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NEW SOLUTION OF THE GENERALIZED ELLIPSOIDAL WAVE EQUATION

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Certain aspects and a contribution to the theory of new forms of solutions of an algebraic form of the generalized ellipsoidal wave equation are deduced by considering the Laplace transform of a soluble system of linear differential equations. An ensuing system of non-linear algebraic equations is shown to be consistent and is numerically implemented by means of the computer algebra package MAPLE V. The main results are presented as series of hypergeometric type of there and four variables which readily lend themselves to numerical handling although this does not indicate all of the detailed analytic properties of the solutions under consideration.

1. Introduction.

Considerable interest has recently arisen in Heun's equation and its confluent forms. A detailed general overview has been compiled by Ronveaux [4] to which the reader should refer. Such equations have recently been tackled successfully by considering the inverse Laplace transform of soluble systems of linear differential equations. See Exton [3], for example.

The purpose of this paper is to present a contribution to the study of new forms of solutions of the generalized ellipsoidal wave equation (GEWE), a

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second-order differential equation with four singularities. Three of these are regular and one is irregular of the first type, usually at infinity. The ellipsoidal wave equation, a special case of the GEWE, is the most recondite member of the class of linear differential equations of the second order which arise when the three-dimensional wave equation is separated in ellipsoidal coordinates. The standard form of the GWEW can be written as

$$X(X-1)(X-A)Z" + [\alpha X^2 + \beta X + \gamma]Z' + [\phi X^2 + \Psi X + \chi]Z = 0. \tag{1.1}$$

Complicated solutions of this equation of perturbation type have been discussed by Exton [2]. More compact solutions with triple and quadruple series representations of hypergeometric type are obtained in this study by the application of the inverse Laplace transform to a soluble system of linear differential equations This is followed by a process of matching the parameters with (1.1). The ensuing system of non-linear algebraic equations is shown to be consistent by the use of the computer algebra package MAPLE V.

In what follows, any values of parameters leading to results which do not make sense are tacitly excluded and indices of summation are taken to run over all of the non-negative integers. The Pochhammer symbol $(a, n) = \Gamma(a = n)/\Gamma(a)$ is frequently used and the interchanging of the operations of summation and integration is justified in all cases on account of the convergence of the series or integral representations involved. Constant multipliers which have no bearing on any final results are often left out for convenience.

2. A soluble system of differential equations.

Consider the differential equations

$$(at + b)v' + cv = u \tag{2.1}$$

and

$$(t/a + d)u''' + (ft + g)u'' + (ht + j)u' + ku = 0, \tag{2.2}$$

which are equivalent to the fourth-order equation

$$\begin{aligned} &[t^2 + (b/a + ad)t + bd]v'''' + \\ &+ [aft^2 + (ag + bf + 3 + c/a)t + cd + bg + 3ad]v''' + \\ &+ [akt^2 + (aj + bh + 2af + cf)t + cg +'' ag + bj]v'' + \end{aligned} \tag{2.3}$$

$$+ [(ah + ch + ak)t + aj + cj + bk]v' + ckv = 0.$$

Let the inverse Laplace transform be given by

$$v(t) = \int \exp(-xt)y(x)\,dx\,, \tag{2.4}$$

where the contour of integration is a simple closed path on the Riemann surface of the integrand, such that this integrand remains unchanged after the completion of one circuit.

The function $y(x)$ is found to be given by the differential equation

$$\begin{gathered}(x^4 - afx^3 + ahx^2)y'' + [(b/a + ad)x^4 + \\ + (5 - ag - bf - c/a)x^3 + (aj + bh + cf - 4af)x^2 + \\ + (3ah - ch - ak)x]y'[(4b/a + 4ad)x^3 + (3 - 3ag - 3bf - 3c/a)x^2 + \\ + (2aj + 2bh + 2cf - 2af)x + ah - ch - ak]y = 0.\end{gathered} \tag{2.5}$$

