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CLOSED FORM REPRESENTATION OF BINOMIAL SUMS AND SERIES

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This paper deals with the classical quest for 'closed form' expressions of binomial sums and series. We shall consider a generalised Binomial sum and some relations and investigate several methods for its representation in closed form. In the process of our analysis we shall 'discover' several new identities and the closed form representation of a related series depending on a parameter.

1. Introduction.

Many methods and techniques are available for identity representations. Residue theory and contour integration can be gainfully employed to express certain sums in closed form. For example, Flajolet and Salvy [4] apply contour integral methods to obtain an identity, originally given by Ramanujan,

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700},$$

and currently Borwein [1] and his coworkers are carrying out a great deal of exciting work on symbolically discovered identities. Finally, recently Efthimiou

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[3] uses an elegant method, related to Laplace transforms, and originally given by Wheelon [12], to obtain closed form expressions for sums of the form

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{j} (an+k)}.$$

We shall investigate in this paper, the generalised binomial sum

$$f_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} (-a)^k \binom{bk}{k} \tag{1.1}$$

and some of its variations. Finally we find the closed form representation of the series

$$\sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}((m+k)!)^2},$$

where m is a non negative integer.

2. Closed form representation of binomial sums.

Consider the generalised sum (1.1),

$$f_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} (-a)^k \binom{bk}{k} = \sum_{k=0}^{n} T_k \tag{2.1}$$

for a real and b integer. The ratio of consecutive terms is

$$\frac{T_{k+1}}{T_k} = \frac{ab^b (k-n)}{(b-1)^{b-1}(k+1)^2} \frac{\prod_{j=1}^{b-1}\left(k+\frac{b-j}{b}\right)}{\prod_{j=2}^{b-1}\left(k+\frac{b-j}{b-1}\right)}, \tag{2.2}$$

$T_0 = 1$, and hence from (2.2), in terms of the generalised hypergeometric function ${}_xF_{x-1}\left[\begin{matrix} ., ..., . \\ ., ..., . \end{matrix} \middle| . \right]$, we have

$$f_n(a,b) = {}_bF_{b-1}\left[\begin{matrix} \frac{b-1}{b}, \frac{b-2}{b}, \frac{b-3}{b}, ..., \frac{1}{b}, -n \\ 1, \frac{b-2}{b-1}, \frac{b-3}{b-1}, ..., \frac{1}{b-1} \end{matrix} \middle| \frac{ab^b}{(b-1)^{b-1}} \right], \tag{2.3}$$

for $b \geq 2$. For the relatively simple case of $b = 1$, from (2.1)

$$f_n\,(a, 1) = {}_1F_0\left[\begin{matrix}-n\\ -\end{matrix}\,\middle|\, a\right] = (1-a)^n .$$

Now, we concentrate on the case of $b = 2$; from (2.3)

$$f_n\,(a, 2) = {}_2F_1\left[\begin{matrix}\frac{1}{2}, -n\\ 1\end{matrix}\,\middle|\, 4a\right] \tag{2.4}$$

and a recurrence relation for (2.4) obtained from the Zb algorithm in Mathematica, is

$$\left.\begin{matrix}(n+2)\, f_{n+2} + (2n+3)\,(2a-1)\, f_{n+1} + (n+1)\,(1-4a)\, f_n = 0,\\ f_0\,(a, 2) = 1,\ f_1\,(a, 2) = 1 - 2a.\end{matrix}\right\} \tag{2.5}$$

We can see from (2.5) that for two special cases of $a = 1/2$ and $a = 1/4$ the recurrence relation (2.5) becomes manageable. From (2.4) let $a = 1/2$ such that

$$f_n\left(\frac{1}{2}, 2\right) = {}_2F_1\left[\begin{matrix}\frac{1}{2}, -n\\ 1\end{matrix}\,\middle|\, 2\right] \tag{2.6}$$

and replacing k with $n-k$ we have

$$f_n\left(\frac{1}{2}, 2\right) = T_0\, {}_2F_1\left[\begin{matrix}-n, -n\\ \frac{1}{2} - n\end{matrix}\,\middle|\, \frac{1}{2}\right],\ T_0 = \left(\frac{-1}{2}\right)^n \binom{2n}{n}. \tag{2.7}$$

