THE ASYMPTOTIC REPRESENTATION OF THE FUNDAMENTAL MATRIX OF A DISCRETE SYSTEM

RAÚL NAULIN

In this paper by using the notion of discrete dichotomies, an asymptotic representation of $\Phi$, the fundamental matrix of the linear difference equation $y(n + 1) = (A(n) + B(n))y(n)$ is given.

1. Introduction.

An important problem in the theory of difference equations [1] is the description of the solutions of the perturbed system

$\hspace{1cm}$ (1) $y(n + 1) = (A(n) + B(n))y(n), \hspace{1cm} n \in \mathbb{N},$

where the solutions of the system

$\hspace{1cm}$ (2) $x(n + 1) = A(n)x(n), \hspace{1cm} n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\},$

or equivalently, the fundamental matrix of (2)

$\hspace{1cm}$ $\Phi(n) = \prod_{m=0}^{n-1} A(m) = A(n - 1) \ldots A(2)A(1), \hspace{1cm} \Phi(0) = I,$

Entrato in Redazione il 9 marzo 1999.

Key words and phrases: Difference equations, Asymptotic integration of nondiagonal systems.

1991 AMS subject classification: Primary 39A11, 39A10
is assumed to be known. Among the existing methods in solving this problem
[2], [3], [6] we emphasize the theory of asymptotic integration, developed
by Levinson for diagonal systems of ordinary differential equations [5], and
laterly adapted to difference equations by Devinatz and Benzaid-Lytz [2]. This
method applies to (1), where $A(n) = \text{diag}(\lambda_1(n), \lambda_2(n), \ldots, \lambda_r(n))$, when
the coefficients of these diagonal matrices satisfy the so termed Levinson
dichotomic conditions (2). A second method relies on the notion of asymptotic
under the assumption that (2) has some discrete dichotomy [10], [11].

**Definition 1.** [7], [11]. Let $(h, k)$ be a pair of sequences of positive numbers.
We shall say that (1) has a discrete $(h, k)$-dichotomy if there exists an orthog-
onal projection $P$ and a positive constant $K \geq 1$ such that

\[
|\Phi(n)P(\Phi^{-1}(m)| \leq Kh(n)h(m)^{-1}, \quad n \geq m, \quad (3)
\]

\[
|\Phi(n)(I - O)\Phi^{-1}(m)| \leq Kk(n)k(m)^{-1}, \quad m \geq n.
\]

The case when $h = k$, is sample called an $h$-dichotomy.

If (2) yields a discrete dichotomy, and the sequence $\{B(n)\}$ is summable,
then the $h$-bounded solutions (respectively the $k$-bounded solutions) of (1) and
(2) are biunivocal correspondence. This correspondence is obtained by means
of the solutions of the integral equation

\[
y(n) = x(n) + \sum_{m=n_0}^{n-1} \phi(n)P\phi^{-1}(m)b(m)
\]

\[
- \sum_{m=n}^{\infty} \phi(n)(I - P)\phi^{-1}(m)B(m).
\]

The main result on the asymptotic integration of $h$-bounded solutions estab-
lishes that if solutions $\{y(n)\}, \{x(n)\}$ of (1), (2) respectively are in correspon-
dence, then

\[
y(n) = x(n) + \rho_h(n),
\]

where $\rho_h$ denotes a sequence satisfying

\[
\lim_{n \to \infty} h(n)^{-1}\rho_h(n) = 0, \quad h(n)^{-1} := 1/h(n).
\]

In this paper we will extend this result in the following respect: we
will assume that (2) has a family of $(h_i, k_i)$-dichotomies. Then, for $\Psi$, the
fundamental matrix of (1), we will obtain the asymptotic formula
\[
\Psi(n)E = \sum_{i=1}^{p} ((\Phi(n) + \rho_{h}(n))R_{i} + (\Phi(n) + \rho_{k}(n))S_{i}),
\]
where \( E \) is an invertible matrix, \( R_{i} \) and \( S_{i} \) are orthogonal projections satisfying \( \sum_{i=1}^{p}(R_{i} + S_{i}) = I \).

2. Preliminaries.

In what follows \( \mathcal{V} \) will denote the vector space \( \mathbb{R}^{r} \) or \( \mathbb{C}^{r} \), where a norm \( |\cdot| \) is defined. If \( A \) is \( r \times r \) matrix, then \( |A| \) will denote the corresponding operator norm. By \( x(n, \xi) \) and \( y(n, \xi) \) we will denote the solutions of (2) and (1) satisfying \( x(0, \xi) = y(0, \xi) = \xi \).

**Definition 2.** We shall say that the ordered pair of sequences \((h, k)\) are compensated iff \( h(n)k(m) \leq Ck(n)h(m), n \geq m \) for some constant \( C \).

