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THE STIELTJES INTEGRAL ALONG FRACTAL CURVE

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The present paper is dealing with the integral $\int_\Gamma f\,dz$ over fractal non-rectifiable curve Γ on the complex plane. In the introduction we observe its known indirect definitions. The Section 2 treats a direct definition, the existence theorems and some properties of the integral, and the next one concerns the Cauchy type integral along fractals. The final section consists of conjectures and open questions.

1. Introduction.

As known, the modern mechanics and physics consider fractal curves as adequate model of real boundaries, trajectories, cracks and so on (see [18], [4]). The property of self-similarity of the fractals is of special significance, because it simulates homogeneity of their real prototypes.

The various boundary value problems of mechanics and physics under classical assumptions on smoothness of the boundaries are resolvable in terms of integrals along these boundaries. For example, the classical investigations of the plane problems of elasticity theory are based on using of the Cauchy type integrals along the boundaries and cracks (see [20]). But according the modern concepts the rough boundaries and cracks have fractal features, and we obtain the Cauchy type integral along fractal curves. In this connection there arises initially a question on existence and properties of the integral

$$\int_\Gamma f(z)\,dz, \tag{1}$$

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where Γ is a fractal curve on the plane of complex variable z, and $f(z)$ is a function defined on this curve.

A number of works treats the integral $\int_\Gamma f\, dm_\alpha$ over fractal fractal set Γ of dimension α with respect to its Hausdorff measure m_α (for instance, see [6], where A. Jonsson and H. Wallin study certain analogs of the Besov spaces with respect to that measures). But the properties of this integral and the integral (1) have important dissimilarities. For instance, if the Hausdorff dimension of $\Gamma \subset \mathbb{C}$ is $\alpha \in (1, 2)$ and $0 < m_\alpha(\Gamma) < \infty$ then the function $F_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{dm_\alpha(\zeta)}{\zeta - z}$ is continuous on the whole complex plane $\mathbb{C}$, but the corresponding integral $F_2(z) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z}$ has unit gap on the curve Γ in all known cases of its existence. This opposition is related with the fact that the formula (1) does not determine a measure in customary sense.

As we mention above, the analysis of certain mechanical problems needs just the integral (1). Its classical theory is based on rectifiability of the curve, but the fractal curve Γ cannot be rectifiable. Therefore all publications on this integral are dealing mainly with various ways for definition of the integral (1) as a new kind of integral.

One of that definitions is based on the Stokes formula

$$\int_\Gamma f(z)\, dz = -\iint_{D^+} \frac{\partial f}{\partial \overline{z}}\, dz d\overline{z}, \tag{2}$$

which is valid if the closed rectifiable curve Γ bounds domain D^+, the function $f(z)$ is continuous in $\overline{D}^+$, and its first partial derivatives are integrable in D^+. Probably, Whitney first noted that the right side of equality (2) can be used as definition of the left one for non-rectifiable curve Γ. This definition is correct only if the right side of (2) has the same value for all functions f with common trace on Γ. This question is considered in the papers [10], [8] for functions f satisfying the Hölder condition with exponent ν on the curve Γ:

$$\sup\left\{\frac{|f(t') - f(t'')|}{|t' - t''|^\nu} : t', t'' \in \Gamma, t' \neq t''\right\} = h_\nu(f, \Gamma) < \infty.$$

There is shown that the equality (2) generates correct definition of integral along non-rectifiable curve under assumption

$$\nu > d - 1, \tag{3}$$

where d is the box dimension of the curve Γ (see its definitions in the books [18] and [4]).

In what follows $H_\nu(\Gamma)$ stands for the Hölder space consisting of defined on Γ functions with finite Hölder coefficient $h_\nu(f, \Gamma)$.

Seemingly, the minimal condition for applicability of the formula (2) for determining of the integral (1) is representability of the function f as the trace on Γ of some function from the Sobolev class $W_1^1(D^+)$. Thus, the space of all functions, which are integrable on Γ in the sense of (2), has to be certain trace space. In the papers [11], [12] this conjecture is proved for the version of the Besov spaces introduced by A. Jonsson and H. Wallin [6].

Another approach to definition of the integral (1) can be characterized as geometrical approximation. If a function $f(z)$ is defined in certain neighborhood of Γ then we can approximate the curve by polygons Γ_n, which converge to it in some sense, and put $\int_\Gamma f(z)\,dz = \lim \int_{\Gamma_n} f(z)\,dz$. This scheme is studied in the papers [8], [9], [7]. Its validity is shown for $f \in H_\nu(\Gamma)$ under the condition (3). In the work [21] the same result is obtained by means of non-standard analysis: the curve Γ is approximated by infinitely close polygon with infinite number of corners.