There is an identity, due to Gauss, see Graham, Knuth and Patashnik [6], which states

$${}_2F_1\left[\begin{matrix}\alpha_1, \alpha_2\\ \alpha_1 + \alpha_2 + \frac{1}{2}\end{matrix}\,\middle|\, 1\right] = {}_2F_1\left[\begin{matrix}2\alpha_1, 2\alpha_2\\ \alpha_1 + \alpha_2 + \frac{1}{2}\end{matrix}\,\middle|\, \frac{1}{2}\right], \tag{2.8}$$

hence from (2.8) and (2.7)

$$f_n = \left(\frac{-1}{2}\right)^n \binom{2n}{n}\, {}_2F_1\left[\begin{matrix}\frac{-n}{2}, \frac{-n}{2}\\ \frac{1}{2} - n\end{matrix}\,\middle|\, 1\right]. \tag{2.9}$$

Using the classical Gauss formula

$${}_2F_1\left[\begin{matrix}\alpha_1, \alpha_2\\ \alpha_3\end{matrix}\,\middle|\, 1\right] = \frac{\Gamma\,(\alpha_3)\,\Gamma\,(\alpha_3 - \alpha_1 - \alpha_2)}{\Gamma\,(\alpha_3 - \alpha_2)\,\Gamma\,(\alpha_3 - \alpha_1)}$$

we obtain from (2.9)

$$f_n = \left(\frac{-1}{2}\right)^n \binom{2n}{n} \frac{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{2}-\frac{n}{2}\right)}, \tag{2.10}$$

(for n even only), such that when n is odd $f_n = 0$ and when n is even, from (2.10)

$$f_{2n} = \binom{4n}{2n} \frac{1}{4^n} \left(\frac{(2n)!}{2^n n!}\right)^2 \frac{4^n (2n)!}{(4n)!} = 2^{-2n} \binom{2n}{n}.$$

Also, from (2.5) for $a = 1/2$ we have that $(n+2) f_{n+2} - (n+1) f_n = 0$ and hence the Reed Dawson identities follow, namely

$$f_{2n}\left(\frac{1}{2}, 2\right) = \sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k \binom{2k}{k} = 2^{-2n} \binom{2n}{n} \tag{2.11}$$

and

$$f_{2n+1}\left(\frac{1}{2}, 2\right) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} \left(-\frac{1}{2}\right)^k \binom{2k}{k} = 0. \tag{2.12}$$

Both (2.11) and (2.12) have been considered by Riordan [10]. The sums (2.11) and (2.12), or their generic representation (2.1) for $a = \frac{1}{2}$ and $b = 2$, arise in the work of Jonassen and Knuth [8] in an algorithm known as tree search and insertion and were also investigated by Greene and Knuth [7] and Rousseau [7].

For $a = 1/4$, from (2.4)

$$f_n\left(\frac{1}{4}, 2\right) = {}_2F_1\left[\left.\begin{matrix}\frac{1}{2}, -n \\ 1\end{matrix}\right| 1\right],$$

and from (2.5)

$$(n+2) f_{n+2} - \frac{1}{2}(2n+3) f_{n+1} = 0,$$

hence

$$f_n = 2^{-n} \prod_{j=0}^{n-1} \left(\frac{2j+1}{j+1}\right) = \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\sqrt{\pi}} = 2^{-2n} \binom{2n}{n},$$

also from (2.1), $f_{2n}(1/2, 2) = f_n(1/4, 2)$. For $b = 3$, a recurrence relation, using the Zb algorithm in Mathematica, $f_n(a, 3) = f_n$, of (2.1) is

$$\left.\begin{matrix} 2(n+3)(2n+5) f_{n+3} + \big(n^2(27a-12) + \\ n(135a-56) + 168a - 66\big) f_{n+2} + \\ 2(n+2)(3n(2-9a) + 11 - 54a) f_{n+1} + \\ (27a-4)(n+1)(n+2) f_n = 0, \\ f_0 = 1,\ f_1 = 1-3a,\ f_2 = 1-3a+15a^2 \end{matrix}\right\} \tag{2.13}$$

