LE MATEMATICHE
Vol. LXII (2007) – Fasc. II, pp. 147–162

COMPLETENESS THEOREMS: FICHERA'S FUNDAMENTAL RESULTS AND SOME NEW CONTRIBUTIONS

ALBERTO CIALDEA

After recalling Fichera's fundamental results in the study of the problem of the completeness of particular solutions of a partial differential equation, we give some new completeness theorem. They concern the Dirichlet problem for a general elliptic operator of higher order with real constant coefficients in any number of variables.

1. The problem of the completeness and Fichera's results

Let E be an elliptic partial differential operator of order $2m$ with real constant coefficients and no lower order terms:

$$Eu = \sum_{|\alpha|=2m} a_\alpha D^\alpha u$$

with

$$\sum_{|\alpha|=2m} a_\alpha \xi^\alpha \geqslant C|\xi|^{2m} \qquad \forall\, \xi \in \mathbb{R}^n.$$

Denote by $\{\omega_k\}$ a complete system of polynomial solutions of the equation $Eu = 0$. Completeness Theorems:

Entrato in redazione 1 gennaio 2007

AMS 2000 Subject Classification: 42C30, 35J30, 31B10
Keywords: Completeness theorems, higher order elliptic equations, potential theory.

Fichera's fundamental results Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is connected. The boundary $\Sigma = \partial\Omega$ is supposed to be C^1.

The aim of the present paper is to prove that the system

$$\{(\omega_k, \partial_\nu \omega_k, \ldots, \partial_\nu^{m-1} \omega_k)\}$$

is complete in the space $[L^p(\Sigma)]^m$ ($1 \leqslant p < \infty$), where ∂_ν denotes the normal derivative on Σ. Recently such a result was proved for the polyharmonic operator [11].

In this way we shall give a general result concerning the completeness of polynomial solutions in the study of the Dirichlet problem for an elliptic operator of higher order without lower order terms. The case in which there are lower order terms is much more delicate and it will be discussed in a forthcoming paper [10].

The problem of the completeness of particular solutions of a partial differential equations is an old problem in the theory of approximation and it can be formulated in two different ways.

The classical one is the following: suppose we have an elliptic partial differential operator E with complex coefficients, say

$$Eu = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha u, \qquad a_\alpha \in C^\infty(\mathbb{R}^n) \tag{1.1}$$

where

$$\sum_{|\alpha|=m} \alpha(x)\, \xi^\alpha \neq 0 \qquad \forall\, x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n.$$

Let K be a compact set in $\mathbb{R}^n$ and define

$$\Omega(K) = \{f \in C(K) \cap C^\infty(K \setminus \partial K) \mid Ef = 0 \text{ in } K \setminus \partial K\}$$

equipped with the norm

$$\|f\| = \max_K |f(x)|.$$

Let A be an open set such that $K \subset A$ and S be a particular class of functions in $\{u \in C^\infty(A) \mid Eu = 0 \text{ in } A\}$. The problem is to give conditions under which

$$\overline{S} = \Omega(K). \tag{1.2}$$

In the case of holomorphic functions of one complex variable, i.e. $n = 2$, $E = \partial_x + i\partial_y$ and S equal to the system of polynomials in the variable z, Mergelyan [25, 26] proved that (1.2) holds if and only if $\mathbb{R}^2 \setminus K$ is connected.

Even if a definite answer like the Mergelyan one is not known for the partial differential operators (1.1), several general results are available. We mention, in particular, the important contributions of Lax [21], Malgrange [23] and Browder [4].

For example, suppose that (1.1) and its adjoint

$$E^*u = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x)\, u)$$

possesses in B the unique continuation property (B being an open set such that $\overline{A} \subset B$), $K \setminus \partial K$ satisfies the restricted cone hypothesis and $B \setminus K$ is connected. Under these conditions (1.2) holds, provided that S is one of the following spaces:

$$S = \{p(x) \mid p(x) = \int_H \varphi(y)\, s(x,y)\, dy, \; \forall\, \varphi \in C^\infty(\overline{H})\}$$

$$S = \{p(x) \mid p(x) = \sum_{j=1}^{m} c_j s(x,y_j), \; \forall\, y_1,\ldots,y_m \in H\}$$

where H is compact set in $B \setminus \overline{A}$ and $s(x,y)$ is a fundamental solution for E. For a proof of this result see [4] or [19].

If the coefficients of (1.1) are constant one can ask if (1.2) holds, S being the class of polynomial solutions of the equation $Eu = 0$. This case, which is much more delicate, was deeply investigated by Malgrange [23] and we refer to his paper for the relevant results.