**Definition 3.** We shall denote by \( \ell_{h}^{\infty} \) and \( \ell_{h}^{1} \) the following sequential spaces
\[
x \in \ell_{h}^{\infty} \text{ iff } |x|_{h}^{\infty} = \sup\{|h(n)^{-1}x(n)| : n \in \mathbb{N}\} < \infty.
\]
\[
x \in \ell_{h}^{1} \text{ iff } |x|_{h}^{1} := \sum_{n=0}^{\infty} |h(n)^{-1}x(n)| < \infty.
\]

The elements of the space \( \ell_{h}^{\infty} \) will be called \( h \)-bounded sequences. Further, we define the subspaces of initial conditions \( V_{h} = \{ \xi \in \mathcal{V} : x(\cdot, \xi) \in \ell_{h}^{\infty}\}, \)
\( V_{h,0} = \{ \xi \in V_{h} : \lim_{n \to \infty} h(n)^{-1}x(n, \xi) = 0 \} \). \( \square_{h} \) and \( Q_{h,0} \) will denote projection matrices such that \( \square_{h}[\mathcal{V}^{r} = V_{h}, \ Q_{h,0}[\mathcal{V}^{r} = V_{h,0} \). Similar subspaces and projections defined for Eq. (1) will be distinguished by a tilde: \( \tilde{V}_{h}, \tilde{V}_{h,0}, \tilde{Q}_{h}, \) etc.

In our paper we will deal with the adjoint system:
\[
z(n+1) = A_{*}(n)z(n),
\]
where \( A_{*} = (C^{-1})^{T} \) is the transpose of the complex conjugate of the inverse matrix \( C^{-1} \). It is clear that \( A_{*} \) is the fundamental matrix of (4). If (2) has an \((h, k)\)-dichotomy with an orthogonal projection \( P \), then (4) has a \((k^{-1}, h^{-1})\)-dichotomy with projection \( I - P \):
\[
|\phi_{*}(n)(I - P)\Phi_{*}^{-1}(m)| \leq Kk(n)^{-1}(n)k(m), n \geq m,
\]
\[
|\Phi_{*}(n)P\Phi_{*}^{-1}(m)| \leq kh(n)^{-1}h(m), quadm \geq n. \tag{5}
\]
Its clear that if the pair \((h, k)\) is compensated, then so is the pair \((k^{-1}, h^{-1})\).

The proof of the following theorem is contained in [7].
Theorem A. If (2) has an \((h, k)\)-dichotomy with projection matrix \(P\), then (1) has an \((h, k)\)-dichotomy with projection matrix \(Q\) iff

\[
V_{h,0} \subset V_{k,0} \subset Q[\mathbb{V}^r] \subset V_h \subset V_k.
\]

The projection \(P\) of dichotomy (3) can be chosen with the property

\[
\lim_{n \to \infty} h(n)^{-1}\Phi(n)P = 0
\]

iff \(V_{h,0} = V_{k,0}\).

Theorem A can be applied to (4). Consequently, if (4) has the dichotomy (5), then it has a \((k^{-1}, h^{-1})\)-dichotomy with projection \(q\) iff

\[
V_{h^{-1},0}^* \subset V_{k^{-1},0}^* \subset Q[\mathbb{V}^r] \subset V_{h^{-1}}^* \subset V_{k^{-1}}^*.
\]

where \(V_{h^{-1}}^*\) is the subspace of initial conditions of (4) of \(h^{-1}\)-bounded solutions etc.

The following result was proved in [7].

Theorem B. Let us suppose (2) has an \((h, k)\)-dichotomy with projection \(P\). If \(||A(n)^{-1}|||B(n)|| \in \ell^1\), then (1) has an \((h, k)\)-dichotomy

\[
|\widetilde{\Phi}(n)\widetilde{P}\Phi^{-1}(m)| \leq \tilde{K}h(n)h(m)^{-1}, \quad n \geq m,
\]

\[
|\widetilde{\Phi}(n)(I - \widetilde{P})\Phi^{-1}(m)| \leq \tilde{K}k(n)k(m)^{-1}, \quad m \geq n,
\]

where \(\tilde{P}\) is a projection similar to projection \(P\).

3. \(h\)-bounded solutions.

Before we go ahead, we will establish some correspondence between the \(h\)-bounded solutions of (2) with those of its adjoint (4).

Lemma 1. Assume that (2) has an \(h\)-dichotomy. If for some subsequence \(\{n_j\}\) one has \(\lim_{j \to \infty} h(n_j)^{-1}x(n_j, \xi) = 0\), then \(\lim_{n \to \infty} h(n)^{-1}x(n, \xi) = 0\).
Proof. Following Proposition 2.2 in [4], let us denote \( x(n, \xi) = x_1(n) + x_2(n) \), \( x_1(n) = P\Phi(n)\xi \), \( x_2(n) = (I - P)\Phi(n)\xi \). It is easy to verify that

\[
|h(n)^{-1}x_1(n)| \leq K|h(n_j)^{-1}x(n_j)| \quad n > n_j,
\]

\[
|h(n)^{-1}x_2(n)| \leq K|(n_j)^{-1}x(n_j)|, \quad n < n_j,
\]

from where the proof of the lemma follows. □

Lemma 2. If (2) has an \( h \)-dichotomy, then

\[
V_{h^{-1}}^* = (I - Q_h)[V^r], \quad V_{h^{-1}}^{n,0} = (I - Q_h)[V].
\]