A dual (with respect to the geometrical approximation) definition of the integral (1) is proposed in the paper [13]. It is based on approximation of the function f. If Γ is Jordan arc with beginning a and end-point b and polynomial sequence $\{p_n(z)\}$ converges on Γ to $f(z)$, then we can put $\int_\Gamma f(z)\,dz = \lim(P_n(b) - P_n(a))$ where $P_n'(z) = p_n(z), n = 1, 2, \ldots$. There is shown that the limit exists if the sequence $\{p_n\}$ converges to f in $H_\nu(\Gamma)$, the exponent ν satisfies condition (3) and the arc Γ does not coil into spirals at its ends.

In the papers [14], [15] the integral (1) is defined as a distribution on the plane $\mathbb{C}$ satisfying certain axioms. Existence of the distribution with required properties is proved there under the same condition (3).

All these definitions of the integral (1) are indirect. In what follows we are dealing with its direct Stieltjes definition.

2. Generalizated rectifiability.

First we define an integral which is slightly more general than the integral (1).

Let Γ be Jordan curve on the plane $\mathbb{C}$ with beginning a and end-point b. As we call the point a by beginning and b by end, so we define an intrinsic ordering relations on the points of this curve: if $z_{1,2} \in \Gamma$ and z_1 precedes z_2 on the curve Γ, then we write $z_1 <_\Gamma z_2$; the notation $z_1 \le_\Gamma z_2$ means that either $z_1 <_\Gamma z_2$ or $z_1 = z_2$. Let $\tau = \{z_j\}_{j=0}^{j=n}$ be ordered sequence of points of the curve Γ, i.e. $a = z_0 <_\Gamma z_1 <_\Gamma z_2 <_\Gamma \ldots <_\Gamma z_n = b$. We call it by partition and put $\operatorname{diam}(\tau) = \max\{|z_j - z_{j-1}| : j = 1, 2, \ldots, n\}$. If, in addition, there are given points $\{w_j\}_{j=1}^{j=n}$ such that $z_{j-1} \leq_\Gamma w_j \leq_\Gamma z_j$, $j = 1, 2, \ldots, n$, then we say that the partition τ is pointed.

Definition 1. *Let functions $f(z)$ and $g(z)$ be defined on directed Jordan curve Γ. For any pointed partition τ we put $S(\tau) = \sum_{j=1}^{n} f(w_j)(g(z_j) - g(z_{j-1}))$. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(\tau) - J| < \varepsilon$ for any pointed partition of diameter $\operatorname{diam}(\tau) < \delta$, then the value J is (Riemann)-Stieltjes integral of the function f with respect to $g(z)$ along Γ, and we write $J = \int_\Gamma f\,dg$.*

This definition gives the integral (1) if $g(z) = z$.

One can easily adapt the definition for closed directed Jordan curve, i.e. for the case $a = b$. In what follows we consider closed curves as oriented counter-clockwise.

Let $z = z(x)$ be continuous one-to-one mapping of segment $I = [0, 1]$ on the curve Γ, $z(0) = a$, $z(1) = b$. Obviously, the sums $S(\tau)$ coinside with the Riemann-Stieltjes sums for customary Stieltjes integral $\int_0^1 f(z(x))\,dg(z(x))$. Therefore

$$\int_\Gamma f\,dg = \int_0^1 f(z(x))\,dg(z(x)),$$

and these integrals exist (or do not exist) simultaneously. The homeomorphism $z(x) : I \mapsto \Gamma$ is not unique, but value and existence of the integral $\int_0^1 f(z(x))\,dg(z(x))$ do not depend on the choice of that mapping.

The classical studies of the Stieltjes integral $\int_0^1 f\,dg$ are based on assumption that the function g has finite variation on the segment I (see, for instance, [23]. But if $z(x)$ has finite variation then its image Γ is rectifiable, what is not interesting for fractal theory. Therefore we must use here another results on existence of the Stieltjes integral.

Let $\Phi(x)$ be real continuous increasing function defined for $x \geq 0$ and $\Phi(0) = 0$. The value of the least upper bound $\sup_\tau \sum_{j=1}^{n} \Phi(\left|f(z_j) - f(z_{j-1})\right|) = v_\Phi(f; \Gamma)$, taken over the set of all partitions of Γ, is called Φ-variation of $f(z)$ on Γ. The class $V_\Phi(\Gamma)$ consists of all functions satisfying condition $v_\Phi(f; \Gamma) < \infty$. If $\Phi(x) = x^p$, $p \geq 1$, then we denote this class by $V_p(\Gamma)$ and corresponding variation by $v_p(f; \Gamma)$; it is called by p-variation.