The problem of completeness can be formulated in a deeper way, which-Completeness Theorems:
Fichera's fundamental results was indicated many years ago by Mauro Picone [30]. He posed the following problem: let E be a partial differential operator

$$Eu = \sum_{|\alpha| \leqslant 2m} a_\alpha(x) D^\alpha u$$

defined in $\mathbb{R}^n$ and let $B_1,\ldots,B_s$ some partial differential operators defined on the boundary Σ of a bounded domain Ω. Let us suppose that there exists a solution of the problem

$$\begin{cases} Eu = 0 & \text{in } \Omega \\ B_h u = f_h & \text{on } \Sigma \text{ (h=1,...,s)} \end{cases} \tag{1.3}$$

if and only if $(f_1,\ldots,f_s)$ satisfies a finite number of compatibility conditions

$$\sum_{h=1}^{s} \int_\Sigma f_h \psi_h^{(k)} d\sigma = 0 \qquad k = 1,\ldots,\mu,$$

that is to say that problem (1.3) is an index problem.

Let us denote by $\{\omega_k\}$ a particular sequence of solutions of the equation $Eu = 0$ in A, where A is a domain such that $\overline{\Omega} \subset A$.

The problem posed by Picone is to find under which conditions for Ω, E, B_h and $\{\omega_k\}$ the system $\{(B_1\omega_k, \dots, B_s\omega_k)\}$ is complete in the space

$$\left\{(v_1, \dots, v_s) \in [L^p(\Sigma)]^s \;\middle|\; \sum_{h=1}^{s} \int_\Sigma v_h \psi_h^{(k)} d\sigma = 0,\ k = 1, \dots, \mu \right\}.$$

Let us consider the very particular case: $n = 2$, $E = \Delta_2$, $s = 1$, $Bu = u|_\Sigma$ and $\{\omega_k\}$ is the system of harmonic polynomials, i.e. $\omega_{2k} = \Re z^k$, $\omega_{2k+1} = \Im z^k$ ($k = 0, 1, 2, \dots$). It is interesting to remark that, if Ω is the unit disk, the completeness of $\{\omega_k\}$ in $L^p(\Sigma)$ is nothing but the completeness of the trigonometric system $\omega_{2k} = \cos k\vartheta$, $\omega_{2k+1} = \sin k\vartheta$ ($k = 0, 1, 2, \dots$).

If we allow Σ to be an open manifold and we take $\Sigma = [0, 1]$, the completeness of the system of harmonic polynomials $\{\omega_k\}$ in $C^0(\Sigma)$ is just the Weierstrass theorem.

The reason why the completeness in the sense of Picone is interesting is that it gives informations on how to approximate a solution of a boundary value problem. For example, let us consider the Dirichlet problem for Laplace equation

$$\begin{cases} u \in C^0(\overline{\Omega}) \cap C^2(\Omega) & \\ \Delta_2 u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \Sigma \end{cases} \tag{1.4}$$

($\varphi \in C^0(\Sigma)$). If we know that the system of harmonic polynomials $\{\omega_k\}$ is complete in $C^0(\Sigma)$, we can find a sequence $\{p_m\}$ of harmonic polynomials such that $\|p_m - \varphi\|_{C^0(\Sigma)} \to 0$. Therefore the sequence $\{p_m\}$ will converge also in $C^0(\overline{\Omega})$, because of the maximum principle. It is easy to see that $\{p_m\}$ will converge to the solution u of the Dirichlet problem (1.4).

There are two numerical methods which are based on the completeness in the sense of Picone. For a brief description of these we refer to [19, p.36–37]

Fichera [16] was the first one to prove some completeness theorems in the sense of Picone. The theorems he proved concern the system of harmonic polynomials in any number of variables. To be more precise, denote by $\{\omega_k\}$ such a system. This can be obtained by ordering in one sequence the polynomials

$$|x|^h Y_{hs}\left(\frac{x}{|x|}\right) \qquad (s = 1, \dots, p_{nh}, h = 0, 1, \dots)$$

where $\{Y_{hs}\}$ ($s = 1, \dots, p_{nh}, h = 0, 1, \dots$) is a complete system of ultra-spherical harmonics and $p_{nh} = (2h+n-2)(h+n-3)!/((n-2)!h!)$. It can be proved that any harmonic polynomial can be written as a finite linear combination of ω_k.

Fichera [16] proved the following results, which are related to the main boundary value problems for Laplace equation: the Dirichlet, the Neumann and the mixed problem:

(i) the system $\{\omega_k\}$ is complete in $L^2(\Sigma)$;

(ii) the system $\{\partial_\nu \omega_k\}$ is complete in

$$\left\{ v \in L^2(\Sigma) \;\Big|\; \int_\Sigma v\, d\sigma = 0 \right\};$$

(iii) if $\Sigma = \Sigma_1 \cup \Sigma_2$, the system $\{(\omega_k|_{\Sigma_1}, \partial_\nu \omega_k|_{\Sigma_2})\}$ is complete in $L^2(\Sigma_1) \times L^2(\Sigma_2)$.