The recurrence (2.13) does not lend itself to easy closed form evaluations for any special values of a. A variation of the sum (1.1) is

$$g_n(a,b) = \sum_{k=0}^{n} \frac{\binom{n}{k}(-a)^k}{\binom{bk}{k}} \tag{2.14}$$

and in hypergeometric notation

$$g_n(a,b) = {}_bF_{b-1}\left[\begin{matrix} 1, \frac{b-2}{b-1}, \frac{b-3}{b-1}, \ldots, \frac{1}{b-1}, -n \\ \frac{b-1}{b}, \frac{b-2}{b}, \frac{b-3}{b}, \ldots, \frac{1}{b} \end{matrix} \middle| \frac{a(b-1)^{b-1}}{b^b}\right].$$

For $b = 1$, $g_n(a,1) = f_n(a,1) = (1-a)^n$. For $b = 2$,

$$g_n(a,2) = {}_2F_1\left[\begin{matrix} 1, -n \\ \frac{1}{2} \end{matrix} \middle| \frac{a}{4}\right]$$

which has a recurrence relation

$$2(2n+1)g_{n+1} + (n+1)(a-4)g_n + 2 = 0,\ g_0 = 1.$$

In the specific case of $a = 4$, we obtain the identity

$$g_n(4,2) = \sum_{k-0}^{n} \frac{\binom{n}{k}(-4)^k}{\binom{2k}{k}} = \frac{1}{1-2n},$$

evaluated by Riordan [10] and it may be easily verified, utilizing the procedure described by Petkovšek et al. [9], by the rational certificate function

$$R(n,k) = \frac{k(1-2k)}{(n+1-k)(2n-1)}.$$

For $b = 3$, a recurrence relation of (2.14), using the Zb algorithm in Mathematica, is

$$\left.\begin{matrix} 3(3n+4)(3n+5)g_{n+2} - 2(n+2)(n(27-2a)+27-3a)g_{n+1} + \\ (4a-27)(n+1)(n+2)g_n - 6 = 0,\ g_0 = 1,\ g_1 = 1 - \frac{1}{2}a \end{matrix}\right\},$$

and again it does not lend itself to easy closed form evaluations for any special values of a.

Another variation of (1.1) is the related sum

$$S_n(p,q) = \sum_{r=0}^{qn} (-1)^r \binom{qn}{r}^p \tag{2.15}$$

for p and q integers. For $q=1$ and $p=1$, (2.15) is identical to (2.1) for $a=1$ and $b=1$. From (2.15) we have

$$S_n(p,q) = {}_pF_{p-1}\left[\begin{matrix}-qn,-qn,-qn,...,-qn\\1,1,1,...,1\end{matrix}\,\middle|\,(-1)^{p+1}\right], \tag{2.16}$$

and some special cases, from (2.16), are

$$S_n(1,q) = {}_1F_0\left[\begin{matrix}-qn\\-\end{matrix}\,\middle|\,1\right] = \begin{cases}0 \text{ if } qn \in Z^+\\ 1 \text{ if } qn = 0\end{cases}$$

and

$$S_n(p,2) = {}_pF_{p-1}\left[\begin{matrix}-2n,-2n,-2n,...,-2n\\1,1,1,...,1\end{matrix}\,\middle|\,(-1)^{p+1}\right]. \tag{2.17}$$