On the segment I these classes were introduced by L.C. Young (see [26], [27]).

L.C. Young Theorem. [27]. *Let functions f and g belong to the classes $V_\Phi(I)$ and $V_\Psi(I)$ respectively and have not common discontinuities on I. If inverse* *functions* $\phi(x) = \Phi^{-1}(x)$ *and* $\psi(x) = \Psi^{-1}(x)$ *satisfy condition*

$$\sum_{n=1}^{\infty} \phi(1/n)\psi(1/n) < \infty \tag{4}$$

then the Stielties integral $\int_0^1 f\,dg$ *exists.*

If $\Phi(x) = x^p$, $\Psi(x) = x^q$, then the condition (4) reduces to inequality

$$1/p + 1/q > 1. \tag{5}$$

L.C. Young proved also (see [26]) that

$$\left|\int_0^1 f\,dg\right| \le c_{p,q} v_p^{1/p}(f; I) v_q^{1/q}(g; I)$$

if $f(0) = 0$; here $c_{p,q} = 1 + \zeta(\frac{1}{p} + \frac{1}{q})$.

Thus, the integral (1) exists if functions $f(z)$ and $g(z) = z$ belong to suitable classes $V_\Phi(\Gamma)$ and $V_\Psi(\Gamma)$. The following definition is geometric version of the requirement $z \in V_\Psi(\Gamma)$.

Definition 2. *A curve* Γ *is called* Ψ*-rectifiable if the value*

$$\sigma_\Psi(\Gamma) = \sup_\tau \sum_{j=1}^{n} \Psi(|z_j - z_{j-1}|)$$

is finite; the least upper bound is taken over all partitions $\tau = \{z_j\}_{j=1}^n$ *of the curve* Γ. *For* $\Psi(x) = x^p$ *we call it* p*-rectifiable and write* σ_p *instead of* σ_Ψ.

Now we are able to formulate a theorem on existence of the Stieltjes integral (1) along non-rectifiable curve.

Theorem 1. *Let* Γ *be Jordan curve and function* $f(z)$ *be defined on it. Then the following propositions are valid.*

i. If the curve Γ *is* Ψ*-rectifiable,* $f \in V_\Phi(\Gamma)$ *and the condition* (4) *fulfils, then the integral (1) exists.*

ii. If the curve Γ *is* q*-rectifiable,* $f \in V_p(\Gamma)$ *and the condition* (5) *fulfils, then the integral (1) exists and satisfies inequality*

$$\left|\int_\Gamma f\,dz\right| \le |f(a)(b-a)| + c_{p,q} v_p^{1/p}(f; \Gamma)\sigma_q^{1/q}(\Gamma).$$

iii. If the curve Γ *is* q*-rectifiable,* $f \in H_\nu(\Gamma)$ *and*

$$\nu > q - 1 \tag{6}$$

then the integral (1) exists and

$$\left|\int_\Gamma f\,dz\right| \le |f(a)(b-a)| + c_{q/\nu,q} h_\nu(f;\Gamma)\sigma_q^{\frac{\nu+1}{q}}(\Gamma).$$

iv. If $f \in V_1(\Gamma)$ *then the integral (1) exists for any Jordan curve* Γ.

v. If this integral satisfies inequality

$$\left|\int_\Gamma f\,dz\right| \le C \sup\{|f(z)| : z \in \Gamma\}$$

for any continuous $f \in V_1(\Gamma)$ *and for certain* $C > 0$ *then the curve* Γ *is rectifiable.*

Proof. Obviously, $f(z) \in V_\Phi(\Gamma)$ if and only if $f(z(x)) \in V_\Phi(I)$, and propositions (i) and (ii) follows from mentioned above L.C.Young's results. If $f \in H_\nu(\Gamma)$ then $\sum_{j=1}^n \left|f(z_j) - f(z_{j-1})\right|^{q/\nu} \le h_\nu^{q/\nu}(f;\Gamma)\sum_{j=1}^n |z_j - z_{j-1}|^q \le h_\nu^{q/\nu}(f;\Gamma)\sigma_q(\Gamma)$. Hence, $f \in V_p(\Gamma)$ for $p = q/\nu$ and $v_{q/\nu}(f;\Gamma) \le h_\nu^{q/\nu}(f;\Gamma)\sigma_q(\Gamma)$. Therefore the condition (6) follows from (5) and proposition (iii) from (ii). As known, the Stieltjes integral $\int_0^1 f\,dg$ exists if function g is continuous and $f \in V_1(I)$ (see, for example, [23]); this result proves the proposition (iv). The proposition (v) is a consequence of the Riesz theorem concerning bounded functionals on the space of continuous functions.