Proof. The existence of an \( h \)-dichotomy for (2) with projection \( P \) implies the existence of two \( h \)-dichotomy for this system, respectively with projection \( Q_h \) and \( Q_{h,0} \). Without loss of generality we can assume that \( Q_{h,0} \) and \( Q_h \) have the diagonal forms \( Q_{h,0} = \text{diag}\{I_0, 0, 0\} \), \( Q_h = \text{diag}\{I_0, I_1, 0\} \), where \( I_0 \) is a unit matrix of dimensions \( r_0 \times r_0 \), \( r_0 = \dim[V_{h,0}] \), and \( I_1 \) is a unit matrix of dimensions \( r_1 \times r_1 \), such that \( r_0 + r_1 = \dim[V_h] \). According to Lemma 1 in [9] (see also Lemma 5.2 in [4]), there exists a bounded sequence \( S : \mathbb{N} \to \mathbb{V}^{r+r} \), such that \( S^{-1} : \mathbb{N} \to \mathbb{V}^{r+r} \) exists, is bounded, and the change of variables \( x(n) = S(n)w(n) \) reduces (2) to the form

\[
\begin{pmatrix}
  w_0(n+1) \\
  w_1(n+1) \\
  w_{\infty}(n+1)
\end{pmatrix} =
\begin{pmatrix}
  C^0(n) & 0 & 0 \\
  0 & C^1(n) & 0 \\
  0 & 0 & C^\infty(n)
\end{pmatrix}
\begin{pmatrix}
  w_0(n) \\
  w_1(n) \\
  w_{\infty}(n)
\end{pmatrix}
\]

where

\[
w_0(n) \in V^{n_0}, \quad C^0(n) \in V^{n_0 \times n_0}, \quad w_1(n) \in V^{n_1}, \quad C^1(n) \in V^{n_1 \times n_1},
\]

\[
w_{\infty}(n) \in V^{\infty}, \quad C^\infty(n) \in V^{\infty \times \infty}, \quad r_\infty := r - (r_1 + r_0).
\]

By a straightforward calculation, we may verify that the change of variables \( z(n) = S_e(n)u(n) \) reduces the adjoint equation of (10) to the diagonal form

\[
\begin{pmatrix}
  u_0(n+1) \\
  u_1(n+1) \\
  u_{\infty}(n+1)
\end{pmatrix} =
\begin{pmatrix}
  C^0_e(n) & 0 & 0 \\
  0 & C^1_e(n) & 0 \\
  0 & 0 & C^\infty_e(n)
\end{pmatrix}
\begin{pmatrix}
  u_0(n) \\
  u_1(n) \\
  u_{\infty}(n)
\end{pmatrix}
\]

For (10) we have \( V_{h,0} = Q_{h,0}[V^r], \quad V_h = Q_h[V^r] \). We may write \( \Theta \), the fundamental matrix of (10), in the form

\[
\Theta(n) = \text{diag}\{U_0(n), U_1(n), U_{\infty}(n)\},
\]
where \( U_0(n) \in V^{n \times n_0}, U_1(n) \in V^{r_1 \times r_1} \) and \( U_\infty(n) \in V^{r_\infty \times r_\infty} \).

Regarding (11), let us consider the direct sum \( V^r = V^*_{h^{-1}} \oplus W^*_{h^{-1}} \), where \( V^*_{h^{-1}} \) is a complementary subspace of \( V^*_{h^{-1}} \). For the initial condition \( \xi = \text{column}(\xi_1, 0, 0) \), we have \( \lim_{n \to \infty} h(n)^{-1} \Theta(n) \xi = 0 \), and \( \{ \Theta(n) \xi \} \) is \( h^{-1} \)-bounded if \( \xi = \text{column}(\xi_1, \xi_2, 0) \). From these properties

\[
\lim_{n \to \infty} |h(n) \Theta_+(n) \xi| = \infty, \quad \text{if } \xi_2 = 0, \xi_3 = 0, \xi_1 \neq 0
\]

follows. Otherwise, the boundedness of some subsequence \( \Theta_+(n_j) \xi \) would lead to the contradictory equation \( |\xi|^2 = < h(n_j)^{-1} \Phi(n_j) \xi, h(n_j) \Phi(n_j) \xi > = 0 \), where \( < x, y > = \sum_{i=1}^{r} x_i y_i \). Thus, we have proven

(12)
\[ V_{h,0} \subset W^*_{h^{-1}}. \]

Further, for some constants \( K, M \), one satisfies \( K \geq |h(n)^{-1} U_1(n)| \geq M > 0 \), \( \forall n \in \mathbb{N} \). This implies

(13)
\[ M^{-1} \geq |h(n) U_1(n)| \geq K^{-1}, \quad \forall n \in \mathbb{N}. \]

The boundedness of \( \{ h(n) U_1(n) \} \) and \( \{ h(n) U_{\infty}(n) \} \) (the boundedness of this last sequence follows from the fact that the adjoint (11) has an \( h^{-1} \)-dichotomy with projection \( I - Q_h \) implies

(14)
\[ r_1 + r_\infty \leq \dim V^*_{h^{-1}}. \]

Since \( \dim V_{h,0} = r_0 \), from (12) and (14) we obtain \( r_0 = \dim W^*_{h^{-1}} \). Therefore