It is known that $S_n(2,2) = (-1)^n\binom{2n}{n}$, $S_n(3,2) = (-1)^n\binom{3n}{n}\binom{2n}{n}$ and therefore $S_n(3,2) = \binom{3n}{n}S_n(2,2)$; however for $p \geq 4$, deBruijn [2] showed that (2.17) cannot be expressed as a ratio of products of factorials, and Graham et al. [5] also showed this by an application of the multidimensional saddle point method. We can deduce, from (2.16) and by the use of Gauss' formula, the identity

$$S_n(2,q) = {}_2F_1\left[\begin{matrix}-qn,-qn\\1\end{matrix}\,\middle|\,-1\right] = \frac{2^{qn+1}}{B\left(\frac{2+qn}{2},\frac{1-qn}{2}\right)} \tag{2.18}$$

for $q \neq 1$ and n even, where $B(x,y)$ is the Beta function. From (2.15) and (2.16) we may also deduce that

$$\begin{aligned}S_{2n+1}(p,1) &= \sum_{r=0}^{2n+1}(-1)^r\binom{2n+1}{r}^p\\ &= {}_pF_{p-1}\left[\begin{matrix}-(2n+1),...,-(2n+1)\\1,...,1\end{matrix}\,\middle|\,(-1)^{p+1}\right] = 0,\end{aligned}$$

$$
\begin{aligned}
S_{2n}(p,1) &= \sum_{r=0}^{2n} (-1)^r \binom{2n}{r}^p \\
&= (-1)^n \binom{2n}{n}^p + 2\sum_{r=0}^{n-1} (-1)^r \binom{2n}{r}^p
\end{aligned}
$$

and utilizing (2.18), gives the new result

$$
\sum_{r=0}^{n} (-1)^r \binom{2n}{r}^2 = \frac{2^{2n-1}\sqrt{\pi}}{n!\Gamma\left(\frac{1-2n}{2}\right)} + \frac{(-1)^n}{2}\binom{2n}{n}^2 .
$$

The sum (2.15) may, for specific cases of p and q, be written as a recurrence relation. Another related sum is given by Strehl [11], whom in an informative paper shows that, for all natural numbers n

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \\
&= {}_4F_3\left[\begin{matrix} n+1, n+1, -n, -n \\ 1, 1, 1 \end{matrix}\middle| 1\right].
\end{aligned} \tag{2.19}
$$

Strehl offers six different proofs of (2.19) based on:

- Bailey's bilinear generating function for the Jacobi polynomials in the special case when the Jacobi polynomials reduce to Legendre polynomials,
- A combinatorial approach to the Bailey identity,
- Legendre inverse pairs,
- the Pfaff-Saalschütz identity,
- Zeilberger's algorithm, and
- known recurrences for the Franel and Apèry numbers.

From (2.19), after various manipulations Strehl obtains

$$
\sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{(\lambda+1)^2}{\lambda}\right)^k \tag{2.20}
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda+1}{\lambda}\right)^k \sum_{j=0}^{k} \lambda^j \binom{k}{j}^2
$$

$$
= {}_2F_1\left[\begin{matrix} -n, -n \\ 1 \end{matrix}\middle| 2+\lambda+\frac{1}{\lambda}\right]. \tag{2.21}
$$

Given that $2\lambda_{1,2} = -3 \pm \sqrt{5}$ are the zeros of the quadratic $\lambda^2 + 3\lambda + 1$, then from (2.20)

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda_{1,2}+1}{\lambda_{1,2}}\right)^k \sum_{j=0}^{k} \lambda_{1,2}^j \binom{k}{j}^2. \tag{2.22}$$

Identifying (2.21) with (2.18) for $q = 1$ we may also give the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = {}_2F_1\left[\begin{matrix} -n, -n \\ 1 \end{matrix} \middle| -1\right] = \frac{2^n\sqrt{\pi}}{\Gamma\left(\frac{2+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}$$

for n even, and from (2.22) we can write, the new result

$$S_n(2,1) = \sum_{k=0}^{n} \binom{n}{k} \left(1 + \frac{1}{\lambda_{1,2}}\right)^k \sum_{j=0}^{k} \lambda_{1,2}^j \binom{k}{j}^2$$

where a second order recurrence of (2.22) is

$$(n+2)\, S_{n+2}(2,1) + 4\,(n+1)\, S_n(2,1) = 0,$$

with $S_0(2,1) = 1$ and $S_1(2,1) = 0$; for n odd $S_n(2,1) = 0$, hence $(n+1)\, S_{2n+2}(2,1) + 2\,(2n+1)\, S_{2n}(2,1) = 0$ and by iteration $S_{2n}(2,1) = (-2)^n \prod_{j=0}^{n-1} \frac{2j+1}{j+1}$.

