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ON EXPONENTIAL STABILITY OF C_0-QUASISEMIGROUPS IN BANACH SPACES

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In this paper we study some stability concepts for linear systems the evolution which can be described by a C_0-quasisemigroup.

The results obtained may be regarded as generalizations of well known results of Datko, Pazy, Littman and Neerven about exponential stability of C_o-semigroups.

1. Introduction.

Let X be a real or complex Banach space. The norm on X and on the Banach algebra $\widetilde{X}$ of all bounded linear operators from X onto it self will be denoted by $\|\cdot\|$.

We recall ([1], [3]) that an operator-valued map $S : \mathbb{R}_+^2 \to \widetilde{X}$ is called a C_0-*quasisemigroup on* X if it has the following properties:

$$(q_1) \qquad S(0, t_0) = I$$

(the identity operator on X) for every $t_0 \geq 0$;

$$(q_2) \qquad S(t, s + t_0)S(s, t_0) = S(t + s, t_0)$$

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for all $(t, s, t_0) \in \mathbb{R}_+^3$;

(q_3)
$$\lim_{t \to 0} \|S(t, t_0)x_0 - x_0\| = 0$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$;

(q_4) there exists an increasing function $\omega : \mathbb{R}_+ \to R_+^*$ such that:

$$\|S(t, t_0)\| \leq \omega(t) \text{ for all } (t, t_0) \in \mathbb{R}_+^2.$$

Remark 1.1. If $S : \mathbb{R}_+ \to \widetilde{X}$ is a C_0-semigroup on X then:

$$\widetilde{S} : \mathbb{R}_+^2 \to \widetilde{X} \text{ defined by } \widetilde{S}(t, t_0) = S(t) \text{ for all } (t, t_0) \in \mathbb{R}_+^2$$

is a C_0-quasisemigroup on X.

Example 1.1. If $u : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a continuous function then

$$S(t, t_0) = \frac{u(t_0)}{u(t + t_0} \text{ for all } (t, t_0 \in \mathbb{R}_+^2$$

is a C_0-quasisemigroup on $\mathbb{R}$.

Definition 1.1. The C_0-quasisemigroup S is said to be:

(i) *stable* (and we denote $s.$) if

$$\sup_{t \in \mathbb{R}_+} \|S(t, t_0)\| < \infty \text{ for all } t_0 \geq 0$$

(ii) *uniformly stable* (and we denote $u.s.$) if

$$\sup_{(t, t_0) \in \mathbb{R}_+^2} \|S(t, t_0\| < \infty.$$

Definition 1.2. The C_0-quasemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ is said to be:

(i) *exponentially stable* (and we denote $e.s.$) if there is $a > 0$ such that:

$$\sup_{t \geq 0} e^{at} \|S(t, t_0)\| < \infty \text{ for all } t_0 \geq 0;$$

(ii) *uniformly exponentially stable* (and we denote $u.e.s.$) if there is $a > 0$ such that:

$$\sup_{(t, t_0) \in \mathbb{R}_+^2} e^{at} \|S(t, t_0)\| < \infty$$

Remark 1.2. It is obvious that

$$\begin{array}{ccc} u.e.s. & \Rightarrow & e.s. \\ \Downarrow & & \Downarrow \\ u.s. & \Rightarrow & s. \end{array}$$

Example 1.2. If $u(t) = 1 + t^2$ then

$$S(t, t_0) = \frac{u(t_0)}{u(t + t_0)} = \frac{1 + t_0^2}{1 + (t + t_0)^2}$$

is C_0-quasisemigroup on $\mathbb{R}$ which is $u.s.$ and it is not $e.s.$

Indeed, from $|S(t, t_0)| \leq 1$ it follows that S is $u.s.$ and if we suppose that S is $e.s.$ then are $a > 0$ and $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:

$$e^{at} \|S(t, t_0)\| \leq M(t_0) \text{ for all } (t, t_0) \in \mathbb{R}_+^2.$$

For $t_0 = 0$ we obtain that

$$\frac{e^{at}}{t^2 + 1} \leq M(0) \text{ for all } t \geq 0$$

which is a contradiction.