(15)
\[ V_{h,0} = W^*_{h^{-1}}. \]

From (14) and (15) we obtain \( (I - Q_{h,0})[V^r] = V^*_{h^{-1}}. \)

Let us prove that

(16)
\[
\lim_{n \to \infty} |(n) \Theta_+(n) \xi| = 0, \quad \text{if } \xi_1 = 0, \xi_2 = 0.
\]

Assuming the contrary, from Lemma 1, we would have for all values of \( n \) the estimate \( K \geq |h(n) U_{\infty}(n)| \geq M > 0 \), for some constant \( M \). This implies the \( h^{-1} \)-boundedness of the sequence \( \{ U_{\infty}(n) \} \). But this contradicts

\[
\lim_{n \to \infty} |\Theta(n) \xi| = \infty, \quad \text{if } \xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0.
\]

From the assertion (16) we have \( (I - Q_{h,0})[V^r] \subset V^*_{h^{-1},0} \). This content and (13) imply \( |I - Q_{h,0}|[V^r] \subset V^*_{h^{-1}} \). From (15) we obtain \( \dim (I - Q_{h,0})[V^r] = \dim V^*_{h^{-1}} \). Therefore \( |I - Q_{h,0}|[V^r] = V^*_{h^{-1}} \). \( \square \)

**Definition 4.** The dichotomy (3) is said to be exhaustive if \( V_k = V \), and precise if \( V_{h,0} = \{0\} \).
From Lemma 2 it follows

**Theorem 1.**

A: If (2) has an h-dichotomy, then

\[ Q^*_h[V] = (I - Q_{h,0})[V'], \quad Q^*_{h-1,0}[V'] = (I - Q_h)[V']. \]

B: If (3) is compensated, then the dichotomy (3) is exhaustive iff \( V^*_k,0 = [0] \).

C: If (3) is compensated, then the dichotomy (3) is precise iff \( V^*_h,1 = V' \).

4. Asymptotic formulae.

If (2) has the \((h, k)\)-dichotomy (3) and this dichotomy is compensated, then according to Theorem A the projections \( Q_h, Q_{h,0}, Q_k \) satisfy

\[ Q_h Q_{h,0} = Q_{h,0}, \quad Q_k Q_h = Q_k Q_k = Q_k. \]

We recall that the notation \( \rho_h \) will indicate a sequence with the property

\[ \lim_{h \to \infty} h(n)^{-1} \rho_h(n) = 0. \]

4.1. An asymptotic formula to the h-bounded solutions.

**Theorem 2.** Let us assume that (2) has an h-dichotomy. If

\[ \sum_{m=0}^{\infty} |A(m)^{-1}| |B(m)| < 1, \]

then the h-bounded solutions of (2) and the h-bounded solutions of (1) are in biunivocal correspondence, satisfying

\[ y(n) = x(n) + \rho_h(n). \]

The fundamental matrix \( \Psi \) of (1), \( \Psi(0) = I \), satisfies

\[ \Psi(n) \tilde{Q}_h = \Phi(n)Q_h + \rho_h Q_h. \]
Proof. According to Theorem B, the assumed $h$-dichotomy of (2) can be accomplished with projection $Q_{h,0}$ satisfying (7). Given $\{x(n)\}$, an $h$-bounded solution of (2), we consider the integral equation

$$\begin{align*}
y(n) &= x(n) + \sum_{m=0}^{n} \Phi(n)Q_{h,0}\Phi^{-1}(m + 1)B(m)y(m) \\
&\quad - \sum_{m=n+1}^{\infty} \Phi(n)(I - Q_{h,0})\Phi^{-1}(m + 1)B(m)y(m).
\end{align*}$$

Then, following [7] it is possible to prove that (22) has a unique $h$-bounded solution satisfying Eq. (1) and property (20). If we put $n = 0$ in (22), then

$$y(0) = x(0) - \sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m + 1)B(m)\Psi(m)y(0).$$

The estimate (9) implies $|\Psi(n)\tilde{Q}_h| \leq \tilde{K}h(n)$, for some constant $K$. Henceforth

$$|\sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m + 1)B(m)\Psi(m)\tilde{Q}_h| \leq k\tilde{K}\sum_{m=0}^{\infty} |A(m)|^{-1} ||B(m)|| < 1.$$

From this estimate we obtain $y(0) = \Theta_h x(0)$, where

$$\Theta_h = (I + \sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m + 1)B(m)\Psi(m)\tilde{Q}_h)^{-1}.$$

The (22) implies

$$\Psi(n)\tilde{Q}_h y(0) = \Phi(n)Q_{h}x(0) + \rho_h(n)x(0),$$

where

$$\rho_h(n) = \sum_{m=0}^{n-1} \Phi(n)Q_{h,0}\Phi^{-1}(m + 1)B(m)\Psi(m)\Theta_h q_h$$

$$- \sum_{m=n+1}^{\infty} \Phi(n)(I - Q_{h,0})\Phi^{-1}(m + 1)B(m)\Psi(m)\Theta - hQ_h.$$

From (7) and (24) it follows that $\rho_h$ satisfies the property (18).
4.2. An asymptotic formula to the \( k \)-bounded solutions.

