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ON SOME LOCAL PROPERTIES OF FUZZY MANIFOLD AND ITS FOLDING

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This paper is a continuation of [5]. It introduces the local properties of the fuzzy manifold. the fuzzy vector field in a fuzzy tangent space will be defined. Theorems governing the relation between the folding of the fuzzy tangent space and the folding of the fuzzy manifold are deduced.

Introduction.

Manifolds were first introduced into mathematics by Riemann in 1950s in the form of Riemann surfaces. Let M and N be two smooth connected manifolds of dimension m and n respectively, a map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic $\gamma : J \to M$ the induced path $f\,0\,\gamma : J \to N$ is picewise geodesic and of the same length as γ, $J = [0, 1]$. If f does not preserves length, then f is a topological folding. The folding of a manifold introduced in 1977 by S. A. Robertson [9], also this subject was discussed in [8,2]. The exponential map at $m \in M$ is a mapping of a neighbourhood U of $0 \in T(M)$ into M,

$$\exp_m : U \to M.$$

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For $t \in T(M)$ for which $\exp_m(t)$ is defined as follows. Let γ be the unique geodesic in M such that $\gamma(0) = m$ and $\gamma^*(0) = t$, $\exp_m(t) = \gamma(1)$, $\exp_m(ut) = \gamma(u)$, u is real number. A point x in $T(M)$ is conjugate point if $\exp_m$ is singular at x.

A fuzzy differentiable manifold is a c^∞ - manifold which has a physical character this character represented by the density function μ, $\mu \in [0, 1]$, like the electric current, the temperature inside molten glass, etc. The fuzzy folding of a fuzzy manifolds and graphs are introduced in [1], [3], [4], [6], [7], [10]. This work is the study of the local characters of fuzzy submanifold and cartesian product of two fuzzy manifold. The relations between fuzzification of the manifold and the curvature, the torsion are discussed. Theorems governing these relations are deduced . The exponential folding of a fuzzy tangent spaces into a fuzzy manifold are obtained.

The main results.

We will introduce the following definitions:

(1) *The cartesian product of two fuzzy manifolds.*
Let $(l_1; \mu_1), (l_2; \mu_2)$ *be two fuzzy edges then*

$$(l_1; \mu_1).(l_2; \mu_2) = (l_1.l_2; \mu_1 \odot \mu_2) = (A; \max \mu_i)$$

or

$$= (A; \min \mu_i)$$

or

$$= (A; \frac{\mu_1 + \mu_2}{n}),$$

n is a positive integer > 1, $\mu_i \in [0, 1]$, $i = 1, 2$.

The generalization of the above definition is the definition of the fuzzy volume for the fuzzy submanifold.

(2) *The fuzzy first fundamental form* $\widetilde{I} = (d\widetilde{X}; \mu_e)$.

$$(d\widetilde{X}; \mu_e) = ((d\widetilde{X} \cdot d\widetilde{X}); \mu_e \odot \mu_e)$$

The second fundamental form $\widetilde{\Pi} = -d\widetilde{X}.d\widetilde{N} = (d\widetilde{X}; \mu_i).(d\widetilde{N}, \mu_2)$.

(3) *The fuzzy tangent space of a manifold is defined as follows.*

(a) *If at every point p of a fuzzy manifold $\widetilde{M}$ there are different charactaristic membership μ, for a point (p, μ_p) we obtain a tangent space every point in it has μ_p, all points have the same μ_p (see figure 1).*

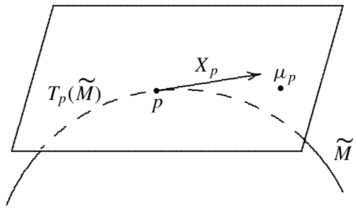

Figure 1

(b) *In this case for any point p the tangent $T_p(M)$ has μ_p but μ_p decreases continuously when d increases such thalt $\lim_{d\to\infty} \mu_{pn} = 0$ (see figure 2).*

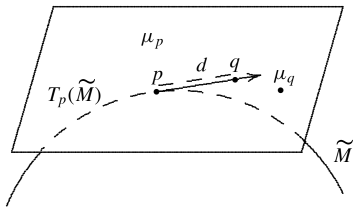

Figure 2

In the case of fuzzy submanifold $\widetilde{M}$ such that $\forall\, p \in \widetilde{M}$, $\mu_p = a$, $a \in [0, 1]$. The fuzzy tangent spaces at different points carry a physical character of equal appearance (dizziling). But in the second case the physical character will be different at different points.