Note 1. The bounds for the integral (1) in the case where the curve Γ is Ψ-rectifiable and $f \in V_\Phi(\Gamma)$ can be obtained from results of the paper [27] (see also [17]).

Note 2. A.M. Dyachkov [2] describes the class of all functions $f(x)$ such that the Stieltjes integral $\int_0^1 f(x)\,dg(x)$ exists for a fixed function $g \in V_\Psi(I)$. This result enables us to desribe the class of all functions $f(z)$ such that the integral

$$\int_\Gamma f(z)\,dz = \int_0^1 f(z(x))\,dz(x)$$

exists for a given Ψ-rectifiable curve Γ with fixed mapping $z(x) : I \mapsto \Gamma$. It is interesting to make this description independent on the choice of the homeomorphism $z(x)$.

Note 3. In connection with proposition (v) of the theorem and cited above result of [13] the following question arises: if $|P(b) - P(a)| \leq C \sup\{|P'(z)| : z \in \Gamma\}$ for any polynomial P and certain $C > 0$ then the curve Γ is rectifiable, isn't it?

Now we describe certain properties of p-rectifiable curves.

Theorem 2. *A Jordan curve Γ is p-rectifiable if and only if there exists homeomorphism $z = z(x) : I \mapsto \Gamma$ belonging to Hölder class $H_{1/p}(I)$. The box dimension d of p-rectifiable curve Γ does not exceed p.*

Proof. Let us consider arc Γ_t of the curve Γ with beginning a and end $t \in \Gamma$. The function $s(t) = \sigma_p(\Gamma_t)$ continuously increases (in the sense of ordering relation on Γ which is defined at the beginning of the section). It maps Γ on the segment $[0, \sigma]$ where $\sigma = \sigma_p(\Gamma)$. Then the function $x(t) = s(t)/\sigma$ is homeomorphism of Γ onto I. If $t \leq_\Gamma t'$ then $x(t') - x(t) \geq |t' - t|^p/\sigma$. Therefore the inverse function $z(x) : I \mapsto \Gamma$ satisfies inequality $|z(x) - z(x')| \leq (\sigma|x - x'|)^{1/p}$, i.e. it belongs $H_{1/p}(I)$.

In order to prove the second proposition of the theorem we fix $\varepsilon > 0$ and divide the complex plane into grid of squares with mesh ε. Let $N(\varepsilon)$ be number of squares Q such that intersection $Q \cap \Gamma$ is not empty. Now we consider a special partition of Γ. Let $z_0 = a$. We define z_1 as the minimal (with respect to ordering $\leq_\Gamma$) point satisfying conditions $|z_1 - z_0| = \varepsilon$ and $z_0 <_\Gamma z_1$; if that point does not exist then we put $z_1 = b$. Analogously, z_2 is minimal point of the set $\{z \in \Gamma : |z_2 - z_1| = \varepsilon, z_1 <_\Gamma z_2\}$, and $z_2 = b$ if this set is empty, and so on. As a result we obtain partition $\tau = \{z_j\}_{j=1}^m$ such that $|z_j - z_{j-1}| = \varepsilon$ for $j = 1, \dots, m-1$, and $|z_m - z_{m-1}| \leq \varepsilon$. Hence, $(m-1)\varepsilon^p < \sum_{j=1}^m |z_j - z_{j-1}|^p \leq \sigma$ and $m < 1 + \sigma\varepsilon^{-p}$. On the other hand, any arc of Γ with beginning z_{j-1} and end z_j, $j = 1, \dots, m$, is contained in a disk of radius ε which intersects no more than 12 squares. Thus, $N(\varepsilon) \leq 12m < c + c\varepsilon^{-p}$, where constant c does not depend on ε, and $d = \limsup \frac{\log N(\varepsilon)}{-\log\varepsilon} \leq p$ by definition of the box dimension. Theorem is proved.

Note 4. Analogously, Γ is Ψ-rectifiable if and only if there exists homeomorphism $z = z(x) : I \mapsto \Gamma$ such that $|z(x) - z(x')| \leq \psi(c|x - x'|)$ for any $x, x' \in I$ and certain $c > 0$, and box dimension d of Ψ-rectifiable curve Γ does not exceed $\limsup \frac{\log \Psi(\varepsilon)}{\log \varepsilon}$.

