LE MATEMATICHE
Vol. LIV (1999) – Fasc. I, pp. 123–138

THE ASYMPTOTIC REPRESENTATION OF THE FUNDAMENTAL MATRIX OF A DISCRETE SYSTEM

RAÚL NAULIN

In this paper by using the notion of discrete dichotomies, an asymptotic representation of Φ, the fundamental matrix of the linear difference equation $y(n+1) = (A(n) + B(n))y(n)$ is given.

1. Introduction.

An important problem in the theory of difference equations [1] is the description of the solutions of the perturbed system

$$y(n+1) = (A(n) + B(n))y(n), \quad n \in \mathbb{N}, \tag{1}$$

where the solutions of the system

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}, \tag{2}$$

or equivalently, the fundamental matrix of (2)

$$\Phi(n) = \prod_{m=0}^{n-1} A(m) = A(n-1)\dots A(2)A(1), \quad \Phi(0) = I,$$

Entrato in Redazione il 9 marzo 1999.
Key words and phrases: Difference equations, Asymptotic integration of nondiagonal systems.
1991 AMS subject classification: Primary 39A11, 39A10

is assumed to be known. Among the existing methods in solving this problem [2], [3], [6] we emphasize the theory of asymptotic integration, developed by Levinson for diagonal systems of ordinary differential equations [5], and lately adapted to difference equations by Devinatz and Benzaid-Lytz [2]. This method applies to (1), where $A(n) = \operatorname{diag}\{\lambda_1(n), \lambda_2(n), \dots, \lambda_r(n)\}$, when the coefficients of these diagonal matrices satisfy the so termed Levinson dichotomic conditions (2). A second method relies on the notion of asymptotic equivalence [11] allowing the asymptotic integration of the solutions of (1) under the assumption that (2) has some discrete dichotomy [10], [11].

Definition 1. [7], [11]. *Let (h, k) be a pair of sequences of positive numbers. We shall say that* (1) *has a discrete (h, k)-dichotomy iff there exists an orthogonal projection P and a positive constant $K \geq 1$ such that*

$$
\begin{aligned}
|\Phi(n)P(\Phi^{-1}(m)| &\leq Kh(n)h(m)^{-1}, \quad n \geq m, \\
|\Phi(n)(I - O)\Phi^{-1}(m)| &\leq Kk(n)k(m)^{-1}, \quad m \geq n.
\end{aligned} \tag{3}
$$

The case when $h = k$, is sample called an h-dichotomy.

If (2) yields a discrete dichotomy, and the sequence $\{B(n)\}$ is summable, then the h-bounded solutions (respectively the k-bounded solutions) of (1) and (2) are biunivocal correspondence. This correspondence is obtained by means of the solutions of the integral equation

$$
\begin{aligned}
y(n) = x(n) &+ \sum_{m=n_0}^{n-1} \phi(n)P\phi^{-1}(m)b(m) \\
&- \sum_{m=n}^{\infty} \phi(n)(I - P)\phi^{-1}(m)B(m).
\end{aligned}
$$

The main result on the asymptotic integration of h-bounded solutions establishes that if solutions $\{y(n)\}$, $\{x(n)\}$ of (1), (2) respectively are in correspondence, then

$$y(n) = x(n) + \rho_h(n),$$

where ρ_h denotes a sequence satisfying

$$\lim_{n\to\infty} h(n)^{-1}\rho_h(n) = 0, \quad h(n)^{-1} := 1/h(n).$$

In this paper we will extend this result in the following respect: we will assume that (2) has a family of (h_i, k_i)-dichotomies. Then, for Ψ, the fundamental matrix of (1), we will obtain the asymptotic formula

$$\Psi(n)E = \sum_{i=1}^{p}((\Phi(n) + \rho_{h_i}(n))R_i + (\Phi(n) + \rho_{k_i}(n))S_i),$$

where E is an invertible matrix, R_i and S_i are orthogonal projections satisfying $\sum_{i=1}^{p}(R_i + S_i) = I$.

2. Preliminaries.

In what follows $\mathbb{V}^r$ will denote the vector space $\mathbb{R}^r$ or $\mathbb{C}^r$, where a norm $|\cdot|$ is defined. If A is $r \times r$ matrix, then $|A|$ will denote the corresponding operator norm. By $x(n,\xi)$ and $y(n,\xi)$ we will denote the solutions of (2) and (1) satisfying $x(0,\xi) = y(0,\xi) = \xi$.

Definition 2. *We shall say that the ordered pair of sequences (h,k) are compensated iff $h(n)k(m) \le Ck(n)h(m)$, $n \ge m$ for some constant C.*

Definition 3. *We shall denote by ℓ_h^∞ and ℓ_h^1 the following sequential spaces*

$$x \in \ell_h^\infty \quad \textit{iff} \quad |x|_h^\infty = \sup\{|h(n)^{-1}x(n)| : n \in N\} < \infty.$$

$$x \in \ell_h^1 \quad \textit{iff} \quad |x|_h^1 := \sum_{n=0}^{\infty}|h(n)^{-1}x(n)| < \infty.$$

The elements of the space ℓ_h^∞ will be called h-bounded sequences. Further, we define the subspaces of initial conditions $V_h = \{\xi \in \mathbb{V}^r : x(\cdot,\xi) \in \ell_h^\infty\}$, $V_{h,0} = \{\xi \in V_h : \lim_{n\to\infty} h(n)^{-1}x(n,\xi) = 0\}$. Q_h and $Q_{h,0}$ will denote projection matrices such that $Q_h[\mathbb{V}^r] = V_h$, $Q_{h,0}[\mathbb{V}^r] = V_{h,0}$. Similar subspaces and projections defined for Eq. (1) will be distinguished by a tilde: $\widetilde{V}_h$, $\widetilde{V}_{h,0}$, $\widetilde{Q}_h$, etc.