Let

$$x = pX \ \text{ and } \ y = \exp(qX)X^{\Gamma}Z, \tag{2.6}$$

when (2.5) becomes

$$\begin{gathered}(X^3 - afp^{-1}X^2 + ahp^{-2}X)Z'' + \\ + [(2q + bp/d + adp)X^3 + (5 - ag - bf - c/a - 2afq/p + 2r)X^2 + \\ + (aj/p + bh/p + cf/p - 4af/p + 2qah/p^2 - 2raf/p)X + \\ + (2rah/p^2 + 3ah/p^2 - ch/p^2 - ak/p^2)]Z' + \\ + [(q^2 + bqp/a - adpq)X^3 + (-afq^2/p + 5q - agq - bfq - cq/a - 4bp/a + \\ + 4adp + 2qr + bpr/a + adpr)X^2 + \\ + (r^2 + 4r - agr - bfr - cr/a - 2afqr/p + ahq^2/p^2 + ajq/p + bhq/p + \\ + chq/p - 4afq/p + 3 - 3ag - 3bf - 3c/a)X - \\ -af/p + ajr/p + bhr/p + cfr/p - 4afr/p + 2qahr/p^2 + 3ahq/p^2 - chq/p^2 - \\ - akq/p^2 + 2af/p - 2aj/p - 2aj/p - 2bh/p - 2cf/p + \\ + (3ahr/p^2 - chr/p^2 - akr/p^2 + ah/p^2 - ch/p^2 = ak/p^2)/X]z = 0.\end{gathered} \tag{2.7}$$

On comparing (2.7) with (2.1), a system of eleven non-linear algebraic equations involving the twelve quantities a, b, c, d, f, g, h, j, k, p, q, and r, any one of which is arbitrary. This system can be shown to be consistent and also numerically implemented by means of the computer algebra package MAPLE V. This process, the main theoretical point of which is to establish the existence of the solutions of this system of non-linear algebraic equations is very lengthy and it is impracticable to record the details here. Ultimate numerical implementation follows from the application of the MAPLE V package itself. Hence, the solution of the GEWE depends upon the solution of (2.1) and (2.2).

3. The solution of (2.1) and (2.2).

From (2.2),

$$(t + ad)u''' + (aft + ag)u'' + (aht + aj)u' + aku = 0. \tag{3.1}$$

Let

$$u = \int \exp(st)\eta(s)\,ds, \tag{3.2}$$

where

$$\eta'/\eta = [ads^3 + (ag - 3)s^2 + (aj - 2af)s + ak - ah]/[s(s^2 + afs + ah)] \tag{3.3}$$

For convenience, put

$$af = -\xi - \xi \ \text{ and } \ ah = -\xi\xi, \tag{3.4}$$

and we can then write (3.3) in the form

$$\eta'/\eta = ad + \lambda/s + \mu/(s - \xi) + \nu/(s - \zeta). \tag{3.5}$$

Hence,

$$\eta = \exp(ads)s^{\lambda}(s - \xi)^{\mu}(s - \zeta)^{\nu}. \tag{3.6}$$

More than one relevant form of u can then be obtained.

4. A solution of the GEWE relative to the origin.

Bearing in mind that the contour of integration of (3.2) is the same as that of (2.4), the sign of λ is immaterial and we expand the function $\eta(s)$ in ascending powers of s:

$$\eta(s) = \exp(ads) \sum_{m,n} [(-\mu, m)(-\nu, n)\xi^{-m} \zeta^{-n} s^{\lambda+m+n}]/[m!n!], \tag{4.1}$$

so that, apart from a constant multiplier, we have, formally,

$$u \sim (ad+t)^{-\lambda-1} \sum_{m,n} [(-\mu, m)(-\nu, n)(-\xi)^{-m}(-\zeta)^{-n} \cdot \\ \cdot (1+\lambda, m+n)(ad+t)^{-m-n}]/[m!n!]. \tag{4.2}$$

Any formal processes used below can always be justified by noting that instead of divergent series, corresponding convergent integrals can always be used to obtain equivalent results.
From (2.1), $(t + b/a)v' + ca^{-1}v$

$$(ad+t)^{-\lambda-1} \sum_{m,n} [(-\mu, m)(-\nu, n)(1+\lambda, m+n)(-\xi)^{-m} \cdot \\ \cdot (-\zeta)^{-n}(ad+t)^{-m-n}]/[m!n!]. \tag{4.3}$$