As far as the theorems (i) and (ii) are concerned, you could repeat Fichera's proof word by word and obtain the corresponding completeness theorem in $L^p(\Sigma)$ for any $1 \leqslant p < \infty$. By small changes, one can prove that the same results hold in $C^0(\Sigma)$ (see [19]).

In the case of (iii) the situation is different. It is clear that the result for $p = 2$ implies the completeness in $L^p(\Sigma)$ for any $1 \leqslant p \leqslant 2$, but the completeness of $\{(\omega_k|_{\Sigma_1}, \partial_\nu \omega_k|_{\Sigma_2})\}$ in $L^p(\Sigma_1) \times L^p(\Sigma_2)$ for $p > 2$ is still an open problem.

The idea of Fichera's proof is to show at first that the completeness property is equivalent to the corresponding uniqueness theorem in a certain class of functions, which was introduced by Amerio [1]. Namely let us denote by $\mathscr{A}^p$ the following class

$$\mathscr{A}^p = \Big\{ u \in L^p(\Omega) \mid \exists\, \alpha, \beta \in L^p(\Sigma) :$$
$$\int_\Omega u\, \Delta_2 w\, dx = \int_\Sigma (\alpha\, \partial_\nu w - \beta\, w)\, d\sigma, \quad \forall\, w \in C^\infty(\mathbb{R}^n) \Big\}.$$

In view of Caccioppoli-Weyl lemma, a function $u \in \mathscr{A}^p$ is harmonic. Fichera [16] obtained the uniqueness results in $\mathscr{A}^p$ by means of a useful and ingenious representation theorem. He proved that u belongs to $\mathscr{A}^p$ if and only if there exists $\varphi \in L^p(\Sigma)$ such that

$$u(x) = \int_\Sigma \varphi(y)\, s(x-y)\, d\sigma_y$$

s being the fundamental solution for Laplace equation. With this representation formula he was able to prove the following uniqueness results:

(i) if $u \in \mathscr{A}^p$ and $\alpha = 0$, then $u = 0$;

(ii) if $u \in \mathscr{A}^p$ and $\beta = 0$, then $u = \text{const.}$;

(ii) if $u \in \mathscr{A}^2$ and $\alpha|_{\Sigma_1} = 0$, $\beta|_{\Sigma_2} = 0$, then $u = 0$.

After Fichera's results, several completeness theorems have been obtained for particular partial differential equations. We mention the biharmonic equation [3, 14, 27], the elasticity system [7, 15, 17], the heat equation [22] and general

2nd order elliptic equations [2, 6, 18, 28]. We also mention [8] where the laplacian in any number of variables is considered and completeness theorems for the oblique derivative problem are proved.

All of these results are proved on smooth boundary, namely on Lyapunov boundaries. Very few results are known on non smooth boundaries (see [5, 12, 13]).

2. A result of potential theory

We recall that the function h is said to be *essentially homogeneous of degree* α if $h(x) = h_1(x)\log x + h_2(x)$ where $h_2(\rho x) = \rho^\alpha h(x)$, $x \neq 0$, $\rho > 0$ and $h_1(x)$ is a homogeneous polynomial of degree α if α is a nonnegative integer, $h_1(x) \equiv 0$ otherwise.

In [11], by extending the results proved in [9] for Lyapunov boundaries, the following theorem was proved

Theorem 2.1. *Let $\Sigma \in C^1$. Let $h \in C^2(\mathbb{R}^n \setminus \{0\})$ be even and essentially homogeneous of degree $2-n$. If $\varphi \in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ, then*

$$\lim_{x \to x_0} \left(\int_\Sigma \varphi(y)\, \partial_{x_k}[h(x-y)]\, d\sigma_y - \int_\Sigma \varphi(y)\, \partial_{x_k}[h(x'-y)]\, d\sigma_y \right) =$$

$$2\, \nu_k(x_0)\, \gamma(x_0)\, \varphi(x_0)$$

where x is a point on the inner normal to Σ at x_0, x' is its symmetric with respect to x_0 and $\gamma(x_0)$ is given by

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2} \displaystyle\int_{|\xi|=1} \Delta h_2(\xi)\, \log|\xi \cdot \nu_{x_0}|\, d\sigma_\xi & \text{if } n = 2 \\[2ex] \dfrac{1}{2} \displaystyle\int_{|\xi|=1} [(2-n)h(\xi) - \Delta h(\xi)\, \log|\xi \cdot \nu_{x_0}|]\, d\sigma_\xi & \text{if } n \geqslant 3. \end{cases}$$

Moreover, as remarked in [11], the function γ can be expressed by means of the Fourier transform of the kernel h:

$$\gamma(x_0) = \frac{1}{2} \mathscr{F}(\Delta h)(\nu_{x_0}) = -2\pi^2 \mathscr{F}(h)(\nu_{x_0})$$

where the Laplacian Δ has to be understood in the sense of distributions and $\mathscr{F}$ denotes the Fourier transform.