In our paper we will deal with the adjoint system:

$$z(n+1) = A_*(n)z(n), \tag{4}$$

where $C_* = \overline{(C^{-1})}^t$ is the transpose of the complex conjugate of the inverse matrix C^{-1}. Is clear that ϕ_* is the fundamental matrix of (4). If (2) has an (h,k)-dichotomy with an orthogonal projection P, then (4) has a (k^{-1},h^{-1})-dichotomy with projection $I - P$:

$$\begin{aligned} &|\phi_*(n)(I-P)\Phi_*^{-1}(m)| \le Kk(n)^{-1}(n)k(m) \quad , n \ge m, \\ &|\Phi_*(n)P\Phi_*^{-1}(m)| \le kh(n)^{-1}h(m), quadm \ge n. \end{aligned} \tag{5}$$

Its clear that if the pair (h,k) is compensated, then so is the pair (k^{-1},h^{-1}).

The proof of the following theorem is contained in [7].

Theorem A. *If* (2) *has an* (h,k)*-dichotomy with projection matrix* P*, then* (1) *has an* (h,k)*-dichotomy with projection matrix* Q *iff*

$$V_{h,0} \subset V_{k,0} \subset Q[\mathbb{V}^r] \subset V_h \subset V_k. \tag{6}$$

The projection P *of dichotomy* (3) *can be chosen with the property*

$$\lim_{n\to\infty} h(n)^{-1}\Phi(n)P = 0 \tag{7}$$

iff $V_{h,0} = V_{k,0}$.

Theorem A can be applied to (4). Consequently, if (4) has the dichotomy (5), then it has a (k^{-1}, h^{-1})-dichotomy with projection q iff

$$V^*_{k^{-1},0} \subset V^*_{h^{-1},0} \subset Q[\mathbb{V}^r] \subset V^*_{k^{-1}} \subset V^*_{h^{-1}}, \tag{8}$$

where $V^*_{h^{-1}}$ is the subspace of initial conditions of (4) of h^{-1}-bounded solutions etc.

The following result was proved in [7].

Theorem B. *Let us suppose* (2) *has an* (h,k)*-dichotomy with projection* P*. If* $\{|A(n)^{-1}||B(n)|\} \in \ell^1$*, then* (1) *has an* (h,k)*-dichotomy*

$$\begin{aligned} |\widetilde{\Phi}(n)\widetilde{P}\widetilde{\Phi}^{-1}(m)| &\le \widetilde{K}h(n)h(m)^{-1}, \quad n \ge m,\\ |\widetilde{\Phi}(n)(I-\widetilde{P})\widetilde{\Phi}^{-1}(m)| &\le \widetilde{K}k(n)k(m)^{-1}, \quad m \ge n, \end{aligned} \tag{9}$$

where $\widetilde{P}$ *is a projection similar to projection* P.

3. *h*-bounded solutions.

Before we go ahead, we will establish some correspondence between the h-bounded solutions of (2) with those of its adjoint (4).

Lemma 1. *Assume that* (2) *has an* h*-dichotomy. If for some subsequence* $\{n_j\}$ *one has* $\lim_{j\to\infty} h(n_j)^{-1}x(n_j,\xi) = 0$*, then* $\lim_{n\to\infty} h(n)^{-1}x(n,\xi) = 0$.

Proof. Following Proposition 2.2 in [4], let us denote $x(n,\xi) = x_1(n) + x_2(n)$, $x_1(n) = P\Phi(n)\xi$, $x_2(n) = (I - P)\Phi(n)\xi$. It is easy to verify that

$$|h(n)^{-1}x_1(n)| \le K|h(n_j)^{-1}x(n_j)| \quad n > n_j,$$
$$|h(n)^{-1}x_2(n)| \le K|(n_j)^{-1}x(n_j)|, \quad n < n_j,$$

from where the proof of the lemma follows. □

Lemma 2. *If* (2) *has an* h*-dichotomy, then*

$$V^*_{h^{-1}} = (I - Q_{h,0})[\mathbb{V}^r]\ ,\ V^*_{h^{-1},0} = (I - Q_h)[\mathbb{V}].$$

