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UNIQUENESS RESULT FOR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Following the stream of ideas in two recent papers ([1], [8]), one can establish a uniqueness result for the Dirichlet problem for a class of elliptic second order differential equations with discontinuous coefficients in unbounded domains of $\mathbb{R}^n$, $n \geq 3$.

Introduction.

Let us assign an open subset Ω of $\mathbb{R}^n$, $n \geq 3$, and consider the uniformly elliptic differential operator

$$L_o := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where $a_{ij} = a_{ji}$ belong to $L^\infty(\Omega) \cap VMO(\Omega)$.

In [4] it is shown that if Ω is bounded, $\partial\Omega$ is $C^{1,1}$, $p \in]1, +\infty[$, $q \in]1, p]$, $f \in L^p(\Omega)$, $u \in W^{2,q}(\Omega) \cap \overset{o}{W}{}^{1,p}(\Omega)$ and $L_o u = f$ a.e. in Ω, then $u \in W^{2,p}(\Omega)$ and one has

$$\|u\|_{W^{2,p}(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

with c independent of u and f.

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Using this regularization result and the Aleksandrov-Pucci maximum principle, it is proved the following uniqueness result: the solution of the Dirichlet problem

$$\begin{cases} L_o u = 0 & \text{a.e. in} \quad \Omega \\ u \in W^{2,p}(\Omega) \cap \overset{o}{W}{}^{1,p}(\Omega), \end{cases}$$

$1 < p < +\infty$, is zero in Ω.

In [7], [8], the same results are obtained, still with Ω bounded, for operator with lower order terms

$$L := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$

under suitable summability assumptions on a_i and a.

In [1] it is considered the case Ω unbounded. It is shown that if Ω is sufficiently regular, the coefficients a_i and a are in suitable spaces of Morrey type, $p \in]1, +\infty[$, $q_o, q \in]1, p]$, $f \in L^p(\Omega)$, $u \in W^{2,q}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,q}_{\text{loc}}(\overline{\Omega}) \cap L^{q_o}(\Omega)$ and $Lu = f$ a.e. in Ω, then $u \in W^{2,p}(\Omega)$ and one has

$$\|u\|_{W^{2,p}(\Omega)} \leq c(\|f\|_{L^p(\Omega)} + \|u\|_{L^{q_o}(\Omega)}),$$

with c independent of u and f.

In this paper our purpose is to prove that if Ω is unbounded and sufficiently regular and the coefficients a_i, a are in suitable local Lebesgue spaces, then the solution of the Dirichlet problem

$$(\mathcal{D}) \qquad \begin{cases} Lu = 0 & \text{a.e. in} \quad \Omega \\ u \in W^{2,p}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,p}_{\text{loc}}(\overline{\Omega}) \\ \lim_{|x| \to +\infty} u(x) = 0, \end{cases}$$

$1 < p < +\infty$, is zero in Ω.

The result follows on some results of [1], that we quote and precise, and of [8].

1. Notation and function spaces.

Throughout this paper we use the following notation: E, a generic Lebesgue measurable subset of $\mathbb{R}^n$; $\Sigma(E)$, the Lebesgue σ-algebra on E; $|A|$, the Lebesgue measure of $A \in \Sigma(E)$; $\mathcal{D}(A)$, the class of restrictions to A of the functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\zeta \cap \overline{A} \subset A$; $L^p_{\mathrm{loc}}(A)$, the class of the functions g, defined on A, such that $\zeta g \in L^p(A)$ for all $\zeta \in \mathcal{D}(A)$; $|g|_{p,A}$, the $L^p(A)$-norm of g; $B(x,r)$, the open ball centered at x with radius r and $B_r := B(0,r)$; Ω, an unbounded open subset of $\mathbb{R}^n$ and $\Omega(x,r) := \Omega \cap B(x,r)$; $W^{k,p}_{\mathrm{loc}}(\overline{\Omega})$ (resp. $\overset{o}{W}{}^{k,p}_{\mathrm{loc}}(\overline{\Omega})$), the set of the functions $u : \Omega \to \mathbb{R}$ such that $\zeta u \in W^{k,p}(\Omega)$ (resp. $\overset{o}{W}{}^{k,p}(\Omega)$) for all $\zeta \in \mathcal{D}(\overline{\Omega})$.

Now we recall the definitions of the function spaces we deal with.