By virtue of Theorem 2 the condition (6) is more restrictive than (3). But there exist a number of interesting curves such that $p = d$. For example, the von Koch snowflake is p-rectifiable for $p = \frac{\log 4}{\log 3}$ and its box dimension equals to $\frac{\log 4}{\log 3}$, too. On the other hand, one can easily find curves such that $p > d$ and even $p > 2$. An interest example of that curve is graph of the function

$$y(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} h(2^n x),\ 0 \leq x \leq 1,$$

where $1-$periodic hat function $h(x)$ equals to x for $0 \leq x \leq 1/2$ and to $1-x$ for $1/2 < x \leq 1$. The immediate calculation shows that the graph is $p-$rectifiable if and only if $p \geq 1/\alpha$, and its box dimension is $2-\alpha$ (here $0 < \alpha < 1$).

Now we consider relations between the Stieltjes integral along Γ and inderect definitions of the integral along this curve (see introduction).

Let the curve Γ be q-rectifiable and the function f be defined in neighborhood of Γ and satisfy there the Hölder condition with exponent $\nu > q-1$. If Γ_n is polygonal line with beginning a, end b and with corner points $z_{n,1}, z_{n,2}, \dots, z_{n,k_n}$ on the curve Γ, then it is image of piecewise linear function $z_n(x)$ defined on the segment I. We denote by $x_{n,j}$ pre-image of $z_{n,j}$. Let us choose the sequence $\{\Gamma_n\}$ so that all corner points of Γ_n are corner points of Γ_{n+1}, $n = 1, 2, \dots$, and the sets $\cup_{n=1}^{\infty} \cup_{j=1}^{k_n} x_{n,j}$ and $\cup_{n=1}^{\infty} \cup_{j=1}^{k_n} z_{n,j}$ are dense everywhere on I and Γ respectively. Then the sequences of functions $\{z_n(x)\}$ and $\{f(z_n(x))\}$ converge to $z(x)$ and $f(z(x))$ and have bounded in common q-variations and $\frac{q}{\nu}$-variations respectively (see [17]). Therefore $\lim \int_0^1 f(z_n(x))\,dz_n(x) = \int_0^1 f(z(x))\,dz(x)$ (see [26], [27], [17]), and, consequently, $\lim \int_{\Gamma_n} f(z)\,dz(x) = \int_\Gamma f(z)\,dz$.

Let Γ and Γ_n be closed curves. Then we apply the Stokes' formula to polygons Γ_n and obtain

Theorem 3. *Let D be finite domain on the complex plane with q-rectifiable Jordan boundary Γ. If a function f is defined in $\overline{D}$, belongs to the Hölder class $H_\nu(\overline{D})$ and has integrable in $\overline{D}$ derivative $\frac{\partial f}{\partial \bar{z}}$, then under condition* (6) *the Stokes' formula* (2) *is valid for these domain and function.*

Thus, the Young-Stieltjes integration under condition (6) gives the same result as the integrations in terms of geometrical approximation and by means of the Stokes' formula. As we has noted above, this conditions coincides with condition (3) for classical fractal curves such as von Koch snowflake.

3. The Cauchy type integral and self-similarity.

Let Γ be closed Ψ-rectifiable curve dividing the plane $\mathbb{C}$ into two domains D^+ and D^- so that $\infty \in D^-$, and $f(z) \in V_\Phi(\Gamma)$. Then the integral

(7) $$K_f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)dt}{t-z}$$

exists under condition (4) if the point z does not lie on the curve Γ. This expression is known as the Cauchy type integral. It is holomorphic in D^+ and D^- and vanishes at the point ∞. This integral is applicable in a number of boundary value problems (see [20], 45]), and one of its main properties in this connection is existence of boundary values on the curve Γ.

If the curve Γ is piecewise-smooth then the function $K_f(z)$ is continuous in $\overline{D}^+$ and $\overline{D}^-$ if f satisfies the Hölder condition with any exponent $\nu > 0$. This result was proved by Harnak, Morera and Sokhotskii (see [20]). Therefore we call it by *HMS-theorem*. If Γ is rectifiable but non-smooth then the HMS-theorem is not valid in general. E.M. Dynkin [3] proved that the Cauchy type integral along non-smooth rectifiable curve with density $f \in H_\nu$ is continuous in $\overline{D}^+$ and $\overline{D}^-$ under condition $\nu > 1/2$. Simultaneously and independently this result was obtained by T. Salimov [24].