Proof. The existence of an h-dichotomy for (2) with projection P implies the existence of two h-dichotomy for this system, respectively with projection Q_h and $Q_{h,0}$. Without loss of generality we can assume that $Q_{h,0}$ and Q_h have the diagonal forms $Q_{h,0} = \mathrm{diag}\{I_0, 0, 0\}$, $Q_h = \mathrm{diag}\{I_0, I_1, 0\}$, where I_0 is a unit matrix of dimensions $r_0 \times r_0$, $r_0 = \dim[V_{h,0}]$, and I_1 is a unite matrix of dimensions $r_1 \times r_1$, such that $r_0 + r_1 = \dim[V_h]$. According to Lemma 1 in [9] (see also Lemma 5.2 in [4]), there exists a bounded sequence $S : \mathbb{N} \to \mathbb{V}^{r+r}$, such that $S^{-1} : \mathbb{N} \to \mathbb{V}^{r+r}$ exists, is bounded, and the change of variables $x(n) = S(n)w(n)$ reduces (2) to the form

$$\begin{pmatrix} w_0(n+1) \\ w_1(n+1) \\ w_\infty(n+1) \end{pmatrix} = \begin{pmatrix} C^0(n) & 0 & 0 \\ 0 & C^1(n) & 0 \\ 0 & 0 & C^\infty(n) \end{pmatrix} \begin{pmatrix} w_0(n) \\ w_1(n) \\ w_\infty(n) \end{pmatrix} \tag{10}$$

where

$$w_0(n) \in V^{r_0},\ C^0(n) \in V^{r_0\times r_0},\ w_1(n) \in V^{r_1},\ C^1(n) \in V^{r_1\times r_1},$$
$$w_\infty(n) \in V^{r_\infty},\ C^\infty(n) \in V^{r_\infty\times r_\infty},\ r_\infty := r - (r_1 + r_0).$$

By a straightforward calculation, we may verify that the change of variables $z(n) = S_*(n)u(n)$ reduces the adjoint equation of (10) to the diagonal form

$$\begin{pmatrix} u_0(n+1) \\ u_1(n+1) \\ u_\infty(n+1) \end{pmatrix} = \begin{pmatrix} C^0_*(n) & 0 & 0 \\ 0 & C^1_*(n) & 0 \\ 0 & 0 & C^\infty_*(n) \end{pmatrix} \begin{pmatrix} u_0(n) \\ u_1(n) \\ u_\infty(n) \end{pmatrix} \tag{11}$$

For (10) we have $V_{h,0} = Q_{h,0}[\mathbb{V}^r]$, $V_h = Q_h[\mathbb{V}^r]$. We may write Θ, the fundamental matrix of (10), in the form

$$\Theta(n) = \mathrm{diag}\{U_0(n), U_1(n), U_\infty(n)\},$$

where $U_0(n) \in V^{r_0 \times r_0}$, $U_1(n) \in V^{r_1 \times r_1}$ and $U_\infty(n) \in V^{r_\infty \times r_\infty}$.

Regarding (11), let us consider the direct sum $\mathbb{V}^r = V^*_{h^{-1}} \otimes W^*_{h^{-1}}$, where $V^*_{h^{-1}}$ is a complementary subspace of $V^*_{h^{-1}}$. For the initial condition $\xi = \text{column}\{\xi_1, 0, 0\}$, we have $\lim_{n\to\infty} h(n)^{-1}\Theta(n)\xi = 0$, and $\{\Theta(n)\}\xi$ is h^{-1}-bounded if $\xi = \text{column}\{\xi_1, \xi_2, 0\}$. From these properties

$$\lim_{n\to\infty} |h(n)\Theta_*(n)\xi| = \infty, \quad \text{if } \xi_2 = 0,\ \xi_3 = 0,\ \xi_1 \neq 0$$

follows. Otherwise, the boundedness of some subsequence $\Theta_*(n_j)\xi$ would lead to the contradictory equation $|\xi|^2 =< h(n_j)^{-1}\Phi(n_j)\xi, h(n_j)\Phi_*(n_j)\xi >= 0$, where $< x, y >= \sum_{i=1}^r x_i \bar{y}_i$. Thus, we have proven

$$V_{h,0} \subset W^*_{h^{-1}}. \tag{12}$$

Further, for some constants K, M, one satisfies $K \geq |h(n)^{-1}U_1(n)| \geq M > 0$, $\forall n \in \mathbb{N}$. This implies

$$M^{-1} \geq |h(n)U_{1^*}(n)| \geq K^{-1},\ \forall n \in \mathbb{N}. \tag{13}$$

The boundedness of $\{h(n)U_{1^*}(n)\}$ and $\{h(n)U_{\infty^*}(n)\}$ (the boundedness of this last sequence follows from the fact that the adjoint (11) has an h^{-1}-dichotomy with projection $I - Q_h$ implies

$$r_1 + r_\infty \leq \dim V^*_{h^{-1}}. \tag{14}$$

Since $\dim V_{h,0} = r_0$, from (12) and (14) we obtain $r_0 = \dim W^*_{h^{-1}}$. Therefore

$$V_{h,0} = W^*_{h^{-1}}. \tag{15}$$

From (14) and (15) we obtain $(I - Q_{h,0})[\mathbb{V}^r] = V^*_{h^{-1}}$.

