alpha}{\partial U^i} - \lambda^A \frac{\partial \mathcal{A}^\alpha_A}{\partial U^i};$$

consequently, equation (A.6) holds also for $i > n$.
This equation can be considered as a system of $4N$ equations for the determination of the $4n$ unknowns $\omega^{I\alpha}$. We have just proved that this system has a solution; it is also expressed in covariant form, because this property is satisfied by system (A.6).
Let us consider equation (A.6), written with $i = B'$, and let us take its derivative with respect to U^B; we obtain

$$\frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}^\alpha_A}{\partial U^{B'}} = \frac{\partial^2 h^\alpha}{\partial U^B \, \partial U^{B'}} - \lambda^A \frac{\partial^2 \mathcal{A}^\alpha_A}{\partial U^B \, \partial U^{B'}} - \\ - \omega^{I\alpha} \frac{\partial^2 \Phi_I}{\partial U^B \, \partial U^{B'}} - \frac{\partial \omega^{I\alpha}}{\partial U^B} \frac{\partial \Phi_I}{\partial U^{B'}};$$

consequently, we have that

$$X_C^B \frac{\partial \lambda^A}{\partial U^B} \frac{\partial \mathcal{A}_A^\alpha}{\partial U^{B'}} X_{C'}^{B'}$$

is symmetric , because

$$\frac{\partial \Phi_I}{\partial U^{B'}} X_{C'}^{B'} = 0.$$

From this result, it follows that the matrices $\mathcal{A}_{CC'}$ and $\mathcal{B}_{CC'}$ in equation (4.12), are also symmetric.

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