Here we prove non-rectifiable versions of both these results. We obtain certain analog of the Dynkin-Salimov theorem for general q-rectifiable curves, and a version of the HMS-theorem for self-similar ones.

Theorem 4. *Let Γ be closed q-rectifiable curve with box dimension d and $f \in H_\nu(\Gamma)$. If the values q, d and ν satisfy conditions* (6) *and*

$$\nu > d/2 \tag{8}$$

then the Cauchy type integral (7) *has continuous limit values on Γ from both domains D^+ and D^-.*

Proof. Let $f^w(z)$ be the Whitney extension of f from Γ on the whole complex plane. As shown in [16], the derivative $\frac{\partial f^w}{\partial \overline{z}}$ is integrable in D^+ with any exponent less than $\frac{2-d}{1-\nu}$. By virtue of Theorem 2 this derivative is integrable under condition (6). Hence, by virtue of Theorem 3 the Stokes formula is valid for the function f^w and the domain D^+. This formula enables us to represent the Cauchy type integral in the form

$$K_f(z) = \chi(z) f^w(z) - \frac{1}{2\pi i} \iint_{D^+} \frac{\partial f^w}{\partial \overline{t}} \frac{dt d\overline{t}}{t - z}, \tag{9}$$

where function $\chi(z)$ equals to 1 in D^+ and 0 in D^-. If $\nu > d/2$ then $\frac{2-d}{1-\nu} > 2$ and $\frac{\partial f^w}{\partial \overline{z}}$ is integrable in D^+ with exponent exceeding 2. Consequently (see, for instance, [25]), the integral in the right side of the representation is continuous in the whole complex plane. Theorem is proved.

If Γ is rectifiable then $q = d = 1$, and conditions (6), (8) reduce to inequality $\nu > 1/2$. Thus, Theorem 4 is q-rectifiable version of the Dynkin-Salimov theorem.

If $\nu = 1$ then the derivative $\frac{\partial f^w}{\partial \bar{z}}$ is bounded in D^+ for any d. The condition (6) reduces to inequality $q < 2$, and we obtain

Theorem 5. *Let* Γ *be closed* q*-rectifiable curve,* $q < 2$ *and* $f \in H_1(\Gamma)$. *Then the Cauchy type integral* (7) *has continuous limit values on* Γ *from both domains* D^+ *and* D^-.

For $q = 1$ this result was proved by N.A. Davydov [1].

Note 5. In connection with the proposition (iv) of the Theorem 1 the following conjecture arises: the Davydov theorem keeps its validity for arbitrary closed Jordan curve Γ if $f \in H_1(\Gamma)$ has bounded variation.

The proof of Theorem 4 contains certain additional information. The product $\chi(z) f^w(z)$ has the gap $f(z)$ on the curve Γ. Therefore the difference of limit values of the integral (7) from D^+ and D^- at any point $z \in \Gamma$ equals to $f(z)$. This property can be applied for resolvation of the Riemann boundary value problem in just the same way as analogous property of the Cauchy type integral along piecewise-smooth curve (see [20], [5]). Furthermore, the known estimations of integral term of the right side of (9) (see, for instance, [25]) togather with mentioned above result from [16] imply that the function $K_f(z)$ satisfies the Hölder condition in the sets $\overline{D}^+$ and $\overline{D}^-$ with exponent $\frac{2\nu-d}{2-d} - \varepsilon$ for arbitrarily small $\varepsilon > 0$.

Now we consider a version of HMS-theorem for self-similar curves. Its proof in the classical (i.e. piecewise smooth) case consists of the following three steps (see [20]).

Step 1. The improper integral

$$k_{f,\Gamma}(t_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t) - f(t_0)}{t - t_0} dt \tag{10}$$

converges for any point $t_0 \in \Gamma$ if $f \in H_\nu(\Gamma)$, $\nu > 0$.

Step 2. If λ iz non-tangential path connecting a point $w \in \mathbb{C} \setminus \Gamma$ with t_0 then

$$\frac{|z - t_0|}{\operatorname{dist}(z, \Gamma)} \leq C, \;\; z \in \lambda, \tag{11}$$

where constant C depends only on the angle between Γ and λ. Therefore the difference

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(t) - f(t_0)}{t - z} dt - k_{f,\Gamma}(t_0) = \frac{1}{2\pi i} \int_\Gamma \frac{z - t_0}{z - t} \frac{f(t) - f(t_0)}{t - t_0} dt$$

vanishes for $\lambda \ni z \to t_0$. Consequently,

$$K_f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)-f(t_0)}{t-z}dt + f(t_0)\chi(z)$$

has non-tangential limits at any point $t_0 \in \Gamma$ from the both sides.