3. Closed form representation of a binomial related series with a parameter.

The WZ pairs method certifies a given identity as well as having some spin-offs. Given the identity (1.11) for $a = \frac{1}{2}$ and $b = 2$, or from (2.11) we may write

$$\sum_{k\geq 0}^{2n} \frac{\binom{2n}{k}\left(-\frac{1}{2}\right)^k \binom{2k}{k} 2^{2n}}{\binom{2n}{n}} = 1, \tag{3.1}$$

and let

$$F(n,k) = \frac{\binom{2n}{k}\left(-\frac{1}{2}\right)^k \binom{2k}{k} 4^n}{\binom{2n}{n}} = \frac{\left(-\frac{1}{2}\right)^k (2k)!\,(n!)^2\, 4^n}{(2n-k)!\,(k!)^3}. \tag{3.2}$$

Calling up the WZ package in "Mathematica" we obtain the certificate function

$$R(n,k) = \frac{k^2}{(2n-k+1)(k-2-2n)}. \tag{3.3}$$

Now, we define

$$G(n,k) = R(n,k)F(n,k) = \frac{-\left(-\frac{1}{2}\right)^k (2k)!\,(n!)^2\,4^n}{k!\,((k-1)!)^2\,(2n-k+2)!} \tag{3.4}$$

such that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ is true. Sum that equation over all integers k, such that the right hand side telescopes to zero and therefore

$$\sum_{k\geq 0} F(n+1,k) = \sum_{k\geq 0} F(n,k). \tag{3.5}$$

The two discrete functions $F(n,k)$ and $G(n,k)$ are termed the WZ pairs. From (3.5) and with initial conditions we obtain the Reed Dawson identity. Petkovšek et al. [9] claim that the WZ pairs method provides extra information because of the existence of a dual WZ pair. To obtain the dual WZ pair make the substitution $(an+bk+c)!$ by $\frac{(-1)^{an+bk}}{(-an-bk-c-1)!}$ for $a+b \neq 0$ in (3.2) and (3.4) to obtain $\overline{F}$ and $\overline{G}$. Next change the variables (n,k) by $F^*(n,k) = \overline{G}(-k-1,-n)$; $G^*(n,k) = \overline{F}(-k,-n-1)$, (this transformation maps WZ pairs to WZ pairs), such that we obtain

$$F^*(n,k) = \frac{(-1)^{n+1}\,2^n\,(n-1)!\,(n!)^2\,(2k-1-n)!}{4^{k+1}\,(2n-1)!\,(k!)^2} \tag{3.6}$$

and

$$G^*(n,k) = \frac{(-1)^{n+1}\,2^{n+1}\,(n!)^3\,(2k-2-n)!}{4^k\,(2n+1)!\,((k-1)!)^2}. \tag{3.7}$$

As previously, we obtain $f_n^* = \sum_{k\geq 0} F^*(n,k)$ and because of the $(2k-1-n)$ term in (3.6) we shall define

$$f_n^* = \sum_{k\geq\left[\frac{n}{2}\right]+1} F^*(n,k), \tag{3.8}$$

where $[x]$ represents the integer part of x. Now, we need to sum over k, the recurrence

$$F^*(n+1,k) - F^*(n,k) = G^*(n,k+1) - G^*(n,k)\,; \tag{3.9}$$

from(3.9) it follows easily that

$$\begin{aligned} &F^*(n+2,k) - F^*(n,k) \\ &= G^*(n+1,k+1) - G^*(n+1,k) + G^*(n,k+1) - G^*(n,k)\,. \end{aligned}$$

For n even, let $n = 2m$, and summing for $k \geq 2+m$, we obtain

$$\begin{aligned} &f^*(2+2m) - f^*(2m) + F^*(2m,m+1) \\ &= -G^*(2m+1,m+2) - G^*(2m,m+2)\,, \end{aligned}$$

and from (3.6) and (3.7) substituting for F^* and G^* we obtain

$$f^*(2+2m) = f^*(2m) + \frac{(3m+2)\,(2m+1)!\,(2m)!^2}{m!\,(4m+3)!\,(m+1)!}. \tag{3.10}$$