We denote by $M^{p,\lambda}(\Omega)$, $p \in [1,+\infty[$, $\lambda \in [0,n[$, the subset of $L^p_{\mathrm{loc}}(\overline{\Omega})$ consisting of the functions g for which

$$\|g\|_{M^{p,\lambda}(\Omega)} := \sup_{\substack{r\in]0,1] \\ x\in\Omega}} r^{-\frac{\lambda}{p}} |g|_{p,\Omega(x,r)} < +\infty, \tag{1.1}$$

endowed with the norm defined in (1.1) and by $M^{p,\lambda}_{\mathrm{loc}}(\overline{\Omega})$ the set of the functions $u : \Omega \to \mathbb{R}$ such that $\zeta u \in M^{p,\lambda}(\Omega)$ for all $\zeta \in \mathcal{D}(\overline{\Omega})$. Moreover $\tilde{M}^{p,\lambda}(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$ and in the following $\tilde{M}^p(\Omega) := \tilde{M}^{p,0}(\Omega)$.

When Ω has a condition *like Campanato*, more precisely

$$A := \sup_{\substack{x\in\Omega \\ r\in]0,1]}} \frac{|B(x,r)|}{|\Omega(x,r)|} < +\infty, \tag{1.2}$$

it is possible to introduce two different function spaces, $BMO(\Omega,t)$ and $VMO(\Omega)$, as follows.

$$BMO(\Omega,t) := \Big\{ g \in L^1_{\mathrm{loc}}(\overline{\Omega}) : [g]_{BMO(\Omega,t)} := = \sup_{\substack{x\in\Omega \\ r\in]0,t]}} \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} |g - g_{\Omega(x,r)}| < +\infty \Big\},$$

where

$$g_{\Omega(x,r)} := \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} g.$$

As in [6], we set

$$BMO(\Omega) := BMO(\Omega, t_A),$$

where

$$t_A := \sup_{t\in\mathbb{R}_+} \Big\{ \sup_{\substack{x\in\Omega \\ r\in]0,t]}} \frac{|B(x,r)|}{|\Omega(x,r)|} \le A \Big\},$$

and define

$$VMO(\Omega) := \Big\{ g \in BMO(\Omega) : [g]_{BMO(\Omega,t)} \to 0 \quad \text{as} \quad t \to 0^+ \Big\}.$$

We point out that (1.2) holds for Ω when it satisfies the regularity assumption $(\mathcal{P}_\Omega)$ below. For more informations about the previous function spaces we refer to [5], [6], [2].

2. Assumption and main results.

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 3$, and $p \in]1, +\infty[$.

We suppose from now on that $\partial\Omega$ satisfies the uniform $C^{1,1}$-regularity property:

$(\mathcal{P}_\Omega)$ there exist a locally finite open cover $(U_i)_{i\in\mathbb{N}}$ of $\partial\Omega$ and corresponding $C^{1,1}$-diffeomorphisms $\Phi_i : U_i \to B_1$ such that:

♡) for some $\delta > 0$, $\{x \in \Omega : dist(x, \partial\Omega) < \delta\} \subset \bigcup_{i\in\mathbb{N}} \Phi_i^{-1}\big(B_{\frac{1}{2}}\big)$;

◇) for each $i \in \mathbb{N}$, $\Phi_i(U_i \cap \Omega) = \{x \in B_1 : x_n > 0\}$;

♣) there is an $m_0 \in \mathbb{N}$ such that any $m_0 + 1$ distinct sets U_i have empty intersection;

♠) the components of Φ_i and Φ_i^{-1} have $C^{1,1}$-norm bounded independently of i.

We consider the operator

$$L := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \tag{2.1}$$

and on the coefficients of L we make the following assumptions:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO(\Omega), \quad i, j = 1, \cdots, n, \\ \exists\, \Lambda > 0 : \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \Lambda|\xi|^2 \quad \text{a.e.} \quad x \in \Omega,\ \forall \xi \in \mathbb{R}^n; \end{cases} \tag{2.2}$$

$$a_i \in L^r_{\text{loc}}(\overline{\Omega}) \quad \text{where} \quad r > n \quad \text{for} \quad p \leq n, \quad r = p \quad \text{for} \quad p > n,\ i = 1, \cdots, n; \tag{2.3}$$

$$\begin{cases} a \in L^s_{\text{loc}}(\overline{\Omega}) \quad \text{where} \quad s > \frac{n}{2} \quad \text{for} \quad p \leq \frac{n}{2},\ \ s = p \quad \text{for} \quad p > \frac{n}{2}; \\ a \leq 0 \quad \text{a.e. in} \quad \Omega. \end{cases} \tag{2.4}$$

In this paper our main results are:

Theorem. *Assume* (2.2), (2.3), (2.4). *Then the solution of the Dirichlet problem*

$$(\mathcal{D}) \qquad \begin{cases} Lu = 0 & a.e. \ in \quad \Omega \\ u \in W^{2,p}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,p}_{loc}(\overline{\Omega}) \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases}$$

is zero in Ω.

