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# HOMOTOPIES USING CONFORMAL TRANSFORMATION WITH INVARIANT TOTAL NORMAL TWIST

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The aim of this paper is to detect a homotopy to a spherical curve with invariant total normal twist 0. Also, we use conformal transformation to prove a theorem that for any immersed closed $C^3$-curve $c$ in the Euclidean 3-space $E^3$ with vanishing total normal twist, there exists a Frenet curve $\tilde{c}$ homotopic to $c$ in an arbitrary neighborhood of $c$ such that the total normal twist is invariant along the homotopy between $c$ and $\tilde{c}$ in that neighborhood.

## 1. Introduction.

Let $c : S^1 \longrightarrow E^3$ be a regular closed smooth curve (at least of class $C^3$) for the subsequent considerations. Parallel transfer of the normal plane along one period of the curve $c$ with respect to the normal connection leads to a rotation of the normal plane which is characterized (up to integer multiples of $2\pi$) by an oriented angle $\alpha(c)$ which we call the *total normal twist* of $c$. For Frenet curves this quantity is given by their total torsion up to integer multiples of $2\pi$ (see [1] and [6]).

It has been shown in [3] that the total normal twist of a closed curve is invariant under similarities (homotheties). In the work of B. Wegner [7] it has

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been shown that parallel sections of the normal bundle may remain parallel after renormalization, if the ambient space is subjected to a conformal transformation and consequently the local and uniform parallel ranks of immersions [2,5] into Euclidean 3-space $E^3$ are preserved under conformal transformations. Also, if the total normal twist of an immersed closed curve in $E^3$ is an integer multiple of $2\pi$, then the same is true for any image of the curve under a conformal transformation of the ambient space.

## 2. A homotopy to spherical curve.

We begin with proving the existence of a homotopy in the Euclidean 3-space $E^3$ from a plane curve to a spherical one preserving the total normal twist.

**Theorem 2.1.** *Let $c$ be a closed plane curve (at least of class $C^3$). Then there exist a curve $c^*$ in $S^2(R)$ and a homotopy from $c$ to $c^*$ with total normal twist* 0.

*Proof.* Let $\phi$ be the stereographic projection from the north pole $x_0$ of the sphere $S^2(R)$ of radius $R$ onto the tangent plane of the south pole $\mathbb{R}^2 \times \{0\}$ (see Fig. 1). Assume that the image of $c$ is contained in the plane. Then the coordinates of the center of the sphere are $p_0 = (0, 0, R)$, and those of the north pole $x_0 = (0, 0, 2R) = 2p_0$.

The mapping $\phi$ takes $x \in S^2(R) - \{x_0\}$ into the intersection of the plane $\mathbb{R}^2 \times 0$ with the line that passes through $x$ and $x_0$. It is clear that $R = \|x - p_0\|$. Then we have

$$R = \|r\phi(x) + (1 - r)x_0 - p_0\|$$

for some $r \in [0, 1]$. This will be

$$R = \|r\phi(x) + (2(1 - r) - 1)p_0\|,$$

i.e.

$$R = \|r\phi(x) + (1 - 2r)p_0\|. \tag{1}$$

Equation (1) gives us

$$R^2 = r^2\|\phi(x)\|^2 + (1 - 2r)^2 R^2 = r^2\|\phi(x)\|^2 + R^2 + 4r^2 R^2 - 4r R^2,$$

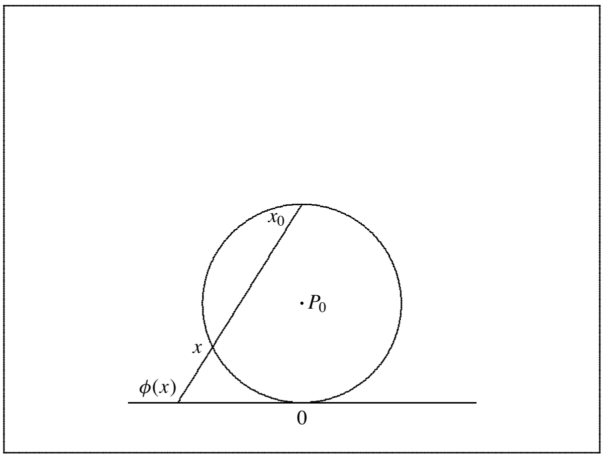


Figure 1.