Let us prove that

$$\lim_{n\to\infty} |(n)\Theta_*(n)\xi| = 0, \quad \text{if } \xi_1 = 0, \xi_2 = 0. \tag{16}$$

Assuming the contrary, from Lemma 1, we would have for all values of n the estimate $K \geq |h(n)U_{\infty^*}(n)| \geq M > 0$, for some constant M. This implies the h^{-1}-boundedness of the sequence $\{U_\infty(n)\}$. But this contradicts

$$\lim_{n\to\infty} |\Theta(n)\xi| = \infty, \quad \text{if } \xi_1 = 0,\ \xi_2 = 0,\ \xi_3 \neq 0.$$

From the assertion (16) we have $(I - Q_h)[\mathbb{V}^r] \subset V^*_{h^{-1},0}$. This content and (13) imply $[I - Q_{h,0}][\mathbb{V}^r] \subset V^*_{h^{-1}}$. From (15) we obtain $\dim[I - Q_{h,0}][\mathbb{V}^r] = \dim V^*_{h^{-1}}$. Therefore $[I - Q_{h,0}][\mathbb{V}^r] = V^*_{h^{-1}}$. □

Definition 4. *The dichotomy* (3) *is said to be exhaustive if* $V_k = V$, *and precise if* $V_{h,0} = \{0\}$.

From Lemma 2 it follows

Theorem 1.
A: *If* (2) *has an* h*-dichotomy, then*

$$Q^*_{h^{-1}}[\mathbb{V}] = (I - Q_{h,0})[\mathbb{V}^r], \;\; Q^*_{h^{-1},0}[\mathbb{V}^r] = (I - Q_h)[\mathbb{V}^r].$$

B: *If* (3) *is compensated, then the dichotomy* (3) *is exhaustive iff* $V^*_{k^{-1},0} = \{0\}$.
C: *If* (3) *is compensated, then the dichotomy* (3) *is precise iff* $V^*_{h^{-1}} = \mathbb{V}^r$.

4. Asymptotic formulae.

If (2) has the (h, k)-dichotomy (3) and this dichotomy is compensated, then according to Theorem A the projections Q_h, $Q_{h,0}$, Q_k satisfy

$$Q_h Q_{h,0} = Q_{h,0}\,, \;\; Q_k Q_h = Q_h Q_k = Q_h. \tag{17}$$

We recall that the notation ρ_h will indicate a sequence with the property

$$\lim_{n\to\infty} h(n)^{-1}\rho_h(n) = 0. \tag{18}$$

4.1. An asymptotic formula to the h-bounded solutions.

Theorem 2. *Let us assume that* (2) *has an* h*-dichotomy. If*

$$K\widetilde{K}\sum_{m=0}^{\infty}|A(m)^{-1}||B(m)| < 1, \tag{19}$$

then the h*-bounded solutions of* (2) *and the* h*-bounded solutions of* (1) *are in biunivocal correspondence, satisfying*

$$y(n) = x(n) + \rho_h(n). \tag{20}$$

The fundamental matrix Ψ *of* (1)*,* $\Psi(0) = I$*, satisfies*

$$\Psi(n)\widetilde{Q}_h = \Phi(n)Q_h + \rho_h Q_h. \tag{21}$$

Proof. According to Theorem B, the assumed h-dichotomy of (2) can be accomplished with projection $Q_{h,0}$ satisfying (7). Given $\{x(n)\}$, an h-bounded solution of (2), we consider the integral equation

$$y(n) = x(n) + \sum_{m=0}^{n} \Phi(n) Q_{h,0} \Phi^{-1}(m+1) B(m) y(m) - \sum_{m=n}^{\infty} \Phi(n)(I - Q_{h,0})\Phi^{-1}(m+1) B(m) y(m). \tag{22}$$

Then, following [7] it is possible to prove that (22) has a unique h-bounded solution satisfying Eq. (1) and property (20). If we put $n = 0$ in (22), then

$$y(0) = x(0) - \sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m+1) B(m) \Psi(m) y(0).$$

The estimate (9) implies $|\Psi(n)\widetilde{Q}_h| \leq \widetilde{K} h(n)$, for some constant K. Henceforth

$$|\sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m+1) B(m) \Psi(m) \widetilde{Q}_h| \leq k\widetilde{K} \sum_{m=0}^{\infty} |A(m)^{-1}||B(m)| < 1.$$

From this estimate we obtain $y(0) = \Theta_h x(0)$, where

$$\Theta_h = \Big(I + \sum_{m=0}^{\infty} (I - Q_{h,0})\Phi^{-1}(m+1) B(m) \Psi(m) \widetilde{Q}_h \Big)^{-1} \tag{23}$$

The (22) implies

$$\Psi(n)\widetilde{Q}_h y(0) = \Phi(n) Q_h x(0) + \rho_h(n) x(0), \tag{24}$$

where

$$\rho_h(n) = \sum_{m=0}^{n-1} \Phi(n) Q_{h,0} \Phi^{-1}(m+1) B(m) \Psi(m) \Theta_h q_h - \sum_{m=n}^{\infty} \Phi(n)(I - Q_{h,0})\Phi^{-1}(m+1) B(m) \Psi(m) \Theta - h Q_h.$$

from (7) and (24) it follows that ρ_h satisfies the property (18). □

4.2. An asymptotic formula to the k-bounded solutions.

Let us now assume that (2) has an (h, k)-dichotomy. Since the pair of functions (h, k) is compensated, then (2) has both an h and a k-dichotomy. If (19) is satisfied, Theorem 2 can be applied to this k-dichotomy, and therefore the k-bounded solutions of (2) and (1) are in biunivocal correspondence with k-bounded solutions of (1); this correspondence is obtained by the k-bounded solution of the integral equation