Theorem 1. *In the parallel fuzzy system $\widetilde{M}$ which is homeomorphic to $\widetilde{S}^n$. For $\overline{\widetilde{M}}$, then $\overline{K}_i$ increases when μ_i decreases. For $\underline{\widetilde{M}}$, then $\underline{K_i}$ decreases when μ_i decreases. If $K = 0$ then μ_i decreases when d increases, where d is the distance from $\widetilde{M}$.*

Proof. Let $\widetilde{M}$ be a fuzzy manifold which is homeomorphic to $\widetilde{S}^n$, at every point $p \in \widetilde{M}$, then $\mu_p = a \in (0, 1]$, then the system of fuzzy manifolds will be $\cup\overline{\widetilde{M}}_i$

inside $\widetilde{M}$, $\mu(\overline{\widetilde{M}}_i) = \mu_i, \quad i = 1, 2, 3, \cdots$. We find that

$$(1) \qquad \begin{cases} \mu_1 > \mu_2 > \mu_3 > \cdots > \mu_n & \text{and} \\ K_1 < K_2 < K_3 < \cdots < K_n \end{cases}$$

where K_i is the curvature of $\overline{\widetilde{M}}_i$. For outside system $\underline{\widetilde{M}_i}$

$$(2) \qquad \begin{cases} \mu_1 > \mu_2 > \mu_3 > \cdots > \mu_n \\ K_1 > K_2 > K_3 > \cdots > K_n \end{cases}$$

where K_i is the curvature of $\underline{\widetilde{M}_i}$ (see figure 3.a).

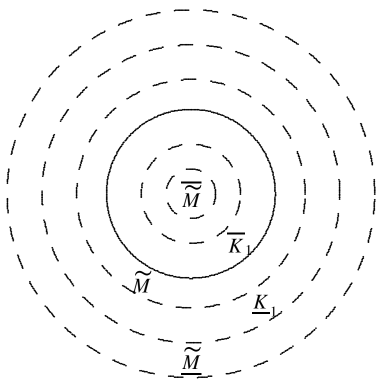

Figure 3.a

In the case of homeomorphic system fig. (3.b) K_i, μ_i satisfies (1) and (2).

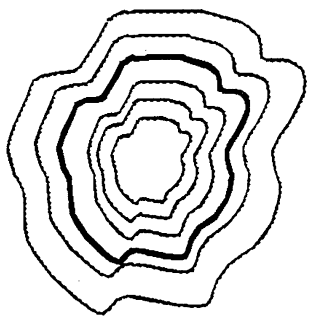

Figure 3.b

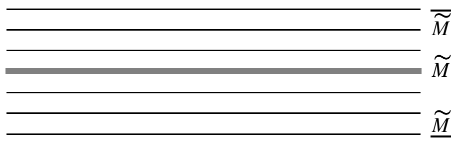

Figure 4

If $K = 0$, then

$$\begin{cases} \overline{\mu}_1 > \overline{\mu}_2 > \cdots > 0 \\ \underline{\mu}_1 > \underline{\mu}_2 > \cdots > 0 \end{cases}$$

i.e. μ_i decreases when d_i increases (see figure 4).

Theorem 2. *In the fuzzy system with a common point $\widetilde{p}$, then K_i increases when $\mu_i =$ constant, or K_i increases when μ_i decreases.*

Proof. Let $\widetilde{M}$ be a fuzzy manifold such that $\mu_p = \max \underset{\sim}{\mu_i}$. In the system $\overline{\widetilde{M}}_i$, then $\mu_1 = \mu_2 \Rightarrow K_1 < K_2$, also for the outside system $\underline{\widetilde{M}}$, $\mu_1 = \mu_2 \Rightarrow K_1 < K_2$ (see figure 5.a).

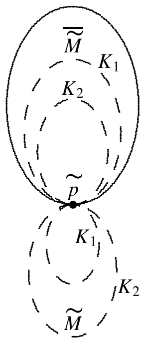

Figure 5.a

In figure 5.b, $\mu_i > \mu_j \Rightarrow K_j > K_i$ i.e. K_i increases but μ_i decreases.