Example 1.3. If $v(t) = \exp[2t - t \sin t]$ then the C_0- quasisemigroup

$$S(t, t_0) = \frac{v(t_0)}{v(t + t_0)} = e^{-2t} e^{(t+t_0)\sin(t+t_0) - t_0 \sin t_0}$$

satisfies the inequality

$$e^t S(t, t_0) \leq e^{2t_0} \text{ for all } (t, t_0) \in \mathbb{R}_+^2$$

and hence S is $e.s.$
Because

$$S\Big(\frac{\pi}{2}, 2n\pi - \frac{\pi}{2}\Big) \to \infty$$

it results that S is not $u.s.$ and hence S is also not $u.e.s.$

Remark 1.3. The C_0-quasisemigroup S is e. s. if and only if there are a constant $a > 0$ and an increasing function $M : \mathbb{R}_+ \to [1, \infty)$ such that

$$e^{at}\|S(t+s, t_0)x_0\| \le M(s+t_0)\|S(s, t_0)x_0\|$$

for all $(t, s, t_0) \in \mathbb{R}_+^3$ and all $x_0 \in X$.

Remark 1.4. The C_0-quasisemigroup S is $u.e.s.$ if and only if there exist two positive constants $a, M > 0$ such that:

$$e^{at}\|S(t+s, t_0)x_0\| \le M\|S(s, t_0)x_0\|$$

for all $(t, s, t_0) \in \mathbb{R}_+^3$ and all $X_0 \in X$.

Definition 1.3. Let p be a real number with $1 \le p < \infty$. The C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ is said to be:

(i) *L^p-stable* (and we denote $L^p.s.$) if there exists $N : \mathbb{R}_+ \to [1, \infty)$ such that:

$$\int_0^\infty \|S(t, t_0)x_0\|^p \, dt \le N(t_0)^p\|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$;

(ii) *uniformly-L^p-stable* (and we denote $u.L^p.s.$) if there exists $N > 0$ such that:

$$\int_0^\infty \|S(t, t_0)x_0\|^p \, dt \le N^p\|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$.

It is obvious that if S is $u.L^p.s.$ then it is $L^p.s.$ The converse is not true and this is illustrated by the following:

Example 1.4. If $X = \mathbb{R}$ and

$$S(t, t_0) = \frac{1+t_0}{1+(t+t_0)^2}$$

then

$$\int_0^\infty |S(t, t_0)x_0| \, dt = (t_0^2+1)\Big(\frac{\pi}{2} - \arctan t_0\Big)|x_0| \le \frac{\pi}{2}(t_0^2+1)|x_0|$$

and hence the C_0-quasisemigroup S is L^1-stable. It is easy to see that S is not $u.L^1.s.$

Remark 1.5. If the C_0-quasisemigroup $S : \mathbb{R}^2_+ \to \widetilde{X}$ is:

(i) $e.s.$ then it is L^p-stable for all $p \in [1, \infty)$;
(ii) $u.e.s.$ then it is $u.L^p.s.$ for all $p \in [1, \infty)$.

Definition 1.4. Let p be a real number with $1 \le p < \infty$. The C_0-quasisemigroup $S : \mathbb{R}^2_+ \to \widetilde{X}$ is said to be:

(i) l^p*-stable* (and we denote $l^p.s.$) if there exists $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\sum_{n=0}^{\infty} \|S(n, t_0)x_0\|^p \le N^p(t_0)\|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$;
(ii) *uniformly-*l^p*-stable* (and we denote $u.l^p.s.$) if there exists $N > 0$ such that:

$$\sum_{n=0}^{\infty} \|S(s, t_0)x_0\|^p \le N^p\|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$.

It is obvious that if S is $u.l^p.s.$ then it is $l^p.s.$

Remark 1.6. If the C_0-quasisemigroup $S : \mathbb{R}^2_+ \to \widetilde{X}$ is :

(i) $e.s.$ then it is l^p-stable for all $p \in [1, \infty)$;
(ii) $u.e.s.$ then it is $u.l^p$-stable for all $p \in [1, \infty)$.

This paper is devoted to the relationships between the stability concepts defined above. Thus we obtain some generalizations of some well-known results of Datko and Pazy. We remark that we consider and asymptotic behaviors that are not uniform and that our proofs are not generalization of Datko's proof.

2. Auxiliary results.

We start with the following:

Lemma 2.1. *If $S : \mathbb{R}^2_+ \to \widetilde{X}$ is a C_0-quasisemigroup on X then there is $m > 0$ such that*

$$\|S(t+1, t_0)x_0\|^p \le m \int_t^{t+1} \|S(s, t_0)x_0\|^p \, ds$$

for all $(t, t_0) \in \mathbb{R}^*_+$, $x_0 \in X$ and all $p \in [1, \infty)$.