Hence, by means of the usual method of solution of the linear differential equation of the first order,

$$v = \tau^{-c/a} \sum_{m,n,M} [(-\mu, m)(-\nu, n)(-\xi)^{-m}(-\zeta)^{-n}(1+\lambda, m+n+M) \cdot \\ \cdot (ad - b/a)^{M}]/[m!n!M!]x\tau^{c/a-lambda a-2-m-n-M} d\tau. \tag{4.4}$$

Thus, v is proportional to

$$\sum_{m,n,M} [(-\mu, m)(-\nu, n)(1+\lambda, m+n+M)(-\xi)^{-M}(-\zeta)^{-n}(ad-b/a)^{M} \cdot \\ \cdot (\lambda+2-c/a, m+n+M)\tau^{c/a-\lambda-1-m-n-M}]/[m!n!M!(\lambda+3-c/a, m+n+M)]. \tag{4.5}$$

From (2.4) by inversion, bearing in mind (4.4),

$$y = \exp(-bx/a) \int \exp(\tau x) v(\tau)\, d\tau \tag{4.6}$$

From (4.5) and (2.6), we have

$$Z = \exp[(bp/a - q)X^{\lambda - c/a - r} \cdot \tag{4.7}$$

$$\cdot \sum_{m,n,M} \{(-\mu, m)(-\nu, n)(1+\lambda, m+n+M)(\lambda - c/a + 2, m+n+M)$$

$$(-\xi pX)^m (-\zeta pX)^n [(ad - b/a)pX]^M\}/$$

$$/\{(\lambda + 3 - c/a, m+n+M)(\lambda + 1 - c/a, m+n+M)m!n!M!\}.$$

Put

$$ad - b/a = 1/p, \tag{4.8}$$

which fixes the parameter d which was previously arbitrary.

Hence, we obtain a solution of the GEWE relative to its regular singularity at the origin. This representation converges within the unit circle as expected.

5. A formal solution of the GEWE relative to the point at infinity.

The function $\eta(s)$ is expanded in descending power of s:

$$\eta(s) = \exp(ads)s^{\lambda+\mu+\nu} \sum_{m,n} [(-\mu, m)(-\nu, n)\xi^m \zeta^n s^{-m-n}]/[m!n!]. \tag{5.1}$$

Hence, from (3.2), apart from a constant multiplier,

$$u(t) = (ad + t)^{-\lambda-\mu-\nu-1} \sum_{m,n} [(-\mu, m)(-\nu, n)\xi^m \zeta^n (ad + t)^{m+n}] \cdot \tag{5.2}$$

$$\cdot [m!n!(-\lambda - \mu - \nu, m+n)]$$

and from (2.1) and (4.4),

$$v' = -cv/(a\tau) + \tag{5.3}$$

$$+ \tau^{-1} \sum_{m,n} [(-\mu, m)(-\nu, n)\xi^m \zeta^n (\tau + ad - b/a)^{-\lambda-\mu-\nu-1+m+n}]/$$

$$/[m!n!(-\lambda - \mu - \nu, m+n)].$$

We then see that

$$(5.4)\qquad v = \tau^{-\lambda-\mu-\nu} \sum_{m,n,M} [(-\mu, m)(-\nu, n)(\lambda + \mu + \nu + 1, M - m - n) \cdot$$

$$\cdot (c/a - \lambda - \mu - \nu - 1, m + n + M)(b/a - ad)^{-M}]/$$

$$/[m!n!M!(c/a - \lambda - \mu - \nu, m + n + M)]x[\xi^m \zeta^n \tau^{m+n+M}]$$

and from (4.6), formally,

$$(5.5)\quad y \sim \exp(-bx/a)x^{\lambda+\nu-1} \sum_{m,n,M} [(-\mu, m)(-\nu, n)(\lambda+\mu+\nu+1, M-m-n) \cdot$$

$$\cdot (c/a - \lambda - \mu - \nu - 1, m + n + M)]/[m!n!M!(c/a - \lambda - \mu - \nu, m + n + M)] \cdot$$

$$\cdot [(-\xi)^m (-\zeta)^n (ad - b/a)^{-M} x^{-m-n-M}.$$

If this expression is combined with (2.6), a formal representation of a solution of the GEWE relative to the irregular singularity at the point at infinity is obtained. As expected, the series does not converge.