Let us now assume that (2) has an \((h, k)\)-dichotomy. Since the pair of functions \((h, k)\) is compensated, then (2) has both an \( h \) and a \( k \)-dichotomy. If (19) is satisfied, Theorem 2 can be applied to this \( k \)-dichotomy, and therefore the \( k \)-bounded solutions of (2) and (1) are in biunivocal correspondence with \( k \)-bounded solutions of (1); this correspondence is obtained by the \( k \)-bounded solution of the integral equation

\[
y(n) = x(n) + \sum_{m=0}^{n} \Phi(n) Q_{k,0}^{-1}(m + 1) B(m) y(m) - \sum_{m=n}^{\infty} \Phi(n)(I - Q_{k,0}) \Phi^{-1}(m + 1) B(m) y(m),
\]

from where it follows the asymptotic formula

\[
y(n) = x(n) + \rho_k(n), \tag{25}
\]

where

\[
R_k(n) = \sum_{m=0}^{n-1} \Phi(n - 1) Q_{k,0}^{-1}(m + 1) B(m) \Psi(m) \Theta_k Q_k - \sum_{m=n}^{\infty} \Phi(n)(I - Q_{k,0}) \Phi^{-1}(m + 1) B(m) \Psi(m) \Theta_k Q_k,
\]

and

\[
\Theta_k = \left\{ I + \sum_{m=0}^{\infty} (I - Q_{k,0}) \Phi^{-1}(m + 1) B(m) \Psi(m) \tilde{Q}_k \right\}^{-1}.
\]

The asymptotic formula (25) yields the following asymptotic correspondence

\[
\Psi(n) \tilde{Q}_k = \Phi(n) Q_k + \rho_k(n) Q_k, \tag{26}
\]
4.3. An asymptotic formula to the fundamental matrix.

Let us denote $W_k = (Q_k - Q_h)[V']$ and $S_k = Q_k - Q_h$. From (17) we obtain $V_k = V_h \oplus W_k$. From (18) we point out to the important property

$$(27) \quad S_k Q_k = S_k = Q_k S_k.$$

**Theorem 3.** If (19) is satisfied and (2) has an $(h, k)$-dichotomy, with a compensated pair $(h, k)$, then the fundamental matrix $\Psi(n)$ of (1) satisfies

$$(28) \quad \Psi(n)(\tilde{Q}_h Q_h + \tilde{Q}_k S_k) = (\Phi(n) + \rho_h(n)) Q_h + (\Phi(n) + \rho_k(n)) S_k.$$ 

Moreover, if the $(h, k)$-dichotomy is exhaustive, then the matrix $(\tilde{Q}_h Q_h + \tilde{Q}_k S_k)$ is invertible.

**Proof.** The first part of the theorem is obtained by adding (21) and (26), previously being multiplied by $Q_h$ and $S_k$.

Let us denote $E = \tilde{Q}_h Q_h + \tilde{Q}_k S_k$. Let $E \xi = 0$. From (27) we may write the decomposition $\xi = \xi_1 + \xi_2$, $\xi_1 \in Q_h[V']$, $\xi_2 \in (Q_k - Q_h)[V']$. Therefore

$$0 = \Phi(n)\xi_1 + R_h(n)\xi_1 + \Phi(n)\xi_2 + R_k(n)\xi_2.$$ 

This identity shows that $\xi_2 \in V_{k,0}$. Theorem A implies $\xi_2 \in V_h$. Thus $\xi_2 = 0$. Since $\Phi(n)\xi_1 + R_h(n)\xi_1 = 0$ is an $h$-bounded solution of Eq. (1) and the $h$-bounded solutions of (1), (2) are in biunivocal correspondence given by the nonsingular matrix (23). Therefore $\xi_1 = 0$ implies that $E$ is invertible. \hfill \Box

It is clear that if the $(h, k)$-dichotomy is exhaustive and $\tilde{Q}_h = Q_h$, $\tilde{Q}_k = Q_k$, then

$$\tilde{Q}_h Q_h + \tilde{Q}_k S_k = I.$$

4.4. An asymptotic formula to the inverse matrix.

Condition (19) can be replaced by

$$(29) \quad K \tilde{K} \sum_{m=n_0}^{\infty} |A(m)^{-1}| |B(m)| < \infty,$$

implying

$$K \tilde{K} \sum_{m=n_0}^{\infty} |A(m)^{-1}| |B(m)| < 1,$$
for an $n_0$ sufficiently large. It is clear that the results of the previous section will remain valid on the interval $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \ldots\}$.

In this section we wish to find an asymptotic formula to the inverse of the fundamental matrix $\Psi$.