$$\begin{aligned} y(n) = x(n) &+ \sum_{m=0}^{n} \Phi(n) Q_{k,0} \Phi^{-1}(m+1) B(m) y(m) \\ &- \sum_{m=n}^{\infty} \Phi(n)(I - Q_{k,0}) \Phi^{-1}(m+1) B(m) y(m), \end{aligned}$$

from where it follows the asymptotic formula

$$y(n) = x(n) + \rho_k(n), \tag{25}$$

where

$$\begin{aligned} R_k(n) = &\sum_{m=0}^{n-1} \Phi(n-1) Q_{k,0} \Phi^{-1}(m+1) B(m) \Psi(m) \Theta_k Q_k \\ &- \sum_{m=n}^{\infty} \Phi(n)(I - Q_{k,0} \Phi^{-1}(m+1) B(m) \Psi(m) \Theta_k Q_k, \end{aligned}$$

and

$$\Theta_k = \left\{ I + \sum_{m=0}^{\infty} (I - Q_{k,0}) \Phi^{-1}(m+1) B(m) \Psi(m) \widetilde{Q}_k \right\}^{-1}.$$

The asymptotic formula (25) yields the following asymptotic correspondence

$$\Psi(n) \widetilde{Q}_k = \Phi(n) Q_k + \rho_k(n) Q_k. \tag{26}$$

4.3. An asymptotic formula to the fundamental matrix.

Let us denote $W_k = (Q_k - Q_h)[\mathbb{V}^r]$ and $S_k = Q_k - Q_h$. From (17) we obtain $V_k = V_h \oplus W_k$. From (18) we point out to the important property

$$S_k Q_k = S_k = Q_k S_k. \tag{27}$$

Theorem 3. *If* (19) *is satisfied and* (2) *has an* (h, k)*-dichotomy, with a compensated pair* (h, k)*, then the fundamental matrix* $\Psi(n)$ *of* (1) *satisfies*

$$\Psi(n)(\widetilde{Q}_h Q_h + \widetilde{Q}_k S_k) = (\Phi(n) + \rho_h(n))Q_h + (\Phi(n) + \rho_k(n))S_k. \tag{28}$$

Moreover, if the (h, k)*-dichotomy is exhaustive, then the matrix* $(\widetilde{Q}_h Q_h + \widetilde{Q}_k S_k)$ *is invertible.*

Proof. The first part of the theorem is obtained by adding (21) and (26), previously being multiplied by Q_h and S_k.

Let us denote $E = \widetilde{Q}_h Q_h + \widetilde{Q}_k S_k$. Let $E\xi = 0$. From (27) we may write the decomposition $\xi = \xi_1 + \xi_2$, $\xi_1 \in Q_h[\mathbb{V}^r]$, $\xi_2 \in (Q_k - Q_h)[\mathbb{V}^r]$. Therefore

$$0 = \Phi(n)\xi_1 + R_h(n)\xi_1 + \Phi(n)\xi_2 + R_k(n)\xi_2.$$

This identity shows that $\xi_2 \in V_{k,0}$. Theorem A implies $\xi_2 \in V_h$. Thus $\xi_2 = 0$. Since $\Phi(n)\xi_1 + R_h(n)\xi_1 = 0$ is an h-bounded solution of Eq. (1) and the h-bounded solutions of (1), (2) are in biunivocal correspondence given by the nonsingular matrix (23). Therefore $\xi_1 = 0$ implies that E is invertible. □

It is clear that if the (h, k)-dichotomy is exhaustive and $\widetilde{Q}_h = Q_h$, $\widetilde{Q}_k = Q_k$, then

$$\widetilde{Q}_h Q_h + \widetilde{Q}_k S_k = I.$$

4.4. An asymptotic formula to the inverse matrix.

Condition (19) can be replaced by

$$K\widetilde{K} \sum_{m=n_0}^{\infty} |A(m)^{-1}||B(m)| < \infty, \tag{29}$$

implying

$$K\widetilde{K} \sum_{m=n_0}^{\infty} |A(m)^{-1}||B(m)| < 1,$$

for an n_0 sufficiently large. it is clear that the results of the previous section will remain valid on the interval $\mathbb{N}_{n_0} = \{n_0, n_0+1, n_0+2, \ldots\}$.

In this section we wish to find an asymptotic formula to the inverse of the fundamental matrix Ψ.

We may write $(A(n)+B(n))^{-1} = A^{-1}(n) + \widetilde{B}(n)$, where we have defined $\widetilde{B}(n) = (A(n)+B(n))^{-1} - A^{-1}(n)$. Condition (29) implies for a large n_0:

$$K\widetilde{K}\sum_{m=n_0}^{\infty} |A(m)||\widetilde{B}(n)| < \infty.$$

Let assume that (2) allows a compensated (h,k)-dichotomy. Then (4) has the compensated (k^{-1},h^{-1})-dichotomy (5). If (29) is fulfilled, then from Theorem 3 and $\Psi(n)^{-1} = \Psi_*^t$ we obtain the following asymptotic equation for Ψ^{-1}

$$\begin{aligned}(Q^*_{k^{-1}}\widetilde{Q}^*_{k^{-1}} + S^*_{h^{-1}}\widetilde{Q}^*_{h^{-1}})\Psi^{-1}(n) &= \\ Q^*_{k^{-1}}(\Phi^{-1}(n) + \rho_{k^{-1}}(n)) &+ S^*_{h^{-1}}(\Phi^{-1}(n) + \rho_{h^{-1}}(n)),\end{aligned}$$