(note that $< \phi(x), p_0 >= 0$), i.e.

$$r(r[\|\phi(x)\|^2 + 4R^2] - 4R^2) = 0.$$

The above equation gives us $r = 0$ or

$$r = \frac{4R^2}{\|\phi(x)\|^2 + 4R^2}, \tag{2}$$

and

$$1 - r = \frac{\|\phi(x)\|^2}{\|\phi(x)\|^2 + 4R^2}. \tag{3}$$

Substituting Equations (2), (3) in $x = r\phi(x) + (1 - r)x_0$ gives

$$x = \frac{4R^2}{\|\phi(x)\|^2 + 4R^2}\, \phi(x) + \frac{\|\phi(x)\|^2}{\|\phi(x)\|^2 + 4R^2}\, x_0. \tag{4}$$

Setting $R = \frac{1}{\lambda}$ and denoting more precisely with $\Phi_\lambda$ the stereographic projection belonging to the sphere with radius $\frac{1}{\lambda}$, Equation (4) will be

$$x = \frac{1}{\lambda^2\|\Phi_\lambda(x)\|^2 + 4}\{4\Phi_\lambda(x) + \lambda^2\|\Phi_\lambda(x)\|^2 x_0\}. \tag{5}$$

Hence let $c$ be a closed curve in the plane $\mathbb{R}^2 \times \{0\}$ and consider this as the image of the curve $\Phi_\lambda^{-1} \circ c$ on $S^2(\frac{1}{\lambda})$. Using the formula (5) we can define a homotopy $H$ as follows:

$$H(t, \lambda) = \frac{1}{\lambda^2 \|c(t)\|^2 + 4}\{4c(t) + \lambda^2 \|c(t)\|^2 x_0\}. \tag{6}$$

It is clear that $H(t, 0) = c(t)$ which has total normal twist 0 (because $c$ is a plane curve). $\lambda = 1$ defines a spherical curve

$$c^*(t) = \frac{1}{\|c(t)\|^2 + 4}\{4c(t) + \|c(t)\|^2 x_0\}$$

lying in the sphere $S^2$ (1) with center (0, 0, 1). For any $\lambda \in [0, 1]$, $H(t, \lambda)$ is a spherical curve lying in $S^2(\frac{1}{\lambda})$, and consequently it has total normal twist 0, and this proves the theorem. □

**Remark.** At $\lambda = 1$, we have the sphere $S^2(1)$ of this family of ambient spheres, containing $c^*$. For decreasing values of $\lambda$ from 1 to 0 in the interval [0,1] the corresponding sphere $S^2(\frac{1}{\lambda})$ begins to blow up and finally tends to a sphere passing through infinity (plane) at $\lambda = 0$ ($R = \infty$) where $c$ was located.

Note that not all the notions given above in $E^3$ are straightforward in $E^4$. Hence it is suitable to give some remarks about curves in $E^4$ before giving our main theorem.

**Proposition 2.1.** *Let $c : S^1 \longrightarrow E^4$ be a regular closed smooth curve in $E^4$, then there exists a parallel section $\xi$ of the normal bundle of $c$.*

*Proof.* Let $l$ denote the length of $c$. According to the definition of normal holonomy map,

$$A : \nu_c(s) \subset E^3 \longrightarrow \nu_c(s + l) = \nu_c(s) \subset E^3$$

it is an orientation preserving linear isometry, where $\nu_c(s)$ is a normal vector space of $c$ at $s$. Hence $A$ has 1 as an eigenvalue of multiplicity 1 at least. Let

$$A(\xi_0) = \xi_0,$$

i.e. $\xi_0$ is an eigenvector of $A$ in $\nu_c(s)$. This implies that $\xi_0$ can be extended by parallel transfer in the normal bundle to a globally defined vector field $\xi$. □

To define the total normal twist of $c$ in $E^4$ let $\{T, N_1, N_2, N_3\}$ be an orthonormal frame field along $c$, $T$ denoting the field of unit tangents. As has been shown in Proposition 2.1, we may choose $N_1$ as a parallel section of the normal bundle of $c$. Hence starting parallel transfer of $N_i(0)$, $i = 1, 2, 3$, in $c(0)$ will leave $N_1(0)$ invariant after one period, and it will rotate $N_2(0)$ and $N_3(0)$ by an angle $\alpha(c)$ in the $(N_2(0), N_3(0))$-plane. This may be taken as the total normal twist $\alpha$ of $c$ in $E^4$.

Under these assumptions the calculation of $\alpha$ in terms of the given frame field was done in our work [3], and it will takes the following formula:

$$\alpha\,(c) = \int_0^1 \omega_{32}(s)\,ds,$$

where $\omega_{32} =< \nabla_T\, N_3,\; N_2 >$.
A spherical curve in $E^3$ has total normal twist zero. By using the above formula it is easy to extend this fact to spherical curves in $E^4$. For more details see [4], [6].