Step 3. The convergence of K_f to its non-tangential boundary limits is uniform with respect to t_0. Therefore this function has boundary limits along any paths, i.e. it is continuous in closures of domains $D^{\pm}$.

At least two of these steps are invalid even for non-smooth rectifiable curves. For example, the arc $\gamma = \{t = re^{ir^{-\mu}} : 0 \le r \le 1\}$ is rectifiable for $\mu < 1$ and the function $f(t) = |t|^\nu$ belongs to $H_\nu(\gamma)$, but the integral $k_{f,\gamma}(0) = \frac{1}{2\pi i}\int_\gamma |t|^\nu \frac{dt}{t}$ converges for $\nu > \mu$ only. Moreover, for $\Gamma = \gamma$, $t_0 = 0$ and any $C > 0$ the inequality (11) is not valid for any arc λ with end-point t_0, i.e. there is not any non-tangential path even in the weak sense (11).

But the property of self-similarity ensures convergence of the improper integral (10) under the same condition (6). We shall say that an arc γ with beginning (or end) at a point t_0 is *strictly self-similar at this point* if it is representable in the form $\cup_{j=0}^{\infty}\gamma_j$ where arcs γ_j have not common inner points, $\overline{\gamma}_0$ does not contain the point t_0 and the arc γ_{j+1} is obtained from γ_j by similarity mapping $t \mapsto t_0 + k(t-t_0)$, $|k| < 1$, $j = 0, 1, \ldots,$. The value k is called by coefficient of similarity. For example, if Γ is von Koch snowflake constructed on the base of triangle with corners at the points 0, 1 and $\frac{1+i\sqrt{3}}{2}$, then its arc γ with beginning 0 and end-point 1 is strictly self-similar at the point $t_0 = 0$. As γ_0 we can take its subarc with beginning $\frac{1}{3}$ and end-point 1, and coefficient of similarity is 1/3.

Lemma 1. *If q-rectifiable arc γ is strictly self-similar at its point t_0, $f \in H_\nu(\gamma)$ and the condition* (6) *fulfils, then the integral $k_{f,\gamma}(t_0)$ converges.*

Proof. We can determine the improper integral (8) as the series

$$k_{f,\gamma}(t_0) = \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_{\gamma_j}\frac{f(t)-f(t_0)}{t-t_0}dt.$$

The obvious changes of variable gives

$$k_{f,\gamma}(t_0) = \frac{1}{2\pi i}\int_{\gamma_0}\sum_{j=0}^{\infty}(f(t_0+k^j(t-t_0)) - f(t_0))\frac{dt}{t-t_0}.$$

The series $\sum_{j=0}^{\infty}(f(t_0+k^j(t-t_0))-f(t_0))$ converges uniformly on γ_0 to certain function $f_0(t)$ because

$$|f(t_0+k^j(t-t_0))-f(t_0)| \leq h_\nu(f;\gamma)k^{j\nu}|t-t_0|^\nu.$$

If $t', t'' \in \gamma_0$ then

$$|f_0(t')-f_0(t'')| \leq \sum_{j=0}^{\infty} h_\nu(f;\gamma)|k^j t' - k^j t''|^\nu = \frac{h_\nu(f;\gamma)}{1-|k|^\nu}|t'-t''|^\nu,$$

i.e. $f_0 \in H_\nu(\gamma_0)$. As γ_0 does not contain t_0 so the function $f_0(t)(t-t_0)^{-1}$ belongs to $H_\nu(\gamma_0)$ too, and the integral

$$k_{f,\gamma}(t_0) = \frac{1}{2\pi i}\int_{\gamma_0}\frac{f_0(t)dt}{t-t_0}dt$$

exists by virtue of Theorem 1.

In what follows we say that closed curve Γ is *uniformly self-similar* if any its point t is beginning and end of arcs $\gamma' \subset \Gamma$ and $\gamma'' \subset \Gamma$ such that both these arcs are strictly self-similar at the point t with common coefficient of similarity k which does not depend on t. Under that assumption the function $k_{f,\Gamma}(t)$ is defined for any $t \in \Gamma$. Clearly, the von Koch snowflake is uniformly self-similar.