Let E be the operator

$$Eu = \sum_{|\alpha|=2m} a_\alpha D^\alpha u \tag{2.1}$$

where a_α are real constants.

We suppose that the operator is elliptic, i.e.

$$Q(\xi) \neq 0$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, where

$$Q(\xi) = \sum_{|\alpha|=2m} a_\alpha \xi^\alpha.$$

Let now $S(x-y)$ the following functions:

$$S(x-y) = \frac{1}{4(2\pi i)^{n-1}(2m-1)!} (\Delta_y)^{(n-1)/2} \int_{|\omega|=1} \frac{|(x-y)\cdot\omega|^{2m-1}}{Q(\omega)} d\sigma_\omega \tag{2.2}$$

for n odd, and

$$S(x-y) = \frac{-1}{(2\pi i)^n (2m)!} (\Delta_y)^{n/2} \int_{|\omega|=1} \frac{|(x-y)\cdot\omega|^{2m} \log|(x-y)\cdot\omega|}{Q(\omega)} d\sigma_\omega \tag{2.3}$$

for n even. As it was shown by Fritz John [20, p.65–72], $S(x-y)$ is a fundamental solution for (2.1).

Theorem 2.2. *Let $\Sigma \in C^1$. Let $\varphi \in L^1(\Sigma)$ and $x_0 \in \Sigma$ be a Lebesgue point for φ. For any multi-index α with $|\alpha| = 2m-1$, we have*

$$\lim_{x\to x_0} \left(\int_\Sigma \varphi(y) D_y^\alpha [S(x-y)]\, d\sigma_y - \int_\Sigma \varphi(y) D_y^\alpha [S(x'-y)]\, d\sigma_y \right) = \\ - \frac{\nu^\alpha(x_0)}{Q(\nu(x_0))} \varphi(x_0) \tag{2.4}$$

where x is a point on the inner normal to Σ at x_0 and x' is its symmetric with respect to x_0.

Proof. First write $\alpha = \alpha_0 + \alpha_1$, with $|\alpha_0| = 1$, $|\alpha_1| = 2m-2$ and then

$$\int_\Sigma \varphi(y) D_y^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y \\ = -D_x^{\alpha_0} \int_\Sigma \varphi(y) D_y^{\alpha_1} [S(x-y) - S(x'-y)]\, d\sigma_y. \tag{2.5}$$

Since $S(x)$ is essentially homogeneous of degree $2m-n$, $D^{\alpha_1}S(x)$ is essentially homogeneous of degree $2-n$ and Theorem 2.1 gives

$$\lim_{x\to x_0} D_x^{\alpha_0} \int_\Sigma \varphi(y)\, D_y^{\alpha_1}[S(x-y)-S(x'-y)]\, d\sigma_y = 2\,\nu^{\alpha_0}(x_0)\,\gamma_{\alpha_1}(x_0)\,\varphi(x_0),$$

where

$$\gamma_{\alpha_1}(x_0) = -2\,\pi^2 \mathscr{F}(D^{\alpha_1}S)(\nu_{x_0}).$$

On the other hand, $ES=\delta$ and $(-4\pi^2)^m Q(x)\mathscr{F}(S)(x)=1$. This leads to

$$-2\pi^2\mathscr{F}(D^{\alpha_1}S)(x) = -2\pi^2(2\pi i)^{2m-2}x^{\alpha_1}\mathscr{F}(S)(x) = \frac{1}{2}\,\frac{x^{\alpha_1}}{Q(x)}$$

and (2.4) follows from (2.5). □

Lemma 2.3. *Let α be a multi-index with $|\alpha|=m-1$. Let us denote by u the potential*

$$u(x) = \int_\Sigma \varphi(y)\, D_y^{\alpha}[S(x-y)]\, d\sigma_y\,.$$

If $\varphi\in L^p(\Sigma)$, then $u\in W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

For the proof of this Lemma we refer to [11, Corollary 1].

3. Completeness Theorems

Let $\{\omega_k\}$ denote a complete system of polynomial solutions of the equation $Eu=0$.

