SINGULAR BIELLIPTIC CURVES
AND WEIERSTRASS POINTS

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Here we study the Weierstrass points of singular bielliptic curves in characteristic 0. Most of our results are existence results of the type “there exists a bielliptic curve $Y$ with certain singular points and with a Weierstrass point $P \in Y_{\text{reg}}$ with a prescribed gap sequence”. Another main result is a smoothness one for the set of all genus $g$ bielliptic curves with prescribed singularities.

0. Introduction.

In this paper we study the Weierstrass points of singular bielliptic curves. Unless otherwise stated, we work over an algebraically closed field $K$ with $\text{char}(K) = 0$. For the case $\text{char}(K) > 0$, but $\text{char}(K) \neq 2$, see Remark 1.12. Let $Y$ be an integral projective bielliptic curve with $g := p_a(Y) \geq 6$. Hence there exists a double covering $f : Y \to C$ with $p_a(C) = 1$. We assume that $C$ is smooth. For a remark in the case in which $C$ is singular, i.e. in which $C$ is a rational curve with a unique ordinary node or a unique ordinary cusp as singularities, see Remark 1.11. We want to study simultaneously the smooth points of $Y$ which are Weierstrass points and all the singular points. We list all possible singularities and the possible types of singular points of $Y$ as Weierstrass points (see 1.11). Most of our results are existence results of the

Entrato in Redazione il 9 ottobre 1999.

This research was partially supported by MURST (Italy).
type “there exists a bielliptic curve $Y$ with certain singular points and with a
Weierstrass point $P \in Y_{\text{reg}}$ with a prescribed gap sequence” (see Theorems 0.1,
1.9 and 1.10). Another main result is a smoothness one for the set of all genus
$g$ bielliptic curves with prescribed singularities (see Theorem 0.2). To get the
flavour of our results we state the two main ones.

**Theorem 0.1.** Fix positive integers $g, k, z$ and $\delta_1, \ldots, \delta_k$. For every integer
$i$ with $1 \leq i \leq k$ fix a label: “cusp with invariant $\delta_i$” or “tacnode with
invariant $\delta_i$”. Let $\gamma$ be the number of labels “cusp!”. Assume $z \geq g \geq 6$ and $z + \sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 2$. Fix an elliptic curve $C$ and $M \in \text{Pic}^{g-1}(C)$.

Consider the associated surface cone $T \subset \mathbb{P}^{e-1}$ induced by $M$ and vertex
$v \notin H$. Fix $P \in T \setminus \{v\}$ and then take $k$ general points $Q_1, \ldots, Q_k$ of $T \setminus \{v\}$. Then there exists an integral genus $g$ bielliptic curve $Y$ whose canonical model
is embedded in $T \setminus \{v\}$ and contains $\{P, Q_1, \ldots, Q_k\}$, with $P \in Y_{\text{reg}}$, $P$ a
Weierstrass point of $Y$ of any prescribed in advance type according to the rules
of Lemma 1.7 with respect to the integer $z$ and such that each $Q_i$ is a singular
point of $Y$ whose type is exactly the one prescribed by its label.

**Theorem 0.2.** Let $Y \subset \mathbb{P}^{e-1}$ be a canonically embedded integral bielliptic
curve and $T \subset \mathbb{P}^{e-1}$ the associated elliptic cone with $Y \subset T \setminus \{v\}$. Let
$Q_1, \ldots, Q_k$ be the singular points of $Y$; call $\delta_i$ the invariant associated to $Q_i$
and assume that $Q_i$ is a cusp if $0 \leq i \leq \gamma$ and a tacnode if $\gamma + 1 \leq i \leq k$.
Let $B(Y)$ the subset of $\text{Hilb}(T)$ parametrizing integral bielliptic curves with the
same type of singularities as $Y$. Assume $2g - 2 > \sum_{1 \leq i \leq k} 2\delta_i + \gamma + k$.
Then $B(Y)$ is smooth at $Y$ with the expected dimension

$$3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i).$$

As the reader has certainly noticed, to make sense of the statements of 0.1
and 0.2 we need to introduce several definitions and a few notation. This will
be done (together with their proof and a few related remarks) in the only section
of this paper. The main tool will be the study of the subset $B(Y)$ of the Hilbert
scheme $\text{Hilb}(T)$ parametrizing the equisingular deformations of $Y$. For
the case in which $Y$ is smooth, see [4]. To study $B(Y)$ near $Y$ when $Y$ is singular
we use [7] and [8].
1. Proofs and related remarks.