In order to perform the steps 2 and 3 we must be able to find a family of arcs $\lambda^{\pm}(t)$, $t \in \Gamma$, such that any point $t \in \Gamma$ is end-point of two arcs $\lambda^+ \subset D^+$ and $\lambda^- \subset D^-$, the inequality (11) fulfils for all arcs of the family with the same constant C, and the arcs $\lambda^+(t_1)$ and $\lambda^+(t_2)$ (respectively, $\lambda^-(t_1)$ and $\lambda^-(t_2)$) have a common point in D^+ (respectively, D^-) if $|t_1-t_2| \leq \varepsilon$. Then we say that Γ is *uniformly attainable from the both sides*. The von Koch snowflake satisfies this condition, too. As a result we obtain

Theorem 6. *Let a closed q-rectifiable curve Γ be uniformly self-similar and uniformly attainable from the both sides. If $f \in H_\nu(\Gamma)$ and the values q and ν satisfy condition* (6)*, then the Cauchy type integral* (7) *has continuous limit values on Γ from both domains D^+ and D^-.*

The conditions of the theorem fulfils for all versions of von Koch curves (see [4]), i.e. for all curves which are obtained from equilateral polygons without null angles by means of the Mandelbrot procedure.

If Γ is rectifiable then $q = 1$, and condition (6) reduce to trivial bound $\nu > 0$, i.e., the Theorem 6 is q-rectifiable version of HMS-theorem. Thus, the self-similarity yields the same property of the Cauchy type integral along a curve as the smoothness of this curve.

4. Open questions.

Naturally, a great body of open questions and conjectures is connected with the subject under consideration. Above we have described some questions (see Notes 2, 3, 5). Here we formulate another problems concerning integration along non-rectifiable and fractal curves.

4.1. The Haar expansion.

Lesniewicz and Orlicz [17] apply the Haar and Schauder expansions of f and g respectively as one of main instruments for investigation of the integral $\int_0^1 f\,dg$. Seemingly, the expansions in the Haar functions and another wavelet type systems of functions give us an intrinsic way for definition of the integral $\int_\Gamma f\,dz$ for functions f with rather extensive sets of singularities.

4.2. The Hellinger integral.

V.I. Matsaev and M.Z. Solomiak [19] considered so called Hellinger integral. Its definition can be obtained from the definition of Riemann - Stieltjes integral $\int_0^1 f\,dg$ by means of change of point values $f(y_j)$, $x_{j-1} \le y_j \le x_j$, in the integral sums by averages $\frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} f(x)dx$. That integral exists if f belongs to certain Besov space. Analogously, we can replace the values $f(w_j)$ in the definition 1 by averages $\frac{1}{\mu(\gamma_j)} \int_{\gamma_j} f d\mu$ where $\gamma_j \subset \Gamma$ is arc connecting z_{j-1} with z_j, and μ is certain positive measure on the curve Γ (for example, the Hausdorff measure). Probably, that integral exists if the function f belongs to the version of Besov space introduced in [6].

4.3. Extension of the Stokes formula.

Stokes' formula (2) is proved in Theorem 3 under condition (6). This condition is rather restrictive. For instance, if Γ is q-rectifiable with $q \ge 2$ then this condition is satisfied only by constants. On the other hand, the conditions (4) and (5) make a sense in this situation. The author supposes that Stokes' formula (2) is valid if the function $f(z)$ is continuous in $\overline{D^+}$ and differentiable in D^+, its first partial derivatives are integrable in $\overline{D^+}$, its boundary values belong to $V_\Phi(\Gamma)$, the curve Γ is Ψ-rectifiable and condition (4) fulfils.

4.4. Fractal version of the Privalov theorem.

The HMS-theorem establishes that the Cauchy type integral has continuous limit values on the curve Γ if this curve is piecewise-smooth and f satisfies Holder condition with any exponent ν. I.I. Privalov [22] obtained the following sharpening of this result: if the piecewise-smooth curve Γ has not cusps and $\nu < 1$ then the limit values of the Cauchy type integral (7) on the curve satisfy the Hölder condition with the same exponent ν. Certain versions of the Privalov theorem seem to be valid for self-similar curves. Particularly, the author supposes that the limit values satisfy the Hölder condition with exponent $\nu - d + 1$.

4.5. Singularities of the Cauchy type integral.

If $d/2 \geq \nu > q - 1$ (we use notation of Theorem 4) then the Cauchy type integral exists but it has not limit values at some points of the curve Γ (cf. [16]). There is of interest to describe the set of boundary singularities of this integral. Probably, its Hausdorff dimension cannot exceed $q - 1$.

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