We may write $(A(n) + B(n))^{-1} = A^{-1}(n) + \tilde{B}(n)$, where we have defined $\tilde{B}(n) = (A(n) + B(n))^{-1} - A^{-1}(n)$. Condition (29) implies for a large $n_0$:

$$K \tilde{K} \sum_{m=n_0}^{\infty} |A(m)||\tilde{B}(n)| < \infty.$$ 

Let assume that (2) allows a compensated $(h, k)$-dichotomy. Then (4) has the compensated $(k^{-1}, h^{-1})$-dichotomy (5). If (29) is fulfilled, then from Theorem 3 and $\Psi(n)^{-1} = \Psi_s^{-1}$ we obtain the following asymptotic equation for $\Psi^{-1}$

$$(Q_{k^{-1}}^* \tilde{Q}_{k^{-1}}^* + S_{h^{-1}}^* \tilde{Q}_{h^{-1}}^*)\Psi^{-1}(n) =$$

$$(Q_{k^{-1}}^* (\Phi^{-1}(n) + \rho_{k^{-1}}(n)) + S_{h^{-1}}^* (\Phi^{-1}(n) + \rho_{h^{-1}}(n)),$$

where $S_{h^{-1}}^* = Q_{h^{-1}}^* - Q_{h^{-1}}^*$, this equation can be simplified if we assume that dichotomy (5) is exhaustive. According to Theorem 1, this condition is accomplished if $Q_{h, 0}^* = 0$, that is, if the dichotomy (3) is precise. In such a case applying Theorem 1 we have $Q_{k^{-1}}^* = I - Q_{k, 0}^*$, $Q_{h^{-1}}^* = I - \tilde{Q}_{k, 0}^*$, $\tilde{Q}_{h^{-1}}^* = I$, from where

$$Q_{k^{-1}}^* \tilde{Q}_{k^{-1}}^* + S_{h^{-1}}^* \tilde{Q}_{h^{-1}}^* = (I - Q_{k, 0}^*)(I - \tilde{Q}_{k, 0}^*) + Q_{k, 0}^*.$$

In this way we have proven the following

**Lemma 3.** If the compensated dichotomy (3) is precise then under condition (29) the fundamental matrix $\Psi$ satisfies

$$((I - Q_{k, 0}^*)(I - \tilde{Q}_{k, 0}^*) + Q_{k, 0}^*)\Psi^{-1}(n) =$$

$$(Q_{k^{-1}}^* (\Phi^{-1}(n) + \rho_{k^{-1}}(n)) + S_{h^{-1}}^* (\Phi^{-1}(n) + \rho_{h^{-1}}(n)),$$

where the matrix $(I - Q_{k, 0}^*)(I - \tilde{Q}_{k, 0}^*) + Q_{k, 0}^*$ is invertible.

**Theorem 4.** Let us assume that the discrete dichotomy (3) is compensated, exhaustive and precise. Moreover, let us assume that $\tilde{Q}_h = Q_h$, $\tilde{Q}_k = Q_k$, $\tilde{Q}_{k, 0} = Q_{k, 0}$, $\tilde{Q}_{h, 0} = Q_{h, 0}$. If condition (29) is satisfied, then the fundamental matrix $\psi$ of (1) and its inverse have the following asymptotic representation

$$\Psi(n) = (\Phi(n) + \rho_{k}(n))Q_h + (\Phi(n) + \rho_{k}(n))(I - Q_h),$$

$$\Psi^{-1}(n) = (I - Q_{k, 0}^*)(\Phi^{-1}(n) + \rho_{k^{-1}}(n)) + Q_{k, 0}^*(\Phi^{-1}(n) + \rho_{h^{-1}}(n)),$$
From this theorem it follows an asymptotic formula for the Cauchy matrix of \((1)\).

**Theorem 5.** Under conditions of Theorem 4, the assumption \(Q_{k,0} = Q_h\) implies the asymptotic representation
\[
\Psi(n)\Psi^{-1}(m) = (\Phi(n) + R_h(n))Q_h(\Phi^{-1}(m) + \rho_{h^{-1}}(m))
+ (\Phi(n) + \rho_h(n))(I - Q_h)(\Phi^{-1}(m) + R_{h^{-1}}(m)), \forall m, n.
\]

5. Dichotomic chains.

Let us consider two ordered sets of positive continuous functions
\[
\mathcal{H} = \{h_1, h_2, \ldots, h_p\}, \quad \mathcal{K} = \{k_1, k_2, \ldots, k_p\},
\]
and a collection of projections matrices
\[
\mathcal{P} = \{P_1, P_2, \ldots, P_p\}.
\]

**Definition 5.** We shall say that the triplet \((\mathcal{H}, \mathcal{K}, \mathcal{P})\) is a dichotomic chain for \((2)\) if

\begin{enumerate}
\item[(L 1)] For \(j = 1, \ldots, r\), \((2)\) has a dichotomy \((h_j, k_j, P_j)\).
\item[(L 2)] \(V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_p}\).
\end{enumerate}

We will employ the abbreviation \((\mathcal{H}, \mathcal{P}) = (\mathcal{H}, \mathcal{H}, \mathcal{P})\). In applications, a convenient algebraic condition implying (L2) is given by

\begin{enumerate}
\item[(L 2')] For some constant \(D\) we have \(k_j(n) \leq Dk_{j+1}(n), \quad j = 1, 2, \ldots, r - 1\).
\end{enumerate}

A more stringent condition than (L 2’) is the uniform condition

\begin{enumerate}
\item[(L 2'')] For some constant \(D\) we have
\[
k_j(n)k_j(m)^{-1} \leq Dk_{j+1}(n)k_{j+1}(m)^{-1}, \quad j = 1, 2, \ldots, r - 1, \quad n \geq m.
\]
\end{enumerate}