Proof. Indeed, if we denote

$$\frac{1}{m} = \int_0^1 \frac{dr}{\omega^p(\tau)}$$

where ω is given by condition (q_4) from the definition of the notion of C_0-quasisemigroup, then

$$\frac{\|S(t+1,t_0)x_0\|^p}{m} = \int_0^1 \frac{\|S(t+1,t_0)x_0\|^p}{\omega^p(\tau)}\, d\tau =$$

$$= \int_t^{t+1} \frac{\|S(t+1,T_0)x_0\|^p}{\omega^p(t-s+1)}\, ds \leq \int_t^{t+1} \|S(s,t_0)x_0\|^p\, ds$$

for all $(t,t_0) \in \mathbb{R}_+^2$, $x_0 \in X$ and all $p \in [1,\infty)$. □

Lemma 2.2. *The C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ is*

(i) stable if and only if

$$\sup_{n\in\mathbb{N}} \|S(n,t_0)\| < \infty$$

for all $t_0 \geq 0$.

(ii) uniformly stable if and only if

$$\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \|S(n,t_0)\| < \infty$$

Proof. *Necessity* is obvious.

Sufficiency. Let $t \geq 0$ and $n \in \mathbb{N}$ with $n \leq t < n+1$. Then

$$\|S(t,t_0)\| \leq \|s(t-n,n+t_0)\|\, \|S(n,t_0)\| \leq \omega(1)\|S(n,t_0)\|$$

and hence

$$\sup_{t\in\mathbb{R}_+} \|S(t,t_0)\| < \omega(1) \sup_{n\in\mathbb{N}} \|S(n,t_0)\|$$

and

$$\sup_{(t,t_0)\in\mathbb{R}_+^2} \|S(t,t_0)\| \leq \omega(1) \sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \|S(n,t_0)\|$$

for all $t_0 \geq 0$. □

Lemma 2.3. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following statements are equivalent:*

(i) *S is u.e.s.;*
(ii) *there exists* $u : \mathbb{R}_+ \to \mathbb{R}_+^*$ *with*

$$\lim_{t\to\infty} u(t) = \infty$$

and

$$\sup_{(t,t_0)\in\mathbb{R}_+^2} u(t)\|S(t,t_0)\| < \infty$$

(iii) *there exists* $v : \mathbb{R}_+ \to \mathbb{R}_+^*$ *with*

$$\lim_{t\to\infty} v(t) = 0 \ \ \textit{and} \ \ \|S(t,t_0)\| \le v(t)$$

for all $(t,t_0)\in\mathbb{R}_+^2$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (i)$. Let $\delta > 0$ and $v(\delta) < 1$ and let $n = [t/\delta]$ where $t \ge 0$. We have that

$$\|S(t,t_0)\| \le \omega(\delta)\|S(n\delta,t_0)\| \le \omega(\delta)v(\delta)^n\omega(\delta)e^{n\ln v(\delta)} \le \frac{\omega(\delta)}{v(\delta)}e^{-at}$$

for all $(t,t_0)\in\mathbb{R}_+^2$, where

$$a = -\frac{\ln v(\delta)}{\delta}\,. \qquad \square$$

Lemma 2.4. *If the* C_0*-quasisemigroup* $S : \mathbb{R}_+^2 \to \widetilde{X}$ *is*

(i) L^p*-stable then it is stable;*
(ii) *uniformly-*L^p*-stable then it is uniformly stable.*

Proof. (i) If S is L^p stable then from Definition 1.3. and Lemma 2.1. it follows that there are $m > 0$ and $N : \mathbb{R}_+ \to [1,\infty)$ such that

$$\|S(t+1,t_0)x_0\|^p \le m\int_t^{t+1} \|S(s,t_0)x_0\|^p\,ds \le mN(t_0)^p\|x_0\|^p$$

and hence

$$\sup_{t\ge 0}\|S(t+1,t_0)\| \le m^{\frac{1}{p}}N(t_0)$$

for all $t_0 \ge 0$, which shows that S is stable. The proof of (ii) is analogous. $\square$

3. Uniform exponential stability.

In this section we give necessary and sufficient conditions for uniform exponential stability of C_0-quasisemigroup in Banach spaces.

Theorem 3.1. *The C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ is uniformly exponentially stable if and only if there exists an increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u(0) = 0$ and*

(i) $\lim_{t\to\infty}(u(t) - t) = \infty$
(ii) there exists $N > 0$ such that

$$\int_t^{u(t)} \|S(s, t_0)x_0\| \, ds \leq N\|x_0\|$$

for all $(t, t_0) \in \mathbb{R}_+^2$ and all $x_0 \in X$.