We use the notation introduced at the beginning of section 0. Let $\pi : X \to Y$ be the normalization map. Since $g \geq 6$ the bielliptic structure of $Y$ is unique ([2], Remark 2.4). For a refined study of case $Y$ smooth and $3 \leq g \leq 5$, see [3], sec. 5. Since $C$ is assumed to be smooth, $Y$ is Gorenstein (see e.g. [5], Ch. 0, sec. 1). Since $Y$ is Gorenstein and not hyperelliptic, the canonical map of $Y$ is an embedding ([10], Th. 15) and we will always see $Y$ canonically embedded in $\mathbf{P}^{g-1}$ as a linearly normal curve of degree $2g - 2$. $Y$ is contained in a cone $T \subset \mathbf{P}^{g-1}$ with vertex $v \notin Y$ and with as base a degree $g - 1$ elliptic curve, $E$, embedded in a hyperplane $H$ of $\mathbf{P}^{g-1}$ as a linearly normal curve ([2], Prop. 4.2). The restriction to $Y$ of the projection of $T \setminus \{v\} \to E$ from the vertex $v$ induces the double covering $f$. In particular there is an isomorphism $j : C \to E$ and we will omit it identifying $C$ and $E$ when there is no danger of misunderstanding. Since $\text{char}(K) \neq 2$ and $C$ is smooth, the double covering $f$ is associated to a unique $M \in \text{Pic}^{g-1}(C)$. The linearly normal embedding $j$ is associated to $M$. Since $\text{char}(K) = 0$, there are exactly $(g - 1)^2$ points $P_i \in E \cong C$, $1 \leq i \leq (g - 1)^2$, such that $\mathcal{O}_C((g - 1)P_i) \cong M$. Let $u : S \to T$ be the blowing-up of $T$ at $v$. We have $S \cong \text{Pic}(\mathcal{O}_C \oplus M)$ and this isomorphism is compatible with the projections $\alpha : \text{Pic}(\mathcal{O}_C \oplus M) \to C$ and $T \setminus \{v\} \to E \cong C$. $\text{Pic}(S) \cong \mathbf{Z}[h] \oplus \alpha^*(\text{Pic}(C))$, where $h := u^{-1}(v) \cong C$. The conormal bundle of $h$ in $S$ is isomorphic to $M$ and hence for all $L$, $L' \in \alpha^*(\text{Pic}(C))$ we have $L \cdot L' = 0$, $L \cdot h = \text{deg}(L)$ and $h^2 = 1 - g$. by the adjunction formula we obtain $\omega_S \cong \mathcal{O}_S(-2h - \alpha^*(M))$. Fix $Q \in \text{Sing}(Y)$ and let $D \subset \mathbf{P}^{g-1}$ be the line $(Q, v)$ spanned by $Q$ and $v$. Since the projection of $Y$ from $v$ as degree 2, we see that the scheme $D \cap Y$ has length 2 and $(D \cap Y)_{\text{reg}} = \{Q\}$. In particular $Q$ is a double point and if it is not unibranch it has exactly two branches, both of them smooth, and with $D$ transversal to the two branches, while if $Y$ is unibranch at $Q$, the line $D$ is not in the tangent cone of $Y$ at $Q$. Let $\delta(Q, Y)$ be the codimension