Definition. *Consider $\widetilde{S}^n$ as a fuzzy sphere, any $\overline{\widetilde{S}}{}_i^n$ with μ_i as a membership degree, μ for $\widetilde{S}^n$, then $T_p(\widetilde{S}^n)$, $T_p(\overline{\widetilde{S}}{}^n)$ such that μ_p is a membership for $T_p(\widetilde{S}^n)$ and μ'_p is a membership for $T_p(\overline{\widetilde{S}}{}^n)$, but $\lim\limits_{d\to\infty} \mu'_p$ (for $\overline{\widetilde{S}}{}^n$) $\to 0$ faster than $\lim_{d\to\infty} \mu_p$ (for $\widetilde{S}^n$) (see figure 6). i.e. the velocity of limit will be depend on $\mu \in [0, 1]$ and we arrive into a system of dizziling tangent spaces.*

Theorem 3. *Let $f : \overline{M}_1 \to \overline{M}_2$ be a fuzzy folding, then there are induced fuzzy foldings $f_T : T\overline{M}_1 \to T\overline{M}_2$.*

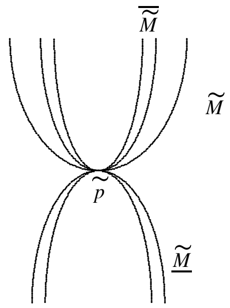

Figure 5.b

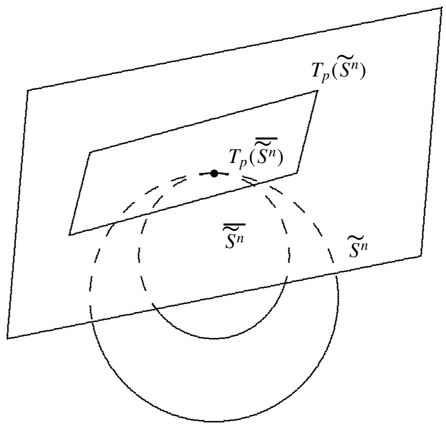

Figure 6

Proof. In figure 7 there are two cases (a) $\mu_1 = \mu_2$ (b) $\mu_1 < \mu_2$.
For the first case $\mu_1(T_p(\overline{M}_1)) = \mu_1(T_p(\overline{M}_2)) = ct$. In the second case there are two overlap tangent vector spaces at p in the first $\mu = \mu_1$ and $\lim_{d\to\infty} \mu_2(\overline{M}_2)$ is faster than $\lim_{d\to\infty} \mu_1(\overline{M}_1)$, and any folding $f : \overline{M}_1 \to \overline{M}_2$ which is a topological folding, then the induced folding $\overline{f}_T : T\overline{M}_1 \to T\overline{M}_2$ s.t. $\overline{f}_T(\alpha_1, \mu_1) = (\beta_1, \widetilde{f}_T(\mu_1)) = (\beta_1, \mu_1)$ or (β, μ_2).

Theorem 4. *For any fuzzy manifold without conjugate points. If* $f_T : T_{p_1}(\widetilde{M}) \to T_{p_2}(\widetilde{M})$, $\exp T_{p_i} : (\widetilde{M}) \to \widetilde{M}_{p_i}$ *then* $\exp f_T = \exp f$, *where* $f : \widetilde{M} \to \widetilde{M}$.

Proof. Let M be a fuzzy manifold and let $f_T : T_{p_1}(\widetilde{M}) \to T_{p_2}(\widetilde{M})$, $T_{p_1}(\widetilde{M})$ is the tangent vector space on which we put μ_{p_1} on it such that for $p_1, p_2 \in T_{p1}(M)$

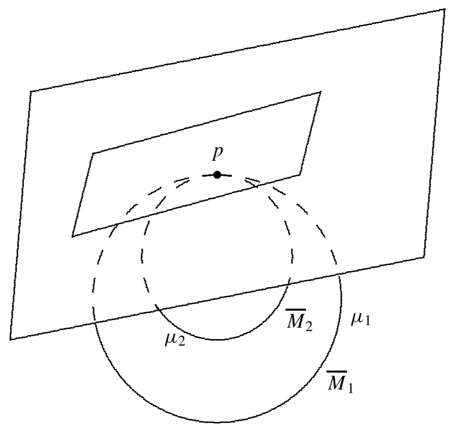

Figure 7

either $\mu_1 = \mu_2$ or $\mu_1 \neq \mu_2$. Then for $f_T : (T_{p_1}(M), \mu_1) \to (T_{p_2}(M), \mu_2)$ then either $f_T(\mu_1) = (\mu_2)$ or one of these cases (a) $f_T(\mu_1) = \mu_1$ (b) $f_T(\max(\mu_1, \mu_2))$. Then there is an induced folding $f : M \to M$ such that $f(g_1, \mu_1) = (g_2, \mu_1)$ or (g_2, μ_2) where g_i is a geodesis, $i = 1, 2$. Let M be a fuzzy manifold without conjugate points ($\exp^{-1}$ not defined), $\exp : T(M) \to M$ we arrive into the following diagram such that $\exp f_T = f \exp$.