**Theorem 6.** Let us assume that \((2)\) has the dichotomic chain \((\mathcal{H}, \mathcal{K}, \mathcal{P})\), then \((2)\) has a dichotomic chain \((\mathcal{H}, \mathcal{K}, \mathcal{P}')\), where the projections \(\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_p\}\) are respectively similar to the projections of the collection \(\mathcal{P} = \{P_1, P_2, \ldots, P_p\}\).
Proof. Applying Theorem B to each \((h_i, k_i)-dichotomy\) of \((\mathcal{H}, \mathcal{K}, \mathcal{P})\) we obtain an \((h_i, k_i)-dichotomy\) for \((2)\) with a projection \(P'_i\) similar to projection \(P_i\). In \([8]\) it is proven that for any \((h, k)-dichotomy\) of System \((2)\) we have the properties dimension \([V_h] = \dim [\tilde{V}_h]\), dimension \([V_k] = \dim [\tilde{V}_k]\). Hence, from \((L2)\) we obtain \(\tilde{V}_h \subset \tilde{V}_k \subset \cdots \subset \tilde{V}_{k_p}\). If \(\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_p\}\), then \((\mathcal{H}, \mathcal{K}, \mathcal{P}')\) is required dichotomic chain. \(\Box\)

**Remark 1.** In what follows we will assume that all \((h, k)-dichotomies\) of the dichotomic chain \((\mathcal{H}, \mathcal{K}, \mathcal{P})\) (respectively \((\mathcal{H}, \mathcal{K}, \mathcal{P}')\)) are defined with a same constant \(K\) (respectively \(\tilde{K}\)).

Assume that \((2)\) has the dichotomic chain \((\mathcal{H}, \mathcal{K}, \mathcal{P})\). We will perform the following construction: Let us define \(U_{h_1} = V_{h_1}\). Further, if \(V_h = V_{k_i}\) we define \(W_{k_i} = \{0\}\). If \(V_{h_1}\) is properly contained in \(V_{h_1}\), then we define \(W_{k_i}\) as a complementary subspace to \(V_{h_1}\) in the space \(V_{k_i}\). In both cases we can write the disjoint sum \(V_{k_i} = \{0\} + U_{h_1} + W_{k_i}\), thus in the space \(U_{h_1}\) we keep all the initial conditions corresponding to the \(h_1\)-bounded solutions of \((2)\). To the space \(W_{k_i}\) we assign the initial conditions of \(k_1\)-bounded solutions that are not \(h_1\)-bounded. We repeat this process for the space \(V_{k_2}\), in the following manner: If \(V_{k_1} = V_{k_1}\), we define \(U_{h_2} = W_{k_2} = \{0\}\). If \(V_{k_2}\) is properly contained in \(V_{k_2}\), then we define \(U_{h_2}\) as the subspace of the initial condition of the \(h_2\)-bounded solutions not contained in \(V_{h_2}\), and the subspace \(W_{k_2}\) groups the initial conditions of \(k_2\)-solutions not included in \(U_{h_2}\); therefore \(V_{k_2}\) can be written as a disjoint sum \(V_{k_2} = V_{k_2} + U_{h_2} + W_{k_2}\). Carrying out this process further, we obtain the decomposition:

\[
\begin{align*}
V_{k_1} &= \{0\} + U_{h_1} + W_{k_1} \\
V_{k_2} &= V_{k_2} + U_{h_2} + W_{k_2} \\
&\vdots\quad\vdots\quad\vdots\quad\vdots \\
V_{k_p} &= V_{k_{p-1}} + U_{h_p} + W_{k_p}
\end{align*}
\]

(30)

In applications the table (30) does not give a good decomposition of the subspaces of initial conditions corresponding to solutions with different growths; for example if \(k_1 = k_2 = \cdots = k_p\), all subspaces of table (30) would be trivial, except \(U_{h_1}\) and maybe \(W_{k_i}\). This situation can be improved by asking from the dichotomic chain the property defined as follows.

**Definition 6.** We shall say that the LD \((\mathcal{H}, k, \mathcal{P})\) is a stratified iff

\[U_{h_1} \subset V_{k_1} \subset U_{h_2} \subset V_{k_2} \subset \ldots \subset U_{h_p} \subset V_{k_p}\]
This property is given if for some constant $D$ we have $k_j(t) \leq Dh_{j+1}(t)$, $j = 1, 2, \ldots, r - 1$. We emphasize that the dichotomic chain $(\mathcal{H}, \mathcal{P})$ is stratified.