It is possible to extend the classical procedure for obtaining harmonic polynomials to the case of the poly-harmonic operator Δ^m. In fact, a complete system of polyharmonic polynomials $\{\omega_k^{(m)}\}$ is given by

$$|x|^{h+2j}Y_{hs}\left(\frac{x}{|x|}\right) \qquad (j=0,\dots,m-1, s=1,\dots,p_{nh}, h=0,1,\dots)$$

($p_{nh}=(2h+n-2)(h+n-3)!/((n-2)!h!)$), where $\{Y_{hs}\}$ ($s=1,\dots,p_{nh}, h=0,1,\dots$) is a complete system of ultra-spherical harmonics (see, e.g., [11]).

In the particular case of two independent variables, a system of polynomial solutions of the equation $Eu=0$ was constructed in [5] in the following way. The operator E can be written as $E=E_1^{k_1}\dots E_m^{k_m}$, where

$$E_i = a_0^{(i)}\frac{\partial^2}{\partial x^2} + a_1^{(i)}\frac{\partial^2}{\partial x\partial y} + a_2^{(i)}\frac{\partial^2}{\partial y^2}$$

are elliptic operators, and

$$Q_i(w) = a_0^{(i)} w^2 + a_1^{(i)} w + a_2^{(i)}.$$

Let $\lambda_i \in \mathbb{C}$ such that $Q_i(\lambda_i) = Q_i(\overline{\lambda}_i) = 0$ with $\Im \lambda_i < 0$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. If p is a homogenous polynomial of degree $k \geqslant 2m$, we have that $Lp = 0$ if and only if p is a finite linear combination of the following polynomials

$$\begin{cases} \rho_1^h(\lambda_1 x + y)^{k-2h};\ \rho_1^h(\overline{\lambda}_1 x + y)^{k-2h}, & h = 0, 1, \ldots, k_1 - 1 \\ \cdots \\ \rho_m^h(\lambda_m x + y)^{k-2h};\ \rho_m^h(\overline{\lambda}_m x + y)^{k-2h}, & h = 0, 1, \ldots, k_m - 1 \end{cases}$$

where $\rho_i = a_2^{(i)} x^2 - a_1^{(i)} xy + a_0^{(i)} y^2$. These polynomials are complex; if we want real polynomials, it is sufficient to take

$$\begin{cases} \rho_1^h \Re(\lambda_1 x + y)^{k-2h};\ \rho_1^h \Im(\lambda_1 x + y)^{k-2h}, & h = 0, 1, \ldots, k_1 - 1 \\ \cdots \\ \rho_m^h \Re(\lambda_m x + y)^{k-2h};\ \rho_m^h \Im(\lambda_m x + y)^{k-2h}, & h = 0, 1, \ldots, k_m - 1. \end{cases}$$

For the construction of system of polynomial solutions for more general operators, we refer to [29, 31, 32]

For the sake of completeness we give the following result

Theorem 3.1. *There exists one and only one solution $u \in W^{m,p}(\Omega)$ of the Dirichlet problem*

$$Eu = f, \qquad u - g \in \mathring{W}^{m,p}(\Omega)$$

where $f \in W^{-m,p}(\Omega)$, $g \in W^{m,p}(\Omega)$ are given.

Proof. Let us write the operator E in the form

$$Eu = \sum_{|\alpha|=|\beta|=m} (-1)^m a_{\alpha\beta} D^\alpha D^\beta u$$

and set

$$B(u,v) = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\beta u D^\alpha v\, dx.$$

Because of the ellipticity we have

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u D^\beta u\, dx \geqslant K \int_\Omega |\nabla_m u|^2 dx \qquad \forall\, u \in C_0^\infty(\Omega)$$

where ∇_m denotes the gradient of order m. Poincaré's inequality implies the Gårding inequality

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u D^\beta u\, dx \geqslant C\|u\|^2_{W^{m,2}(\Omega)} \qquad \forall\, u \in C_0^\infty(\Omega)$$

and the result follows from a known general existence and uniqueness Theorem (see [24, p.303]). $\square$

Theorem 3.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\Sigma = \partial\Omega$ is C^1. Let $1 \leqslant p < \infty$. The system*

$$\{(\omega_k, \partial_\nu \omega_k, \dots, \partial_\nu^{m-1}\omega_k)\}$$

is complete in $[L^p(\Sigma)]^m$.

Proof. Suppose $1 < p < \infty$. Let $(\varphi_1, \dots, \varphi_m) \in [L^q(\Sigma)]^m$ ($q = p/(p-1)$) be such that

$$\int_\Sigma (\varphi_1 \omega_k + \dots + \varphi_m \partial_\nu^{m-1}\omega_k)\, d\sigma = 0 \qquad \forall\, \omega_k \in \{\omega_k\}. \tag{3.1}$$

We have to show that $\varphi_1 = \dots \varphi_{m-1} = 0$.