$$\begin{array}{ccc}
T_{p_1}(\widetilde{M}) & \xrightarrow{\ f_T\ } & T_{p_2}(\widetilde{M}) \\
{\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\
M_{p_1} & \xrightarrow{\ f\ } & M_{p_2}
\end{array}$$

In the case of the limit of the fuzzy folding of fuzzy manifold we obtain the following diagram.

$$\begin{array}{ccccccc}
T_{p_1}(\widetilde{M}) & \xrightarrow{f_{T_1}} & T_{p_2}(\widetilde{M}) & \xrightarrow{f_{T_2}} & T_{p_3}(\widetilde{M}) & \cdots\cdots \lim_{n\to\infty} f_{T_n} \cdots\cdots & T_{p^{n-1}}(\widetilde{M}) \\
{\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\
\widetilde{M}_{p_1} & \xrightarrow{f_1} & \widetilde{M}_{p_2} & \xrightarrow{f_2} & \widetilde{M}_{p_3} & \cdots\cdots \lim_{n\to\infty} f_n \cdots\cdots & \widetilde{M}_{p^{n-1}}
\end{array}$$

s.t.

$$f_i \exp = \exp f_{T_i}$$

Also

$$f_{T_1}(\mu_i) = \mu_{p_1}(T_{p_1}M) \text{ or } \mu_{p_2}(T_{p_2}M)$$

and

$$f_1(\mu_i) = \mu_{p_1}(M) \text{ or } \mu_{p_2}(M).$$

The most general fuzzy system of fuzzy circles.

Consider the fuzzy circle $\widetilde{S}_1$, $\bigcup \widetilde{S}_i$ such that $\tau_i = 0$ (the torsion). There are another fuzzy system $\bigcup \widetilde{\widetilde{S}}_j$ such that $\tau_j \neq 0$ see figure 8.

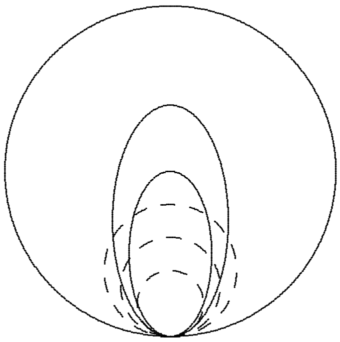

Figure 8

In the case of a parallel fuzzy system of $\widetilde{M}$ in different planes see figure 9.

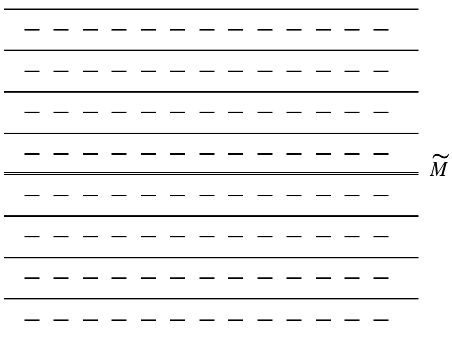

Figure 9

$$M_1 \mathbin{//} M_2 \mathbin{//} M_3 \mathbin{//} \cdots \mathbin{//} M_n, \quad \tau_1 = \tau_2 = \tau_3 \cdots = \tau_n = 0$$

the relation between the fuzzy folding of the system and the folding of tangent vector spaces comes from these chains.

If

$$S_{\tau_1} \xrightarrow{f_1} S_{\tau_2} \xrightarrow{f_2} S_{\tau_3} \cdots\cdots\cdots \xrightarrow{f_n} S_{\tau_0}$$

there are induced sequence

$$T(S_{\tau_1}) \xrightarrow{\overline{f}_1} T(S_{\tau_2}) \xrightarrow{\overline{f}_2} T(S_{\tau_3}) \cdots\cdots\cdots \xrightarrow{\overline{f}_n} T(S_{\tau_0})$$

such that

$$f_1(\mu_1) = \mu_2, \cdots\cdots, \overline{f}_1(\mu_{T_1}) = \mu_{T_2}.$$

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