as $K$-vector space of $\mathcal{O}_{Y,Q}$ in its normalization. Hence $0 < \delta(Q, Y) \leq g$ and $\delta(Q, Y) = g$ if and only if $X \cong \mathbf{P}^1$ and $\{Q\} = \text{Sing}(Y)$. If $Y$ is unibranch at $Q$ with invariant $\delta(Q, Y)$, then it is a cusp formally equivalent to the plane singularity $y^2 = x^{2k+1}$, $k := \delta(Q, Y)$; blowing-up the cusp singularity with invariant $k$ we obtain a smooth germ of plane curve if $k = 1$ and a cusp singularity with invariant $k - 1$ if $k \geq 2$. If has two branches, then it is a tacnode formally equivalent to the plane singularity $y^2 = x^{2k}$, $k = \delta(Q, Y)$ (see e.g. [11], bottom of p. 100); blowing-up the tacnode singularity with invariant $k$ we obtain a smooth germ of plane curve if $k = 1$ and a tacnode singularity with invariant $k - 1$ if $k \geq 2$. Now we fix an integer $\delta$ with $0 < \delta \leq g$ and $Q \in T \setminus \{v\} = S \setminus \{h\}$. Let $\Delta(Q, \delta, 1)$
be the following zero-dimensional scheme with $\Delta(Q, \delta, 1)_{red} = \{Q\}$; we fix a germ at $Q$, $Y''$, of a cusp singularity with invariant $\delta$ and with tangent cone not containing the line $\langle Q, v \rangle$, i.e. the vertical fiber of $S$ through $Q$; let $\Delta(Q, \delta, 1)$ be the generalized singularity scheme associated to $Y''$ in the sense of [8], Def. 2.3; by [8], Lemma 2.6, we have length $(\Delta(Q, \delta, 1)) = 3\delta + 1$; if $m$ is the maximal ideal of the local ring $O_{S, Q}$, then the ideal sheaf of $\Delta(Q, \delta, 1)$ contains $m^{2\delta + 1} + I_{Y''}Q$ and it is contained in $m^{2\delta} + I_{Y''}Q$. Let $\Delta(Q, \delta, 2)$ be the following zero-dimensional scheme with $\Delta(Q, \delta, 1)_{red} = \{Q\}$; we fix a germ at $Q$, $Y''$, of a tacnode singularity with invariant $\delta$ and with tangent cone not containing the line $\langle Q, v \rangle$; let $\Delta(Q, \delta, 2)$ be the generalized singularity scheme associated to $Y''$ in the sense of [8], Def. 2.3; by [8], Lemma 2.6, we have length $(\Delta(Q, \delta, 1)) = 3\delta$; the difference with the cuspidal case is that after $\delta$ blowing-ups the strict transform of $Y''$ is transversal to the tree of exceptional divisors; if $m$ is the maximal ideal of the local ring $O_{S, Q}$, then the ideal sheaf of $\Delta(Q, \delta, 2)$ is $m^{2\delta} + I_{Y''}Q$. We will see in (1.8) that $\Delta(Q, \delta, 1)$ (resp. $\Delta(Q, \delta, 2)$) is related to bielliptic curves, $Y$, with $Q \in Sing(Y)$ and having a cusp (resp. a tacnode) with $\delta(Q, Y) = \delta$.