Iterating the recurrence (3.10) we have

$$f^*(2+2m) = f^*(2) + \sum_{j=1}^{m} \frac{(3j+2)\,(2j+1)!\,(2j)!^2}{j!\,(4j+3)!\,(j+1)!} \tag{3.11}$$

and from (3.6) and (3.8) we have

$$f^*(2) = -\frac{2}{3}\sum_{k\geq 2}\frac{(2k-3)!}{4^k\,(k!)^2}. \tag{3.12}$$

We can put (3.12) in “Mathematica, Algebra, SymbolicSum” and obtain

$$f^*(2) = \frac{1}{3} - \ln\sqrt{2}. \tag{3.13}$$

(We may also obtain (3.13) by starting with identity 2.5.16 in the book by Wilf [13]). Now from (3.13), (3.11), and (3.8) we obtain

$$\begin{aligned} &\frac{4^m\,(2m-1)!\,(2m)!^2}{(4m-1)!}\sum_{k=m+1}^{\infty}\frac{(2k-1-2m)!}{4^{k+1}\,(k!)^2} \\ &= \ln\sqrt{2} - \frac{1}{3} - \sum_{j=1}^{m-1}\frac{(3j+2)\,(2j+1)!\,(2j)!^2}{j!\,(4j+3)!\,(j+1)!}. \end{aligned} \tag{3.14}$$

From (3.10) and (3.13) we also obtain $f^*(0) = -\ln\sqrt{2}$ and from (3.14) putting $k^* = k - m$ and renaming k^* we have the new result

$$\sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}\,((m+k)!)^2}$$
$$= \frac{(4m-1)!}{(2m-1)!\,(2m)!^2} \left\{ \ln 4 - 4 \sum_{j=0}^{m-1} \frac{(3j+2)\,(2j+1)!\,(2j)!^2}{j!\,(4j+3)!\,(j+1)!} \right\}.$$

REFERENCES

[1] J. Borwein, *Three adventures: Symbolically discovered identities for Zeta(4N+3) and like matters,* 9th International conference on: Formal Power Series and Algebraic Combinatorics. Vienna, July, 1997.

[2] N.G. deBruijn, *Asymptotic Methods in Analysis,* North Holland, Amsterdam, 1958.

[3] C.J. Efthimiou, *Finding Exact Values For Infinite Sums,* Mathematics Magazine, 72–1 (1999), pp. 45–51.

[4] P. Flajolet - B. Salvy, *Euler sums and contour integral representations,* Experimental Mathematics, 7 (1998), pp. 15–35.

[5] R.L. Graham - M. Grotschel - L. Lovasz, *Handbook of combinatorics, Vol. II,* North Holland, Elsevier, New York, 1995.

[6] R.L. Graham - D.E. Knuth - O. Patashnik, *Concrete mathematics,* Addison Wesley Publishing Co. Massachusetts, 2nd. Ed., 1994.

[7] D.H. Greene - D.E. Knuth, *Mathematics for the analysis of algorithms,* Birkhäuser, Boston, 3rd. Ed., 1990.

[8] A.T. Jonassen D.E. Knuth, *A trivial algorithm whose analysis isn't,* Journal of computer and system sciences, 16 (1978), pp. 301–322.

[9] M. Petkovšek - H.S. Wilf - D. Zeilberger, *A = B,* A. K. Peters, Wellesley, Massachusetts, 1996.

[10] J. Riordan, *Combinatorial identities,* John Wiley and sons Inc., New York, 1968.

[11] V. Strehl, *Binomial identities- Combinatorial and algorithmic aspects,* Discrete mathematics, 136 (1994), pp. 309–346.

[12] A.D. Wheelon, *On the summation of infinite series in closed form,* Journal of Applied Physics, 25 (1954), pp. 113–118.

[13] H.S. Wilf, *Generatingfunctionology,* Academic Press Inc., New York, 1994.

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