where $S^*_{h^{-1}} = Q^*_{h^{-1}} - Q^*_{k^{-1}}$. this equation can be simplified if we assume that dichotomy (5) is exhaustive. According to Theorem 1, this condition is accomplished if $Q_{h,0} = 0$, that is, if the dichotomy (3) is precise. In such a case applying Theorem 1 we have $Q^*_{k^{-1}} = I - Q_{k,0}$, $Q^*_{h^{-1}} = I$, $\widetilde{Q}^*_{k^{-1}} = I - \widetilde{Q}_{k,0}$, $\widetilde{Q}_{h^{-1}} = I$, from where

$$Q^*_{k^{-1}}\widetilde{Q}^*_{k^{-1}} + S^*_{h^{-1}}\widetilde{Q}^*_{h^{-1}} = (I - Q_{k,0})(I - \widetilde{Q}_{k,0}) + Q_{k,0}.$$

In this way we have proven the following

Lemma 3. *If the compensated dichotomy* (3) *is precise then under condition* (29) *the fundamental matrix* Ψ *satisfies*

$$\begin{aligned}((I - Q_{k,0})(I - \widetilde{Q}_{k,0}) + Q_{k,0})\Psi^{-1}(n) &= \\ Q^*_{k^{-1}}(\Phi^{-1}(n) + \rho_{k^{-1}}(n)) &+ S^*_{h^{-1}}(\Phi^{-1}(n) + \rho_{h^{-1}}(n)),\end{aligned}$$

where the matrix $(I - Q_{k,0})(I - \widetilde{Q}_{k,0}) + Q_{k,0}$ *is invertible.*

Theorem 4. *Let us assume that the discrete dichotomy* (3) *is compensated, exhaustive and precise. Moreover, let us assume that* $\widetilde{Q}_h = Q_h$, $\widetilde{Q}_k = Q_k$, $\widetilde{Q}_{k,0} = Q_{k,0}$, $\widetilde{Q}_{h,0} = Q_{h,0}$. *If condition* (29) *is satisfied, then the fundamental matrix* ψ *of* (1) *and its inverse have the following asymptotic representation*

$$\Psi(n) = (\Phi(n) + \rho_h(n))Q_h + (\Phi(n) + \rho_k(n))(I - Q_h).$$

$$\Psi^{-1}(n) = (I - Q_{k,0})(\Phi^{-1}(n) + \rho_{k^{-1}}(n)) + Q_{k,0}(\Phi^{-1}(n) + \rho_{h^{-1}}(n)),$$

From this theorem it follows an asymptotic formula for the Cauchy matrix of (1).

Theorem 5. *Under conditions of Theorem 4, the assumption* $Q_{k,0} = Q_h$ *implies the asymptotic representation*

$$\Psi(n)\Psi^{-1}(m) = (\Phi(n) + R_h(n))Q_h(\Phi^{-1}(m) + \rho_{h^{-1}}(m)) \\ + (\Phi(n) + \rho_k(n))(I - Q_h)(\Phi^{-1}(m) + R_{k^{-1}}(m)), \ \forall m, n.$$

5. Dichotomic chains.

Let us consider two ordered sets of positive continuous functions

$$\mathcal{H} = \{h_1, h_2, \dots, h_p\}, \quad \mathcal{K} = \{k_1, k_2, \dots, k_p\},$$

and a collection of projections matrices

$$\mathcal{P} = \{P_1, P_2, \dots, P_p\}.$$

Definition 5. *We shall say that the triplet* $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ *is a dichotomic chain for* (2) *iff*

*(*L 1) *For* $j = 1, \dots, r$*,* (2) *has a dichotomy* (h_j, k_j, P_j).
*(*L 2) $V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_p}$.

We will employ the abbreviation $(\mathcal{H}, \mathcal{P}) = (\mathcal{H}, \mathcal{H}, \mathcal{P})$. In applications, a convenient algebraic condition implying (L2) is given by

(L 2′) For some constant D we have $k_j(n) \leq Dk_{j+1}(n)$, $j = 1, 2, \dots, r-1$.

A more stringent condition than (L 2′) is the uniform condition

(L 2″) For some constant D we have

$$k_j(n)k_j(m)^{-1} \leq Dk_{j+1}(n)k_{j+1}(m)^{-1}, \quad j = 1, 2, \dots, r-1, \quad n \geq m.$$

Theorem 6. *Let us assume that* (2) *has the dichotomic chain* $(\mathcal{H}, \mathcal{K}, \mathcal{P})$*, then* (2) *has a dichotomic chain* $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$*, where the projections* $\mathcal{P}' = \{P_1', P_2', \dots, P_p'\}$ *are respectively similar to the projections of the collection* $\mathcal{P} = \{P_1, P_2, \dots, P_p\}$.