**Remark 1.1.** Fix $Q \in S \setminus h \cong T \setminus \{v\}$ and an integer $\delta > 0$. Call $L$ the line $\langle Q, v \rangle$. The residual scheme $Res_L(\Delta(Q, \delta, 1))$ of the scheme $\Delta(Q, \delta, 1)$ with respect to the Cartier divisor $L$ of $S \setminus h$ is just $Q$ with its reduced structure if $\delta = 1$, while $Res_L(\Delta(Q, \delta, 1)) = \Delta(Q, \delta, 1, 1)$ if $\delta \geq 2$. We have $Res_L(\Delta(Q, 1, 2)) = \phi$ and $Res_L(\Delta(Q, \delta, 2)) = \Delta(Q, \delta - 1, 2)$ if $\delta \geq 2$.

**Remark 1.2.** We have $h^0(S, O_S(2h + M^{\otimes 2})) = h^0(C, M^{\otimes 2}) + h^0(C, M) + h^0(C, O_C) = 3g - 2$ and $h^1(S, O_S(2h + M^{\otimes 2})) = 1$.

**Remark 1.3.** Fix a curve $A \in |2h + M^{\otimes 2}|$ on $S$. Since $h \cdot M = -h^2 = 1 - g$, we see that if $A$ has a vertical fiber as component, then $A$ has $h$ as a component. Every curve $B \in |h + \alpha^*(R)|, R \in Pic(C)$, is the union of a smooth curve isomorphic to $C$ and possibly some vertical fibers. Hence we easily see that if $Y \in |2h + M^{\otimes 2}|$ has sufficiently many tacnodes or cusps, then it must be irreducible. Fix positive integers $g, k, \delta_1, \ldots, \delta_k$ with $g \geq 6$. Fix $k$ general distinct points of $Q_1, \ldots, Q_k$ of $S \setminus h$, no two of them contained in the same fiber of the ruling of $S$. For each integer $i$ with $1 \leq i \leq k$ fix one of the following two labels: “tacnode with invariant $\delta_i$” or “cusp with invariant $\delta_i$”. Fix an integral curve $Y \in |2h + M^{\otimes 2}|$ with $\{Q_1, \ldots, Q_k\} \subseteq Sing(Y)$ and such that $Q_i$ is a singularity of $Y$ with the formal isomorphic type prescribed by its label. Both tacnodes and cusps are rational double points and for these type of singularities the invariants considered in [7] are known; the Tyurina and the Milnor numbers of a cusp (resp. tacnode) with invariant $\delta_i$ are $2\delta_i$ (resp.}
2δ_i - 1); the local isomorphism defect (in the sense of [7], 3.4) of the germ of
the normal sheaf of Y at Q_i is ([7], Ex. 4.5). Assume 2( ∑_{1≤i≤k} δ_i) + γ ≤ 2g - 3,
where γ is the number of cusp among the labels. Since ωY ≥ 0S(-2h - α^*(M)),
we have -ωY · Y = 2g - 2 and Y · Y = 4g - 4. By [7], Remark 3.8, part 3,
we obtain that the subset of the Hilbert scheme Hilb(S) of S parametrizing
curves near Y which are equisingular to Y at each point Q_i is smooth of the
expected dimension 3g - 3 + k - γ - 2( ∑_{1≤i≤k} δ_i) (see [7], 3.14, for the case
Sing(Y) ≠ {Q_1, . . . , Q_k}).

The following result is just [2], Prop. 2.3. For reader’s sake we reproduce
its proof

Lemma 1.4. Assume C smooth and g ≥ 3. Fix L ∈ Pic^d(X) with 0 < d ≤
g - 2 and L spanned. Then d is even and there exists a unique R ∈ Pic^{d/2}(C)
with L ≅ f^*(R) and h^0(Y, L) = h^0(C, R).

Proof. The uniqueness of R follows from [2], Lemma 2.2. Fix a general
linear suspace V of H^0(Y, L). Since L is a spanned line bundle, V spans
L. Hence V induces a morphism v : Y → P^1 with L ≅ v^*(O_{P^1}(1)) and
V = v^*(H^0(P^1, O_{P^1}(1))). If the morphism v factors through f, we obtain d
even and the existence of R ∈ Pic^{d/2}(C) with L ≅ f^*(R). By the uniqueness
of R we obtain that the image of the injective linear map γ : H^0(C, R) →
H^0(Y, L) contains a general two-dimensional subspace of H^0(Y, L). Hence γ
is an isomorphism. Hence we may assume that v does not factor through f,
i.e. that the induced morphism h = (f, v) : Y → C × P^1 is birational. Thus
p_*(h(Y)) ≥ g. Since h(Y) is a divisor of C × P^1 of bidegree (2, d), we conclude
using the adjunction formula on the smooth surface C × P^1, exactly as in the
classical case with Y smooth.

The following result was checked in [4] (see [4], Lemma 0.2) if Y is
smooth. The proof in the general case is the same quoting Lemma 1.4 as a
reference for Castelnuovo - Severi inequality.

Lemma 1.5. Let P ∈ Y_{reg} be a Weierstrass point which is not a ramification
point of f.