6. A global solution of the GEWE.

By comparison with the confluent Heun equation, global solution of the GEWE might be expected to exist. Compare Ronveaux [4], p. 100.

With the form of v as given by (5.3), employ a Pochhammer double-loop contour in the integral (4.6). First of all, the expression (5.3) is written as a double series of Gauss hypergeometric functions, namely

$$(6.1)\qquad v = \tau^{-\lambda-\mu-\nu} \sum_{m,n} [(-\mu, m)(-\nu, n)(\lambda + \mu + \nu + 1, m + n) \cdot$$

$$\cdot (c/a - \lambda - \mu - \nu - 1, m + n)][m!n!(c/a - \lambda - \mu - \nu, m + n)] \cdot$$

$$\cdot \xi^m \zeta^n \tau^{m+n} {}_2F_1[\lambda + \mu + \nu + 1 - m - n; c/a - \lambda - \mu - \nu - 1 + m + n,$$

$$c/a - \lambda - \mu - \nu + m + n; (b/a - ad)^{-1}\tau].$$

Euler's transform

$$(6.2)\qquad {}_2F_1[a, b; c; x] = (1 - x)^{c-a-b} {}_2F_1[c - a, c - b; c; x],$$

Erdelyi [1], p. 105, is applied to the inner hypergeometric function on the right of (6.1), so that (4.6) now takes the form

(6.3)
$$y = \exp(-bx/a) \sum_{m,n,M,N} [(-\mu, m)(-\nu, n) \cdot$$
$$\cdot (c/a - 2\lambda - 2\mu - 2\nu - 1 - 2m - 2n, M)x^N]/$$
$$/[m!n!N!]\xi^m \zeta^n (b/a - ad)^{-M} (-1)^N \cdot$$
$$\cdot [1 - (b/a - ad)^{-1}\tau]^{-\lambda-\mu-\nu+m+n} \tau^{-\lambda-\mu-\nu+m+n+M+N} d\tau,$$

where the contour of integration is taken to a Pochhammer double loop slung around the origin and the point $b/a - ad$ in the $\tau-$plane.

Apart from a constant multiplier, it is found that

(6.4)
$$y = \exp(-bx/a) \sum_{m,n,M,N} [(-\mu, m)(-\nu, n) \cdot$$
$$\cdot (c/a - 2\lambda - 2\mu - 2\nu - 1 - 2m - 2n, M)x^N \xi^m \zeta^n]/$$
$$/[m!n!N!(c/a - \lambda - \mu - \nu', m + n + N] \cdot$$
$$\cdot [(b/a - ad)^{m+n+N} (1 - \lambda - \mu - \nu, m + n)(1 - \lambda - \mu - \nu, m + n + M + N)]/$$
$$/[(2 - 2\lambda - 2\mu - 2\nu, 2m + 2n + M + N)].$$

This representation furnishes a global solution of the GEWE which converges throughout the whole of the x-plane.

Additional solutions of the GEWE can be deduced from the above results using the appropriate symmetries of the GEWE as outlined by Exton [3].

REFERENCES

[1] A. Erdelyi, *Higher transcendental functions, Vol. I,* McGraw Hill, New York, 1953.

[2] H. Exton, *The generalized ellipsoidal wave equation* $[0, 3, 1_1]$, Collect. Math., 45 (1995), pp. 217–230.

[3] H. Exton, *New solutions of the confluent Heun equation,* Le Matematiche, 53 (1998), pp. 1–10.

[4] A. Ronveaux, *Heun's differential equations,* Oxford University Press, 1995.

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