There exists $R > 0$ such that, for any $x \in \mathbb{R}^n$, $|x| > R$,

$$S(x-y) = \sum_{|\alpha|=0}^{\infty} c_\alpha(x)\, w_\alpha(y) \tag{3.2}$$

uniformly for $y \in \overline{\Omega}$, where w_α are polynomial solutions of the equation $Eu = 0$. We shall prove that by using an idea which Fichera showed me in a private communication.

Fix ξ such that $|\xi| = 1$. Since S is analytic, there exists $r_\xi > 0$ such that

$$S(t-\xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} [D^\alpha S(v)]_{v=-\xi}\, t^\alpha \tag{3.3}$$

uniformly for $|t| \leqslant r_\xi$. From the compactness of the unit sphere it follows that there exists $r > 0$, which does not depend on ξ, such that (3.3) holds uniformly for $|t| \leqslant r$, for any ξ, $|\xi| = 1$.

Suppose n odd or $n > 2m$ even; in this case S is homogeneous of degree $2m - n$ (see (2.2), (2.3)) and we can write

$$S(x-y) = |x|^{2m-n} S(\xi - y/|x|) = |x|^{2m-n} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} [D^\alpha S(v)]_{v=-\xi} \left(\frac{y}{|x|}\right)^\alpha .$$

On the other hand

$$[D^\alpha S(v)]_{v=-x} = |x|^{2m-n-|\alpha|}[D^\alpha S(v)]_{v=-\xi}$$

and then

$$S(x-y) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!}[D^\alpha S(v)]_{v=-x}\, y^\alpha$$

uniformly for $|y| \leqslant r|x|$.

We have thus obtained, uniformly for $|y| \leqslant r|x|$,

$$S(x-y) = \sum_{k=0}^{\infty} \sum_{j=1}^{n_k} R_j^{(k)}(x)\, P_j^{(k)}(y) \tag{3.4}$$

where $R_1^{(k)}(x), \ldots, R_{n_k}^{(k)}$ form a basis for the functions $[D^\alpha S(v)]_{v=-x}$ ($|\alpha| = k$) and $P_j^{(k)}(y)$ are homogeneous polynomials of degree k.

Let us suppose n even, $n \leqslant 2m$. In this case S is essentially homogeneous of degree $2m-n$ and (see (2.3))

$$S(x-y) = \frac{-\log|x|}{(2\pi i)^n(2m)!}(\Delta_y)^{n/2}\int_{|\omega|=1}\frac{|(x-y)\cdot\omega|^{2m}}{Q(\omega)}\,d\sigma_\omega -$$

$$\frac{|x|^{2m-n}}{(2\pi i)^n(2m)!}(\Delta_y)^{n/2}\int_{|\omega|=1}\frac{\left|\left(\xi-\frac{y}{|x|}\right)\cdot\omega\right|^{2m}\log\left|\left(\xi-\frac{y}{|x|}\right)\cdot\omega\right|}{Q(\omega)}\,d\sigma_\omega.$$

This shows that we have

$$S(x-y) = q(x-y)\log|x| + |x|^{2m-n}S\left(\xi - \frac{y}{|x|}\right)$$

where q is a polynomial of degree $2m-n$, and then

$$S(x-y) = q(x-y)\log|x| + |x|^{2m-n}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{1}{\alpha!}[D^\alpha S(v)]_{v=-\xi}\left(\frac{y}{|x|}\right)^\alpha$$

uniformly for $|y| \leqslant r|x|$.

Therefore also in this case a formula like (3.4) holds uniformly for $|y| \leqslant r|x|$. It is clear that we can derive such expansion term by term and this implies $EP_j^{(k)} = 0$.

Now we can say that (3.2) holds uniformly for $y \in \overline{\Omega}$, provided $|x| \geqslant R$, where $R = (\max_{x\in\overline{\Omega}}|x|)/r$. It is also clear that we can derive (3.2) term by term with respect to the variables y and that the corresponding expansions converge uniformly for $y \in \overline{\Omega}$ ($|x| \geqslant R$).

Keeping in mind (3.1), it follows that

$$\sum_{h=1}^{m} \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} S(x-y)\, d\sigma_y = \sum_{|\alpha|=0}^{\infty} \sum_{h=1}^{m} c_\alpha(x) \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} w_\alpha(y)\, d\sigma_y = 0 \quad (3.5)$$

for any x such that $|x| > R$. Set

$$u(x) = \sum_{h=1}^{m} \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} S(x-y)\, d\sigma_y.$$

The potential u is solution of the equation $Eu = 0$. Since u is analytic in the connected set $\mathbb{R}^n \setminus \overline{\Omega}$, (3.5) implies

$$u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

On the other hand $u \in W^{m,p}_{\mathrm{loc}}(T)$ (see Lemma 2.3) and Ω satisfies the restricted cone hypothesis; then there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that u_n tends to u in $W^{m,p}(\Omega)$ (see [4, p.148–149]). This means that $u \in \mathring{W}^{m,p}(\Omega)$ and then $u = 0$ in Ω, in view of Theorem 3.1.