Then one of the following 3 cases occurs:

Type (a): the sequence of non gaps of P is g - 1 and g + 2 + j for all j ≥ 0;
P has weight w(P) = 2;

Type (b): there is an integer k with 1 ≤ k ≤ g - 2, k ≠ g - 3, such that
the sequence of non gaps of P is g - 1, g + j for all j with 1 ≤ j ≤ k and
g + k + 2 + t for all t ≥ 0; P has weight w(P) = k + 2;
Type (c): there is an integer \( k \) with \( 0 \leq k \leq g - 2 \) such that the sequence of non gaps of \( P \) is \( g + j \) for all \( j \) with \( 0 \leq j \leq k \) and \( g + k + 2 + t \) for all integers \( t \geq 0 \); \( P \) has weight \( w(P) = k + 1 \).

(1.6). Fix an integral curve \( Y \subset T \), with \( v \notin Y \) and such that the projection from \( v \) makes \( Y \) a double covering of \( E \equiv C \), say \( f : Y \to C \). Since \( v \notin Y \), we have \( u^{-1}(Y) \equiv Y \). Assume that the corresponding double covering \( u^{-1}(Y) \to C \) is induced by \( M \in \text{Pic}^{(g-1)}(C) \). Since \( u^*(\mathcal{O}_T(1)) \cong \mathcal{O}_S(h + M) \) (Using additive notation in \( \text{Pic}(S) \)), \( \text{deg}(Y) = 2g - 2 \), \( p_a(Y) = g \) and \( \omega_C \cong \mathcal{O}_C \), the adjunction formula implies that \( u^{-1}(Y) \in |2h + M^{\otimes 2}| \). Call \( Q[H] \) the set of points of \( Y \) which are mapped onto one of the points \( P_i \), \( 1 \leq i \leq (g - 1)^2 \). Fix \( P \in Y_{\text{reg}} \) and an integer \( z \) with \( g \leq z \leq 2g - 3 \); if \( P \in Q[H] \), assume \( z \geq 2g - 4 \). Fix a general hyperplane \( H' \) of \( \mathbb{P}^{g-1} \) with \( P \in H' \) and set \( C' := T \cap H \). Hence \( C' \cong C \).

For every integer \( w > 0 \), let \( \{wP\} \) be the zero-dimensional subscheme of \( C' \) of degree \( w \) supported by \( P \). Assume that \( Y \) contains \( \{zP\} \) but not \( \{(z + 1)P\} \). Then the proof of [4], Lemma 1.1, (in which it was assumed \( Y \) smooth instead of just assuming \( P \in Y_{\text{reg}} \)) works verbatim and gives the following result.

**Lemma 1.7.** Assume that \( P \) is not a ramification point of \( f \). Then we have:

1. \( P \) is a Weierstrass point of \( Y \);
2. if \( P \in Q[H] \), then \( P \) is a Weierstrass point of type (a) or of type (b) of \( Y \);
3. if \( P \notin Q[H] \), then \( P \) is a Weierstrass point of type (c) of \( Y \) with associated integer \( k = z - g \);
4. if \( P \in Q[H] \) and \( z \geq g + 1 \), then \( P \) is a Weierstrass point of type (b) of \( Y \) with associated integer \( k = z - g \);
5. if \( P \in Q[H] \) and \( z = g \), then \( P \) is a Weierstrass point of type (a) of \( Y \).

**Proof of Theorem 0.1.** We use the notation introduced for the statement of Lemma 1.7. Let \( \Gamma \) be the union of the points whose label says “cusp!”. Let \( \Delta \) be the union of the schemes \( \Delta(Q_i, \delta_i, 1) \) if \( Q_i \in \Gamma \) and \( \Delta(Q_i, \delta_i, 2) \) if \( \Delta_i \notin \Gamma \).

Set \( W := \mathbb{P}(H^0(S, \mathcal{O}_S(2h + M^{\otimes 2}) \otimes I_{\{zP\} \cup \Delta})) \). Let \( Y \) be a general element of \( \mathbb{P}(H^0(S, \mathcal{O}_S(2h + M^{\otimes 2}) \otimes I_{\{zP\} \cup \Delta})) \).