Proof. Applying Theorem B to each (h_i, k_i)-dichotomy of $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ we obtain an (h_i, k_i)-dichotomy for (2) with a projection P_i' similar to projection P_i. In [8] it is proven that for any (h, k)-dichotomy of System (2) we have the properties dimension $[V_h] =$ dimension $[\widetilde{V}_h]$, dimension $[\widetilde{V}_h]$, dimension $[V_k] =$ dimension $[\widetilde{V}_k]$. Hence, from (L 2) we obtain $\widetilde{V}_{k_1} \subset \widetilde{V}_{k_2} \subset \cdots \subset \widetilde{V}_{k_p}$. If $\mathcal{P}' = \{P_1', P_2', \ldots, P_p'\}$, then $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$ is required dichotomic chain. □

Remark 1. *In what follows we will assume that all (h, k)-dichotomies of the dichotomic chain $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ (respectively $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$) are defined with a same constant K (respectively $\widetilde{K}$).*

Assume that (2) has the dichotomic chain $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. We will perform the following construction: Let us define $U_{h_1} = V_{h_1}$. Further, if $V_h = V_{k_1}$ we define $W_{k_1} = \{0\}$. If V_{h_1} is properly contained in V_{k_1}, then we define W_{k_1} as a complementary subspace to V_{h_1} in the space V_{k_1}. In both cases we can write the disjoint summa $V_{k_1} = \{0\} + U_{h_1} + W_{k_1}$. thus in the space U_{h_1} we keep all the initial conditions corresponding to the h_1-bounded solutions of (2). To the space W_{k_1} we assign the initial conditions of k_1-bounded solutions that are not h_1-bounded. We repeat this process for the space V_{k_2} in the following manner: If $V_{k_2} = V_{k_1}$, we define $U_{k_2} = W_{k_2} = \{0\}$. If V_{k_1} is properly contained in V_{k_2}, then we define U_{h_2} as the subspace of the initial condition of the h_2-bounded solutions not contained in V_{h_1} and the subspace W_{k_2} groups the initial conditions of k_2-solutions not included in U_{h_2}; therefore V_{k_2} can be written as a disjoint sum $V_{k_2} = V_{k_1} + U_{k_2} + W_{k_2}$. Carrying out this process further, we obtain the decomposition:

$$\begin{array}{ccccccc} V_{k_1} & = & \{0\} & + & U_{h_1} & + & W_{k_1} \\ V_{k_2} & = & V_{k_1} & + & U_{h_2} & + & W_{k_2} \\ \vdots & & \vdots & & \vdots & & \vdots \\ V_{k_p} & = & V_{k_{r-1}} & + & U_{h_p} & + & W_{k_p} \end{array} \tag{30}$$

In applications the table (30) does not give a good decomposition of the subspaces of initial conditions corresponding to solutions with different growths; for example if $k_1 = k_2 = \cdots = k_p$, all subspaces of table (30) would be trivial, except U_{h_1} and maybe W_{k_1}. This situation can be improved by asking from the dichotomic chain the property defined as follows.

Definition 6. *We shall say that the LD $(\mathcal{H}, \mathscr{k}, \mathcal{P})$ is a stratified iff*

$$U_{h_1} \subset V_{k_1} \subset U_{h_2} \subset V_{k_2} \subset \ldots \subset U_{h_p} \subset V_{k_p}.$$

This property is given if for some constant D we have $k_j(t) \le Dh_{j+1}(t)$, $j = 1, 2, \ldots, r-1$. We emphasize that the dichotomic chain $(\mathcal{H}, \mathcal{P})$ is stratified.

6. Asymptotic integration.

In this section we generalize the asymptotic formula (28) under the existence of a dichotomic chain for (2). Let us consider a chain $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. According to the table (30), we define the projections matrices R_j, S_j such that $R_j[\mathbb{V}^r] = U_{h_j}$, $S_j[\mathbb{V}^r] = W_{k_j}$. From the construction of subspaces U_{h_j} and V_{k_j} we have $R_j R_i = 0$, $S_j S_i = 0$, if $i \ne j$, $R_j S_i = 0$ for all indexes i, j. Moreover, since the range of projections R_j and S_j are respectively contained in V_{h_j} and V_{k_j}, we have the identities

$$Q_{h_j} R_j = R_j, \; Q_{k_j} S_j = S_j. \tag{31}$$

Theorem 7. *Let us assume that* (2) *has the dichotomic chain* $(\hbar, \mathcal{K}, \mathcal{P})$ *and condition* (19) *is satisfied (see Remark 1), then the fundamental matrix* Ψ *of* (2), $\Psi(t_0) = I$, *has the property*

$$\Psi(n)E = \sum_{j=1}^{r} (\Phi(n) + \rho h_j(n))) R_j + \sum_{j=1}^{r} (\Phi(n) + \rho k_j(n))) S_j, \tag{32}$$

where some of projections R_j *or* S_j *in* (32) *could be equal zero, and* E *is defined by*

$$E = \sum_{j=1}^{p} (\widetilde{Q}_{h+j} R_j + \widetilde{Q}_{k_j} S_j) \tag{33}$$

Proof. Applying Theorem 2 to each (h_j, k_j)-dichotomy we obtain from (21), (26) the decompositions $\Psi(n)\widetilde{Q}_{h_j} = (\Phi(n) + \rho h_j(n)) Q_{h_j}$ and $\Psi(n)\widetilde{Q}_{k_j} = (\Phi(n) + \rho k(n)) Q_{k_1}$. Respectively multiplying each of these formulas by R_j and S_j and using (31) we obtain $\Psi(n)\widetilde{Q}_{k_j} R_j = (\Phi(n) + o(h_j(n))) R_j$ and $\Psi(n)\widetilde{Q}_{k_j} S_j = (\Phi(n) + o(k(n))) S_j$. From these formulas it follows (32). □

Definition 7. *The chain* $(\mathcal{H}, \hbar, \mathcal{P})$ *is called exhaustive iff* $V_{k_p} = [\mathbb{V}^r]$.