Proof. Necessity. If S is $u.e.s.$ then the conditions (i) and (ii) hold for $u(t) = at$ with $a > 1$.
Sufficiency. From (i) it result that there exists $t_1 > 1$ such that $u(t) \geq t + 1$ for all $t \geq t_1$. By Lemma 2.1 there exists $m > 0$ such that

$$\|S(t+1, t_0)x_0\| \leq m \int_t^{t+1} \|S(s, t_0)x_0\| \, ds \leq$$

$$\leq m \int_t^{u(t)} \|S(s, t_0)\| \, ds \ \leq m\ N\|x_0\|$$

for all $(t, t_0) \in \mathbb{R}_+^2$ and all $x_0 \in X$. Then

$$\sup_{(t,t_0)\in\mathbb{R}_+^2} \|S(t, t_0)\| \leq Nm + \omega(1 + t_1) = M < \infty$$

and hence S is $u.s.$ Because

$$(u(t) - t)\|S(u(t), t_0)x_0\| \leq M \int_t^{u(t)} \|S(s, t_0)x_0\| \, ds \leq MN\|x_0\|$$

and $u_1 : \mathbb{R}_+ \to \mathbb{R}_+^*$ defined by

$$u_1(s) = s - u^{-1}(s) + 1$$

has the properties

$$\lim_{t\to\infty} u_1(t) = \infty \ \text{ and } \ u_1(s)\|S(s, t_0)\| \leq M(1 + N)$$

for all $(s, t_0) \in \mathbb{R}_+^2$. By Lemma 2.3 (ii) it follows that S is $u.e.s.$ $\square$

Theorem 3.2. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following statements are equivalent:*

(i) S is uniformly exponentially stable;

(ii) there is an increasing sequence (u_n) with $u_0 > 0$ and

$$\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} u_n \|S(n, t_0)\| < \infty$$

(iii) there are two increasing sequences (u_n) and (v_n) with the properties

(a) $u_0 > 0$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$;

(b) $r_0 = 0$ and $\lim_{n\to\infty} (v_{n+1} - v_n) < \infty$;

(c) $\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} u_n \|S(v_n, t_0)\| < \infty$.

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial for $u_n = n + 1$ and $v_n = n$

$(iii) \Rightarrow (i)$ The property (a) implies that for every $t \geq 0$ there is $n \in \mathbb{N}$ such that $t \in [v_{n-1}, v_n]$. From (b) it follows that there exists $v \in \mathbb{N}$ such that:

$$v_{n+1} - v_n \leq v$$

for all $n \in \mathbb{N}$. Let $u : \mathbb{R}_+ \to \mathbb{R}_+^*$ be the function defined by

$$u(t) = u_n \text{ for } t \in [v_{n-1}, v_n).$$

The function u is increasing with

$$\lim_{t\to\infty} u(t) = \infty$$

and

$$u(t)\|S(t, t_0)x_0\| \leq u_n\|S(t - v_n, v_n + t_0)\| \, \|S(v_n, t_0)x_0\| \leq$$
$$\leq u_n\omega(v)\|S(v_n, t_0)\| \, \|x_0\|$$

for all $(t, t_0, x_0) \in \mathbb{R}_+^2 \times X$. By Lemma 2.3 it results that S is $u.e.s.$ $\square$

Corollary 3.1. *A C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ is uniformly exponentially stable if and only if*

$$\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} (n + 1)\|S(n, t_0)\| < \infty$$

Proof. *Necessity* is trivial from Definition 1.2
Sufficiency. It follows from Theorem 3.2 for $u_n = n + 1$ and $v_n = n$. □

Theorem 3.3. *For every C_0-quasisemigroup $S : \mathbb{R}^2_+ \to \widetilde{X}$ the following assertions are equivalent:*

(i) S is uniformly exponentially stable;
(ii) there are $N, b > 0$ and $p \in [1, \infty)$ such that

$$\int_t^\infty e^{pbu} \|S(u, t_0)x_0\|^p \, du \leq N^p e^{pbt} \|S(t, t_0)x_0\|^p$$

for all $(t, t_0) \in \mathbb{R}^2_+$ and all $x_0 \in X$;
(iii) there are $N, b > 0$ and $p \in [1, \infty)$ such that

$$\int_0^\infty e^{pbt} \|S(t, t_0)x_0\|^p \, dt \leq N^p \|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$;
(iv) there exists $p \in [1, \infty)$ such that S is uniformly L^p-stable.