First Claim: We have \( h^0(S, \mathcal{O}_S(2h + M^{\otimes 2}) \otimes I_{\{zP\} \cup \Delta}) = h^0(S, \mathcal{O}_S(2h + M^{\otimes 2})) - \text{length}(\{zP\} \cup \Delta) = 3g - 2 - z - \sum_{1 \leq j \leq k} 3\delta_j - \gamma \) and \( h^1(S, \mathcal{O}_S(h + M^{\otimes 2})(- \sum_{1 \leq j \leq k} \delta_j Q_j)) \otimes I_{\{zP\} \cup \Delta} = 1 \).

Proof of the First Claim. Since \( h^1(S, \mathcal{O}_S(2h + M^{\otimes 2}) = 1 \) (Remark 1.2) the last equality is true if and only if the first equality is true, i.e. if the zero-dimensional scheme \( \{zP\} \cup \Delta \) imposes independent conditions to the linear
system $|2h + M^\otimes 2|$. By semicontinuity it is sufficient to prove the result for some special configuration of points $Q_1, \ldots, Q_k$. Let $F$ be vertical fiber containing $P$. We specialize $Q_1$ to a general point of $F$. Let $\Delta'$ the residual scheme $Res_F(\Delta)$ of $\Delta$ with respect to $F$; $\Delta'$ and $\Delta$ have outside $Q_1$ the same connected components; length($\Delta'$) $-$ length($\Delta$) $\geq 2$ and the connected component of $\Delta'$ is empty if $Q_1$ is labelled an ordinary node, $\{Q_1\}$ if $Q_1$ is labelled as an ordinary cusp and $\Delta'(Q_1, \delta_1 - 1, i) (i = 1$ or $2$ according to the label of $Q_1)$ if $\delta_1 \geq 2$.

We have $Res_F([zP] \cup \Delta) = [(z - 1)P] \cup \Delta'$ (Remark 1.1 and 1.3). Since the scheme $F \cap ([zP] \cup \Delta)$ has length $3$, we have $h^0(S, O_S(2h + M^\otimes 2) \otimes I_{[zP] \cup \Delta}) = h^0(S, O_S(2h + M^\otimes 2)(-Q_1 \otimes I_{[(z - 1)P] \cup \Delta})$. If $\delta_1 \geq 2$ we have $\text{length}(F \cap ([z - 1)P] \cup \Delta')) = 3$ and hence we continue $\delta_1 - 1$ times obtaining $h^0(S, O_S(2h + M^\otimes 2) \otimes I_{[zP] \cup \Delta}) = h^0(S, O_S(2h + M^\otimes 2)(-\delta_1 P) \otimes I_{[(z - \delta_1)P] \cup \Delta'}), where $\Delta''$ is the union of the connected components of $\Delta$ not supported by $Q_1$ and $* = \emptyset$ is the empty set if $Q_1$ has label “tacnode”, while $* = \{Q_1\}$ if $Q_1$ has “cusp!” as label). If $Q_1$ has “cusp!” as label, i.e. $* \neq \emptyset$, we just prove the weaker statement $h_1(S, O_S(2h + M^\otimes 2)(-\delta_1 + 1)P) \otimes I_{[(z - \delta_1)P] \cup \Delta'}) = 0$. Now, and only now, we specialize $Q_2$ to a general point of $F$. At the end it is sufficient to check that $h^1(S, O_S(2h + M^\otimes 2)(-\sum_{1 \leq i \leq k} \delta_i + \gamma) P) \otimes I_{[(z - \sum_{1 \leq i \leq k} \delta_i)P] \cup \Delta''} = 0$.

Second Claim: $Y$ is integral, $Sing(Y) = \{Q_1, \ldots, Q_k\}$, and $Y$ has at each $Q_i$ the singularity prescribed by the label of $Q_i$, and with the invariant $\delta_i$.

Proof of the Second Claim. We claim that the assertions on $Sing(Y)$ follow from the last part of the First Claim, the definition of singularity scheme given in [8], sec. 2, and its use made in [8] to construct plane curves with prescribed singularities. To check the claim see in particular [8], Lemma 2.4, and the fact that the bijectivity of a map $H_1(S, A \otimes J) \rightarrow H^1(S, A \otimes J)$, $A \in \text{Pic}(S)$, $J$ ideal of a zero-dimensional scheme, is what needed to obtain that $H^0(S, A \otimes J)$ spans $J$; remember that $h^1(S, O_S(2h + M^\otimes 2)) = 1$ (Remark 1.2). The first assertion follows from the first part of the First Claim and the proof of [4], Th 0.3.