For exhaustive chains the projections defined by the table (30) have the property

$$I = Q_{k_p} = \sum_{j=1}^{p} R_j + \sum_{j=1}^{p} S_j.$$

From this identity, we can establish the following version of the Levinson asymptotic theorem for (1).

Theorem 8. *Under conditions of Theorem 2, if the chain* $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ *is exhaustive, then the matrix* E *defined by* (33) *is invertible.*

Proof. Let $E\xi = 0$. Then

$$0 = \sum_{j=1}^{p} (\Phi(n) + \rho h_j(n)) R_j \xi + \sum_{j=1}^{p} (\Phi(n) + \rho k_j(n)) S_j \xi.$$

From the construction of table (30), we obtain that the solution $\Phi(n) S_p \xi$ of (2) satisfies $\Phi(n) S_p \xi = \rho_{k_p(n)}$; therefore $S_p \xi \in V_{k_p,0}$. Applying Theorem A to the dichotomy (h_p, k_p, P_p) we obtain $S_p \xi \in V_{h_p}$. since $S_p \xi \in W_{k_p}$ the last row of table (30) says that $S_p \xi = 0$. henceforth

$$0 = \sum_{j=1}^{p} (\Phi(n) + \rho h_j(n)) R_j \xi + \sum_{j=1}^{p-1} (\Phi(n) + \rho k_j(n)) S_j \xi.$$

The right hand side of this last equation is an h_p-bounded solution of (2). But under condition (19), the h_p-bounded solutions of (1), (2) are in biunivocal correspondence. Therefore

$$0 = \sum_{j=1}^{r} \Phi(n) R_j \xi + \sum_{j=1}^{r-1} \Phi(t) S_j \xi + \Phi(n) R_p \xi. \tag{34}$$

Since $\sum_{j=1}^{r-1} \Phi(n) R_j \xi + \sum_{j=1}^{r-1} \Phi(n) S_j \xi \in V_{k_{r-1}}$ and $\Phi(n) R_p \xi \in U_{h_p}$, we obtain from the last row of table (30) $R_p \xi = 0$. Inasmuch as $R_p \xi = 0$ and $S_p \xi = 0$, we obtain from (34)

$$0 = \sum_{j=1}^{r-1} (\Phi(n) + \rho h_j(n)) R_j + \sum_{j=1}^{r-1} (\Phi(n) + \rho k_j(n)) S_j.$$

By repeating this reasoning we will obtain $R_j \xi = 0$, $S_j \xi = 0$, $\forall j$ implying $\xi = \sum_{j=1}^{r} (R_j + S_j) \xi = 0$. Therefore, E is an invertible matrix $\square$

Dichotomic chains can be used in obtaining an asymptotic decomposition of the inverse matrix Ψ^{-1}. This can be accomplished in a similar way to the decompositions obtained in Subsection 4.4.

Acknowledgements. Supported by Proyecto CI-5-025-00730/95.

REFERENCES

[1] R.P. Agarwal, *Difference Equations and Inequalities, (Theory, Methods and Aplication),* Pure and Applied Matematics, Marcel Dekker, New York, 1992.

[2] Z. Benzaid - D.A. Lutz, *Asymptotic representation of perturbed systems of linear difference equations,* Studies in Appl. Math., 77 (1978), pp. 195–221.

[3] Ch.V. Coffman, *Asymptotic behavior of solutions of ordinary difference equations,* Trans. Amer. Math. Soc., 10–1 (1964), pp. 22–51.

[4] W.A. Coppel, *Dichotomies in Stability Theory,* Lectures Notes in Mathematics 629, Berlin, 1978.

[5] N. Levinson, *The asymptotic nature of solutions of linear differential equations,* Duke Math. J., 15 (1948), pp. 111–126.

[6] Z-H. Li, *The asymptotic estimates of solutions of difference eqautions,* J. Math. Anal. Appl., 94 (1983), pp. 181–192.

[7] R. Naulin - M. Pinto, *Stability of discretes dichotomies for linear difference systems,* J. of Difference Eqns. and Appl., 3 (1995), pp. 101–123.

[8] R. Naulin - M. Pinto, *Projections for dichotomies in linear differential equations,* Applicable Analysis, 69–3 (1998), pp. 239–255.

[9] G. Papaschinopoulos, *Exponential separation, exponential dichotomy, and almost periodicity of linear difference equations,* J. Math. Anal. appl., 120–1 (1986), pp. 276–287.

[10] M. Pinto, *Asymptotic equivalence of difference systems,* J. Difference Eqns. and Appl., 1 (1995), pp. 249–262.

[11] M. Pinto, *Discrete dichotomies,* Computers Math. Applic., 28 (1994), pp. 259-270.

Departamento de Matemáticas,
Universidad de Oriente,
Cumaná 6101 A-285 (VENEZUELA)