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate.
$(iv) \Rightarrow (i)$ If S is $u.L^p.s.$ then by Lemma 2.4 it follows that there exists $M > 0$ such that

$$\|S(t, t_0)\| \leq M \text{ for all } (t, t_0) \in \mathbb{R}^2_+.$$

From Definition 1.3 it results that there exists $N \geq 0$ such that:

$$t^p \|S(t, t_0)x_0\|^p \leq M^p \int_0^{t^p} \|S(s, t_0)x_0\|^p \, ds \leq M^p N^p \|x_0\|^p$$

and hence

$$\sup_{(t,t_0)\in\mathbb{R}^2_+} (t + 1)\|S(t, t_0)\| \leq M(1 + N).$$

By Lemma 2.3. it results that S is $u.e.s.$ □

Corollary 3.2. *For every C_0-quasisemigroup $S : \mathbb{R}^2_+ \to X$ the following statements are equivalent:*

(i) S is u.e.s.;
(ii) S is u. L^p. s. for all $p \in [1, \infty)$;
(iii) there exists $p \in [1, \infty)$ such that S is u. L^p. s.

Proof. It results from Remark 1.5 and Theorem 3.3. □

A discrete variant of Theorem 3.2. is given by

Corollary 3.3. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following assertions are equivalent:*

(i) S is uniformly exponentially stable;
(ii) there are $N, b > 0$ and $p \in [1, \infty)$ such that:

$$\sum_{k=0}^{\infty} e^{bkp} \|S(k, t_0)x_0\|^p \leq N^p e^{pbn} \|S(n, t_0)x_0\|^p$$

for all $(n, t_0, x_0) \in N \times \mathbb{R}_+ \times X$;
(iii) there exists $p \in [1, \infty)$ such that S is uniformly-l^p-stable.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate.
$(iii) \Rightarrow (i)$ If S is $u.l^p.s.$ then Lemma 2.2. it results that S is $u.s.$ and

$$\int_0^\infty \|S(t, t_0)x_0\|^p \, dt \leq \sum_{n=0}^{\infty} \int_n^{n+1} \|S(t-n, n+t_0)x_0\|^p \, dt \leq$$

$$\leq M^p \sum_{n=0}^{\infty} \|S(n, t_0)x_0\|^p \leq M^p N^p \|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$. Then by Theorem 3.2 it results that S is $u.e.s.$ □

Corollary 3.4. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following statements are equivalent:*

(i) S is u.e.s.;
(ii) S is $u.l^p.s.$ for all $p \in [1, \infty)$;
(iii) there exists $p \in [1, \infty)$ such that S is $u.l^p.s.$

Proof. It results from Corollary 3.2. □

4. Nonuniform exponential stability.

In this section we give necessary and sufficient conditions for exponential stability of C_0-quasisemigroups in Banach spaces. In contrast with the uniform case we observe that L^p stability is not a sufficient condition for exponential stability (see Examples 1.2 and 1.4).

An analogous result for the (nonuniform) exponential stability of Theorem 3.2 is given by:

Theorem 4.1. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following assertions are equivalent:*

(i) S is exponentially stable;
(ii) there exists $b > 0$, $p \in [1, \infty)$ and $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\int_t^\infty e^{pbu} \|S(u, t_0)x_0\|^p \, du \leq N(t + t_0)^p e^{pbt} \|S(t, t_0)x_0\|^p$$

for all $(t, t_0, x_0) \in \mathbb{R}_+^2 \times X$.
(iii) there are $b > 0$, $p \in [1, \infty)$ and $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\int_0^\infty e^{pbt} \|S(t, t_0)x_0\|^p \, dt \leq N(t_0)^p \|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$.

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate.
$(iii) \Rightarrow (i)$ The condition (iii) shows that the C_0-quasisemigroup S_1 defined by

$$S_1(t, t_0) = e^{bt} S(t, t_0)$$

is L^p stable. By Lemma 2.4 (i) it results that there exists

$$M : \mathbb{R}_+ \to [1, \infty)$$

such that

$$e^{bt} \|S(t, t_0)\| \leq M(t_0)$$

for all $(t, t_0) \in \mathbb{R}_+^2$ and hence S is $e.s.$ □

The discrete variant of Theorem 4.1 is given by

Corollary 4.1. *For every C_0-quasisemigroup $S : \mathbb{R}_+^2 \to \widetilde{X}$ the following statements are equivalent:*

(i) S is exponentially stable;
(ii) there are $b > 0$, $p \in [1, \infty)$ and $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\sum_{k=n}^\infty e^{pbk} \|S(k, t_0)x_0\|^p \leq N^p(t_0 + n) e^{pbn} \|S(n, t)x_0\|^p$$

for all $(n, t_0, x_0) \in \mathbb{N} \times \mathbb{R}_+ \times X$;
(iii) there are $b > 0$, $p \in [1, \infty)$ and $N : \mathbb{R}_+ \to [1, \infty)$ such that

$$\sum_{n=0}^\infty e^{pbn} \|S(n, t_0)x_0\|^p \leq N(t_0)^p \|x_0\|^p$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$.

Proof. It is analogous with proof of Corollary 3.3. □

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