Corollary. *Assume* (2.2), (2.3), (2.4), *with* $a_i \in \tilde{M}^r(\Omega)$ *and* $a \in \tilde{M}^s(\Omega)$ *when* $p \le n$. *Then the solution of the Dirichlet problem*

$$(\mathcal{D}_0) \qquad u \in W^{2,p}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,p}(\Omega), \qquad Lu = 0$$

is zero in Ω.

3. Tools.

One of the tools of the proof, as we have already said in the introduction, is the stream of results of [1]. Now we quote and precise some of them.

Let us consider the parameters $k \in \mathbb{N}$, $\tau, \alpha, r \in [1, +\infty[$, $\lambda \in [0, n[$ such that:

$$(\star) \qquad \begin{cases} \alpha\tau \le r, \text{ with } \alpha\tau < r \text{ when } \tau = \frac{n}{k} > 1 \text{ and } \lambda = 0, \ \lambda > n - kr \\ \frac{1}{\alpha} \ge 1 - \tau(\frac{k}{n} - \frac{1}{r}) \text{ if } n < kr; \ \frac{1}{\alpha} > 1 - \frac{\tau}{nr}(\lambda - n + kr) \text{ if } n \ge kr. \end{cases}$$

Proposition 1. *Let* $k \in \mathbb{N}$, $\tau, \alpha, r \in [1, +\infty[$, $\lambda \in [0, n[$ *satisfy* $(\star)$, *then the multiplication operator for a function* $g \in M^{r,\lambda}(\Omega)$, *defined in* $W^{k,\tau}(\Omega)$, *has values in* $L^{\alpha\tau}(\Omega)$ *and there exists* $c \in \mathbb{R}_+$, *independent of g and u, such that*

$$|gu|_{\alpha\tau,\Omega} \le c \ \|g\|_{M^{r,\lambda}(\Omega)} \|u\|_{W^{k,\tau}(\Omega)}.$$

Moreover if $g \in \tilde{M}^{r,\lambda}(\Omega)$, *for every* $\epsilon \in \mathbb{R}_+$ *there exists* $c_\epsilon \in \mathbb{R}_+$, *independent of u, such that*

$$|gu|_{\alpha\tau,\Omega} \le \epsilon \|u\|_{W^{k,\tau}(\Omega)} + c_\epsilon |u|_{\tau,\Omega}.$$

Proof. This follows from Theorem 3.2 and Corollary 3.3 of [1], observing that $\gamma := k/n - (\alpha - 1)/\tau\alpha > 0$ and

$$r\gamma < 1 \quad \Rightarrow \quad n \ge kr \quad \Rightarrow \quad \lambda > n(1 - r\gamma). \qquad \square$$

If $k = 1, 2$ we can consider the numbers $q \in]1, +\infty[$, r_k, λ_k such that

$$(\mathcal{A}) \qquad r_k \ge q, \quad \lambda_k \in [0, n[\cap]n - kr_k, +\infty[.$$

One can show:

Proposition 2. *Assume* $(\mathcal{A})$. *Then for any* $p \in]1, q[$ *there exist* $\nu \in \mathbb{N} \setminus \{1\}$ *and* $\beta \in]1, +\infty[$ *such that*

$$
\begin{array}{lll}
I) & \beta^{\nu} = \frac{q}{p}, \quad \beta^{\nu-1} \neq \frac{n}{kp}; & \\
II_a) & \frac{1}{\beta} \geq 1 - p(\frac{k}{n} - \frac{1}{r_k}) & \text{if } n < kr_k; \\
II_b) & \frac{1}{\beta} > 1 - \frac{p}{nr_k}(\lambda_k - n + kr_k) & \text{if } n \geq kr_k.
\end{array}
$$

Proof. Fixed $p \in]1, q[$, obviously the right sides of $II_a)$ and $II_b)$ are fixed quantities strictly less than 1. So the result follows from a right use of the exponential function with basis p/q. □

Remark 1. In the hypothesis $(\mathcal{A})$, keeping in mind the previous result, we can set

$$p_h := \beta^{h-1} p, \qquad \forall h = 1, \cdots, \nu$$

and observe that

$$
(**) \qquad \begin{cases} \beta p_h < q, \quad \text{if } h < \nu; \quad \beta p_\nu = q; \quad p_\nu \neq \frac{n}{k}; \\ \frac{1}{\beta} \geq 1 - p_h(\frac{k}{n} - \frac{1}{r_k}) \quad \text{if} \quad n < kr_k; \\ \frac{1}{\beta} > 1 - \frac{p_h}{nr_k}(\lambda_k - n + kr_k) \quad \text{if} \quad n \geq kr_k. \end{cases}
$$

Proposition 3. *Fixed* $k = 1$ *or 2, consider the correspondent parameters satysfing* $(\mathcal{A})$ *and the numbers* p_h, *previously defined, then the multiplication operator for a function* $g \in M^{r_k, \lambda_k}(\Omega)$, *defined in* $W^{k, p_h}(\Omega)$, *has values in* $L^{\beta p_h}(\Omega)$ *and there exists* $c \in \mathbb{R}_+$, *independent of* g *and* u, *such that*

$$|gu|_{\beta p_h, \Omega} \leq c \|g\|_{M^{r_k, \lambda_k}(\Omega)} \|u\|_{W^{k, p_h}(\Omega)}.$$

Proof. The result follows from Proposition 1, Proposition 2 and Remark 1. □

4. Proofs.

For the proofs of the main results we need the following lemmas.