6. Asymptotic integration.

In this section we generalize the asymptotic formula (28) under the existence of a dichotomic chain for (2). Let us consider a chain $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. According to the table (30), we define the projections matrices $R_j, S_j$ such that $R_j[\mathbb{V}'], S_j[\mathbb{V}'] = W_{j_i}$. From the construction of subspaces $U_{h_j}$ and $V_{k_j}$ we have $R_j R_i = 0, S_j S_i = 0$, if $i \neq j$, $R_j S_i = 0$ for all indexes $i, j$. Moreover, since the range of projections $R_j$ and $S_j$ are respectively contained in $V_{h_j}$ and $V_{k_j}$, we have the identities

\begin{equation}
Q_{h_j} R_j = R_j, \quad Q_{k_j} S_j = S_j.
\end{equation}

Theorem 7. Let us assume that (2) has the dichotomic chain $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ and condition (19) is satisfied (see Remark 1), then the fundamental matrix $\Psi$ of (2), $\Psi(t_0) = I$, has the property

\begin{equation}
\Psi(n) E = \sum_{j=1}^{r} (\Phi(n) + \rho h_j(n)) R_j + \sum_{j=1}^{r} (\Phi(n) + \rho k_j(n)) S_j,
\end{equation}

where some of projections $R_j$ or $S_j$ in (32) could be equal zero, and $E$ is defined by

\begin{equation}
E = \sum_{j=1}^{p} (\bar{Q}_{h_{j-1}} R_j + \bar{Q}_{k_{j-1}} S_j)
\end{equation}

Proof. Applying Theorem 2 to each $(h_j, k_j)$-dichotomy we obtain from (21), (26) the decompositions $\Psi(n) \bar{Q}_{h_j} = (\Phi(n) + \rho h_j(n)) Q_{h_j}$ and $\Psi(n) \bar{Q}_{k_j} = (\Phi(n) + \rho k(n)) Q_{k_j}$. Respectively multiplying each of these formulas by $R_j$ and $S_j$ and using (31) we obtain $\Psi(n) \bar{Q}_{h_j} R_j = (\Phi(n) + o(h_j(n))) R_j$ and $\Psi(n) \bar{Q}_{k_j} S_j = (\Phi(n) + o(k(n))) S_j$. From these formulas it follows (32). \quad \square

Definition 7. The chain $(\mathcal{H}, k, \mathcal{P})$ is called exhaustive iff $V_{k_p} = [\mathbb{V}']$. 

For exhaustive chains the projections defined by the table (30) have the property

\[ I = Q_{k_r} = \sum_{j=1}^{p} R_j + \sum_{j=1}^{p} S_j. \]

From this identity, we can establish the following version of the Levinson asymptotic theorem for (1).

**Theorem 8.** Under conditions of Theorem 2, if the chain \((\mathcal{H}, \mathcal{K}, \mathcal{P})\) is exhaustive, then the matrix \(E\) defined by (33) is invertible.

**Proof.** Let \(E \xi = 0\). Then

\[ 0 = \sum_{j=1}^{p} (\Phi(n) + \rho h_j(n)) R_j \xi + \sum_{j=1}^{p} (\Phi(n) + \rho k_j(n)) S_j \xi. \]

From the construction of table (30), we obtain that the solution \(\Phi(n) S_p \xi\) of (2) satisfies \(\Phi(n) S_p \xi = \rho k_r(n)\); therefore \(S_p \xi \in V_{k_r,0}\). Applying Theorem A to the dichotomy \((h_p, k_p, P_p)\) we obtain \(S_p \xi \in V_{h_p,0}\), since \(S_p \xi \in W_{k_r}\) the last row of table (30) says that \(S_p \xi = 0\). Henceforth

\[ 0 = \sum_{j=1}^{p} (\Phi(n) + \rho h_j(n)) R_j \xi + \sum_{j=1}^{p-1} (\Phi(n) + \rho k_j(n)) S_j \xi. \]

The right hand side of this last equation is an \(h_p\)-bounded solution of (2). But under condition (19), the \(h_p\)-bounded solutions of (1), (2) are in biunivocal correspondence. Therefore

\[ 0 = \sum_{j=1}^{r} \Phi(n) R_j \xi + \sum_{j=1}^{r-1} \Phi(r) S_j \xi + \Phi(n) R_p \xi. \]

Since \(\sum_{j=1}^{r-1} \Phi(n) R_j \xi + \sum_{j=1}^{r-1} \Phi(n) S_j \xi \in V_{k_{r-1}}\), and \(\Phi(n) R_p \xi \in U_{h_r}\), we obtain from the last row of table (30) \(R_p \xi = 0\). Inasmuch as \(R_p \xi = 0\) and \(S_p \xi = 0\), we obtain from (34)

\[ 0 = \sum_{j=1}^{r-1} (\Phi(n) + \rho h_j(n)) R_j + \sum_{j=1}^{r-1} (\Phi(n) + \rho k_j(n)) S_j. \]

By repeating this reasoning we will obtain \(R_j \xi = 0\), \(S_j \xi = 0\), \(\forall j\) implying \(\xi = \sum_{j=1}^{r} (R_j + S_j) \xi = 0\). Therefore, \(E\) is an invertible matrix. \(\square\)
Dichotomic chains can be used in obtaining an asymptotic decomposition of the inverse matrix $\Psi^{-1}$. This can be accomplished in a similar way to the decompositions obtained in Subsection 4.4.

Acknowledgements. Supported by Proyecto CI-5-025-00730/95.

REFERENCES


Departamento de Matemáticas,
Universidad de Oriente,
Cumaná 6101 A-285 (VENEZUELA)