Therefore we can say that

$$u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma. \quad (3.6)$$

Let α be a multi-index with $|\alpha| = m$; we have

$$D^\alpha u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma,$$

i.e.

$$\sum_{h=1}^{m} \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} D_x^\alpha [S(x-y)]\, d\sigma_y = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma.$$

This implies

$$\lim_{x \to x_0} \sum_{h=1}^{m} \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} D_x^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y = 0$$

for any $x_0 \in \Sigma$, where x, x' have the same meaning as in Theorem 2.2.

On the other hand, due to the weak singularities of the kernels, we have

$$\lim_{x \to x_0} \sum_{h=1}^{m-1} \int_{\Sigma} \varphi_h(y)\, \partial_{\nu_y}^{h-1} D_x^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y = 0$$

and then

$$\lim_{x \to x_0} \int_\Sigma \varphi_m(y)\, \partial_{\nu_y}^{m-1} D_x^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y = 0.$$

In view of (2.4) we have also

$$\begin{aligned}
&\lim_{x \to x_0} \int_\Sigma \varphi_m(y)\, \partial_{\nu_y}^{m-1} D_x^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y \\
&= \lim_{x \to x_0} \sum_{|\beta|=m-1} \int_\Sigma \varphi_m(y)\, \nu^\beta(y)\, D_x^\alpha D_y^\beta [S(x-y) - S(x'-y)]\, d\sigma_y \\
&= (-1)^{m-1} \lim_{x \to x_0} \sum_{|\beta|=m-1} \frac{(m-1)!}{\beta!} \int_\Sigma \varphi_m(y)\, \nu^\beta(y)\, D_y^{\alpha+\beta} [S(x-y) - S(x'-y)]\, d\sigma_y \\
&= (-1)^{m-1} \frac{\nu^\alpha(x_0)}{Q(\nu(x_0))} \left(\sum_{|\beta|=m-1} \frac{(m-1)!}{\beta!} (\nu^\beta(x_0))^2 \right) \varphi_m(x_0) \\
&= (-1)^{m-1} \frac{\nu^\alpha(x_0)}{Q(\nu(x_0))}\, \varphi_m(x_0)
\end{aligned}$$

for almost every $x_0 \in \Sigma$.

This leads to

$$\nu^\alpha(x_0)\, \varphi_m(x_0) = 0$$

almost everywhere on Σ and for any multi-index α with $|\alpha| = m$. Then we have also

$$\varphi_m(x_0) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (\nu^\alpha(x_0))^2 \varphi_m(x_0) = 0$$

i.e. $\varphi_m = 0$ almost everywhere on Σ.

Now (3.6) implies

$$\lim_{x \to x_0} \sum_{h=1}^{m-1} \int_\Sigma \varphi_h(y)\, \partial_{\nu_y}^{h-1} D_x^\alpha [S(x-y) - S(x'-y)]\, d\sigma_y = 0$$

for any multi-index α with $|\alpha| = m+1$. An argument similar to the previous one leads to $\varphi_{m-1} = 0$ a.e. and the result follows by induction.

The completeness for $p = 1$ is a consequence of the completeness for $p > 1$. □

REFERENCES

[1] L. Amerio, *Sull'integrazione dell'equazione $\Delta_{2u} - \lambda^2 u = f$ in un dominio di connessione qualsiasi*, Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9** (1945), no. 78, 79–102.

[2] L. Amerio, *Sul calcolo delle soluzioni dei problemi al contorno per le equazioni lineari del secondo ordine di tipo ellittico*, Amer. J. Math. **69** (1947), 447–489.

[3] R.B. Ancora, *Problemi analitici connessi alla teoria della piastra elastica appoggiata*, Rend. Sem. Mat. Univ. Padova **20** (1951), 99–134.

[4] F.E. Browder, *Approximation by solutions of partial differential equations*, Amer. J. Math. **84** (1962), 134–160.

[5] A. Cialdea, *Teoremi di completezza connessi con equazioni ellittiche di ordine superiore in due variabili in un campo con contorno angoloso*, Rend. Mat. (7) **5** (1985), no. 3-4, 327–344 (1988).

[6] A. Cialdea, *L'equazione $\Delta_2 u + a_{10}(x,y)\frac{\partial u}{\partial x} + a_{01}(x,y)\frac{\partial u}{\partial y} + a_{00}(x,y)u = F(x,y)$. Teoremi di completezza*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **81** (1987), no. 3, 245–257.