**Theorem 2.2.** *Let $c : S^1 \longrightarrow E^3$ be an immersion of class $C^3$ in the Euclidean 3-space $E^3$ such that $\alpha(c) = 0$. Then for each $\delta > 0$, there exist a Frenet curve $\tilde{c}$ with $\|\tilde{c} - c\|_3 < \delta$ and a homotopy $H_\lambda$ between $c$ and $\tilde{c}$, $\lambda \in [0, 1]$, with $\|H_\lambda - c\|_3 < \delta$ such that $\alpha(H_\lambda) = 0$ for all $\lambda \in [0, 1]$.*

*Proof.* Let $c : S^1 \longrightarrow E^3$ be a regular smooth curve of class $C^3$ parametrized by arc length in the Euclidean space $E^3$. Let

$$\phi : S^3(1) - \{x_0\} \longrightarrow E^3$$

be the stereographic projection from the north pole $x_0$ into the hyperplane $E^3 \subset E^4$. Then the spherical curve

$$\hat{c} = \phi^{-1} \circ c : S^1 \longrightarrow S^3(1) \subset E^4$$

is of class $C^3$. Since $\phi$ is a conformal mapping, tangents of $c$ are sent to tangents of $\hat{c}$, and osculating circles of $c$ are sent to osculating circles of $\hat{c}$. It is clear that $\hat{c}$ is a Frenet curve ($\hat{c}$ has non-zero curvature).
Defining a mapping

$$\Psi : S^1 \times S^1 \longrightarrow S^3(1)$$

by

$$\Psi(s, \theta) = m(s) + r(s)(N(s)\cos\theta + T(s)\sin\theta),$$

where $m(s)$ is the center of curvature of $\hat{c}$ at $s$, $r(s)$ is the radius of curvature of $\hat{c}$ at $s$. Then $\Psi$ is differentiable of class $C^1$. Since $\dim(S^1 \times S^1) = 2 < \dim S^3(1) = 3$, we have that $\Psi(S^1 \times S^1)$ is nowhere dense in $S^3(1)$, i.e.

$S^3(1) - \Psi(S^1 \times S^1)$ is dense in $S^3(1)$. Hence for given $\varepsilon > 0$, there exists $\hat{x} \in S^3(1) - (\Psi(S^1 \times S^1) \cup \{x_0\})$ such that

$$\|\hat{x} - x_0\| < \varepsilon.$$

Rotating $\hat{c}$ into $\hat{c}_1$ in $S^3(1)$, $\hat{c}_1 = R_1 \circ \hat{c}$, with the unique rotation $R_1$ mapping $\hat{x}$ to $x_0$ ($\hat{x}$ has been brought into the position of the north pole). Applying the stereographic projection

$$\phi : S^3(1) - \{x_0\} \longrightarrow E^3$$

to the curve $\hat{c}_1$. This yields a curve $\tilde{c}_1$ in $E^3$, $\phi \circ \hat{c}_1 = \tilde{c}_1$. Since $\phi$ is $C^\infty$, there exists for given $\delta > 0$ and $c$ an $\varepsilon > 0$ such that with the notations above

$$\|\phi \circ \hat{c} - \phi \circ \hat{c}_1\|_3 < \delta,$$

i.e.

$$\|c - \tilde{c}_1\|_3 < \delta.$$

But we have that $\hat{x} \notin \Psi(S^1 \times S^1)$, which implies that $\tilde{c}_1$ is a Frenet curve in $E^3$. The spherical curve $\hat{c}_1 \in S^3(1)$ has total normal twist 0 (integer multiple of $2\pi$) as we have shown before and illustrated in [4], [6]. Also, since $\phi$ is a conformal mapping, then $\tilde{c}_1$ has total normal twist 0.

Now we are going to prove that there exists a homotopy between the curves $c$ and $\tilde{c}_1$ in the $\delta$-neighborhood of $c$ preserving the total normal twist 0. Connecting $\hat{x}$ with $x_0$ by a shortest geodesic path in $S^3$ (1). It is clear that for any

$$x_\lambda = \lambda\hat{x} + (1-\lambda)x_0, \quad \lambda \in [0, 1],$$

the rotation $R_\lambda$ mapping $x_\lambda$ to $x_0$ yields a rotation of $\hat{c}$ to $\hat{c}_\lambda$, i.e. $R_\lambda \circ \hat{c} = \hat{c}_\lambda$, where $\hat{c}_0 = \hat{c}$, such that

$$\|x_\lambda - x_0\| < \varepsilon.$$

Then applying $\phi$ to $\hat{c}_\lambda$ gives a $C^3$ curve $\tilde{c}_\lambda$ in the Euclidean space $E^3$ such that

$$\|c - \tilde{c}_\lambda\|_3 < \delta,$$

and $\alpha(\tilde{c}_\lambda) = 0$. Hence the homotopy

$$H : S^1 \times [0, 1] \longrightarrow E^3$$

between $c$ and $\tilde{c}_1$ is given by

$$H(s, \lambda) = (\phi \circ R_\lambda \circ \phi^{-1})(c(s))$$

where $R_\lambda$ is the rotation mapping of $x_\lambda$ to the north pole $x_0$ and satisfies the conditions required.
It is easy to see that

$$H(s, 0) = (\phi \circ Id \circ \phi^{-1})(c(s)) = c(s),$$

$$H(s, 1) = (\phi \circ R_1)(\hat{c}(s)) = \phi(\hat{c}_1(s)) = \tilde{c}_1(s),$$

and $\alpha(H(., \lambda)) = 0$, because all the curves of the family $H(., \lambda)$ are images of spherical curves under conformal transformations. □

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