Lemma 4.1. *Assume (2.2),* $a_i \in M^{r_1, \lambda_1}_{\text{loc}}(\overline{\Omega})$, $a \in M^{r_2, \lambda_2}_{\text{loc}}(\overline{\Omega})$, $p \in]1, q]$ *with* r_k, λ_k, q *satisfy* $(\mathcal{A})$. *Then for any* u *solution of the problem*

$$
\begin{cases} u \in W^{2,p}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,p}_{\text{loc}}(\overline{\Omega}) \\ Lu \in L^q_{\text{loc}}(\overline{\Omega}) \end{cases}
$$

one has $u \in W^{2,q}_{\text{loc}}(\overline{\Omega})$.

Proof. The way of proceeding is analogous to that of Theorem 5.1 of [1]. Since our assumptions on the lower order terms' coefficients of L are weaker than there, it seems better to remake the all proof. We just need to prove that if u is also in $W^{2,p_h}_{\text{loc}}(\overline{\Omega})$ then

$$u \in W^{2,\beta p_h}_{\text{loc}}(\overline{\Omega}), \tag{4.1}$$

where β, p_h are the quantities defined in Proposition 2 and in the Remark 1.

We observe that if $u \in W^{2,p_h}_{\text{loc}}(\overline{\Omega})$ then, for every $\zeta \in \mathcal{D}(\overline{\Omega})$, $\zeta u \in W^{2,p_h}(\Omega)$ and so, by Proposition 3, one has:

$$\sum_{i=1}^{n} a_i (\zeta u)_{x_i} + a \zeta u \in L^{\beta p_h}(\Omega).$$

Since $Lu \in L^q_{\text{loc}}(\overline{\Omega})$, it follows that

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \in L^{\beta p_h}_{\text{loc}}(\overline{\Omega}). \tag{4.2}$$

As consequence of known results (see Theorem 4.2 of [3] and Theorem 3.2 of [4]), the assumption on u together with (4.2) give (4.1). □

Lemma 4.2. *Assume (2.2), $a_i \in \tilde{M}^{r_1,\lambda_1}(\Omega)$, $a \in \tilde{M}^{r_2,\lambda_2}(\Omega)$, $p \in]1, q]$ with r_k, λ_k, q satisfy $(\mathcal{A})$, $p_0 \in [1, q]$. Then for any u solution of the problem*

$$\begin{cases} u \in W^{2,p}_{\text{loc}}(\overline{\Omega}) \cap \overset{o}{W}{}^{1,p}_{\text{loc}}(\overline{\Omega}) \cap L^{p_0}(\Omega) \\ Lu \in L^q(\Omega) \end{cases}$$

one has $u \in W^{2,q}(\Omega)$.

Proof. Once observed that, by Lemma 4.1, u belongs to $W^{2,q}_{\text{loc}}(\overline{\Omega})$, one can go on as in the proof of theorem 5.1 of [1] to obtain the desired result. □

Proof of the Theorem. First we point out that for every p a solution u of $(\mathcal{D})$ belongs to $C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. This follows just from the Sobolev imbedding theorem when $p > \frac{n}{2}$ and also from Lemma 4.1 when $p \in]1, \frac{n}{2}]$, that we apply with $r_1 = r$, $r_2 = s$, $\lambda_1 = \lambda_2 = 0$, $q = \min\{r, s\}$.

Moreover owing to the behaviour of u at infinity, u attains its maximum and minimum in Ω. Then to prove the result one can just follow the Vitanza argument ([8, Theorem 3.1]), which is based on a local analysis. □

Proof of the Corollary. Arguing as in the proof of the Theorem, using Lemma 4.2 instead of Lemma 4.1, one shows that a solution u of $(\mathcal{D}_0)$ belongs to $W^{2,t}(\Omega)$ for a $t > n/2$.

It is well known that this implies

$$u \in \overset{o}{W}{}^{1,t_1}(\Omega) \qquad \text{with} \quad t_1 > n.$$

So u goes to zero at infinity. Hence it is also a solution of $(\mathcal{D})$ and therefore $u = 0$ in Ω. $\square$

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