[7] A. Cialdea, *Formule di maggiorazione e teoremi di completezza relativi al sistema dell'elasticità tridimensionale*, Riv. Mat. Univ. Parma (4) **14** (1988), 283–302.

[8] A. Cialdea, *Sul problema della derivata obliqua per le funzioni armoniche e questioni connesse*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **12** (1988), no. 1, 181–200.

[9] A. Cialdea, *A general theory of hypersurface potentials*, Ann. Mat. Pura Appl. (4) **168** (1995), 37–61.

[10] A. Cialdea, *Completeness theorems for elliptic equations of higher order with constant coefficients*, Georgian Math. J. **14** (2007), no. 1, 81–97.

[11] A. Cialdea - A. Malaspina, *Completeness theorems for the Dirichlet problem for the polyharmonic equation*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **29** (2005), no. 1, 153–173.

[12] M.P. Colautti, *Sul problema di Neumann per l'equazione $\Delta_2 u - \lambda c u = f$*

in un dominio piano a contorno angoloso, Mem. Accad. Sci. Torino Parte I (3) **4** (1959/1960), 1–83.

[13] M.P. Colautti, *Teoremi di completezza in spazi Hilbertiani connessi con l'equazione di Laplace in due variabili*, Rend. Sem. Mat. Univ. Padova **31** (1961), 114–164.

[14] G. Fichera, *Teoremi di completezza connessi all'integrazione dell'equazione* $\Delta_4 u = f$, Giorn. Mat. Battaglini (4) **77** (1947), 184–199.

[15] G. Fichera, *Sui problemi analitici dell'elasticità piana*, Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 1–22.

[16] G. Fichera, *Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni*, Annali di Matematica Pura ed Applicata. Serie Quarta **27** (1948), 1–28.

[17] G. Fichera, *Sull'esistenza e sul calcolo delle soluzioni dei problemi al contorno, relativi all'equilibrio di un corpo elastico*, Ann. Scuola Norm. Super. Pisa (3) **4** (1950), no. 248, 35–99.

[18] G. Fichera, *Alcuni recenti sviluppi della teoria dei problemi al contorno per le equazioni alle derivate parziali lineari*, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 174–227.

[19] G. Fichera, *The problem of the completeness of systems of particular solutions of partial differential equations*, Numerical mathematics (Sympos., Inst. Appl. Math., Univ. Hamburg, Hamburg, 1979), Internat. Ser. Numer. Math., vol. 49, Birkhäuser, Basel, 1979, pp. 25–41.

[20] F. John, *Plane waves and spherical means applied to partial differential equations*, Interscience Publishers, New York-London, 1955.

[21] P.D. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations*, Comm. Pure Appl. Math. **9** (1956), 747–766.

[22] E. Magenes, *Sull'equazione del calore: teoremi di unicità e teoremi di completezza connessi col metodo di integrazione di M. Picone. I, II*, Rend. Sem. Mat. Univ. Padova **21** (1952), 99–123, 136–70.

[23] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355.

[24] V.G. Maz'ya - T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Monographs and Studies in Mathematics, vol. 23, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[25] S.N. Mergelyan, *On the representation of functions by series of polynomials on closed sets*, Amer. Math. Soc. Translation **1953** (1953), no. 85, 287–293, Translated from *Doklady Akad. Nauk SSSR* (N.S.) 78 (1951), 405-408.

[26] S.N. Mergelyan, *Uniform approximations to functions of a complex variable*, Amer. Math. Soc. Translation **1954** (1954), no. 101, 294–391, Translated from *Uspehi Matem. Nauk (N.S.)* 7, no. 2(48) (1952), 31-122.

[27] C. Miranda, *Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche de due variabili*, Giorn. Mat. Battaglini (4) **2 (78)** (1948), 97–118.

[28] C. Miranda, *Partial differential equations of elliptic type*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Springer-Verlag, New York, 1970, Second revised edition.

[29] P.S. Pedersen, *A basis for polynomial solutions to systems of linear constant coefficient PDE's*, Adv. Math. **117** (1996), no. 1, 157–163.

[30] M. Picone, *Nuovo metodo d'approssimazione per la soluzione del problema di Dirchlet.*, Rom. Acc. L. Rend. (5) **31** (1922), 357–359.

[31] B. Reznick, *Homogeneous polynomial solutions to constant coefficient PDE's*, Adv. Math. **117** (1996), no. 2, 179–192.

[32] S.P. Smith, *Polynomial solutions to constant coefficient differential equations.*, Trans. Am. Math. Soc. **329** (1992), no. 2, 551–569.

ALBERTO CIALDEA
Dipartimento di Matematica e Informatica,
Università della Basilicata,
Viale dell'Ateneo Lucano 10, 85100, Potenza, Italy.
e-mail: `cialdea@email.it`.