By Lemma 1.7 $P$ is a Weierstrass point of $Y$ with the type we want. Hence we conclude the proof of Theorem 0.1.

(1.8). Fix $P \in Sing(Y)$ and set $\delta := \delta(P, Y) > 0$. By [6], Prop. 3.5, $P$ is a Weierstrass point of $Y$ with weight $w(P) \geq g(g - 1)\delta$. The non-negative integer $E(P) := w(P) - g(g - 1)\delta$ was called the extraweight of $P$ and it is the real measure of how much $P$ is a Weierstrass point of $Y$, not just how singular is $Y$ at $P$. By [6], Prop. 5.5, it is possible to compute $E(P)$ looking at the gap sequences of all points of $\pi^{-1}(P)$ with respect to a suitable linear system, $V$, on $X$ with $V \cong P(\pi^*(H^0(Y, \omega_Y)))$. We distinguish four cases.
Here we assume that $P$ has two branches and that $f(P)$ is not one of the points $P_i$, $1 \leq i \leq (g - 1)^2$. Set $\{P', P''\} := \pi^{-1}(P)$. Since the line $\{P, v\}$ is transversal to each of the two branches of $Y$ at $P$ and $f(P)$ is not a osculating point of $E$, we see that $P'$ and $P''$ are not Weierstrass point of the linear system $V$ on $X$. Thus $E(P) = 0$ ([6], Prop. 5.5). Following the terminology of [4] for smooth ramification points, we will say that $P$ is a tacnode of type $I$ of $Y$.

Here we assume that $P$ has two branches and that $f(P)$ is one of the points $P_i$, $1 \leq i \leq (g - 1)^2$. Set $\{P', P''\} := \pi^{-1}(P)$. Since the line $\{P, v\}$ is transversal to each of the two branches of $Y$ at $P$ and $f(P)$ is a osculating point of $E$ with weight 1, we see that the Hermite invariants, $\{h_i\}_{0 \leq i \leq g-1}$ of $P'$ and $P''$ with respect to $V$ are the same and $h_i = i$ for $i \leq g-2$, $h_{g-1} = g$. Thus $P'$ and $P''$ are Weierstrass points with weight 1 for the linear system $V$ on $X$ (see [9], Th. 15). Thus $E(P) = 2$ ([6], Prop. 5.5). Following the terminology of [4] for smooth ramification points, we will say that $P$ is a tacnode of type $\Pi$ of $Y$.

Here we assume that $P$ has one branch and that $f(P)$ is not one of the points $P_i$, $1 \leq i \leq (g - 1)^2$. Set $\{Q\} := \pi^{-1}(P)$. Since the line $\{P, v\}$ is not in the tangent cone of $Y$ at $P$, we see that the sequence of non gaps of $Q$ for the linear system $V$ on $X$ is given by the integers $2t$ ($2 \leq t \leq g - 1$), $2g - 1, 2g, \ldots$. Hence $Q$ has weight $(g^2 - 5g + 6)/2$. By [6], Prop. 5.5, $P$ has extraweight $E(P) = (g^2 - 5g + 6)/2$. In particular the extraweight does not depend from $\delta$. Following the terminology of [4] for smooth ramification points, we will say that $P$ is a cusp of type $I$ of $Y$.

Here we assume that $P$ has one branch and that $f(P)$ is one of the points $P_i$, $1 \leq i \leq (g - 1)^2$. Set $\{Q\} := \pi^{-1}(P)$. Since the line $\{P, v\}$ is not in the tangent cone of $Y$ at $P$, we see that the sequence of non gaps of $Q$ for the linear system $V$ on $X$ is given by the integers $2t$ ($2 \leq t \leq g - 2$), $2g - 3, 2g - 2, 2g, \ldots$. Hence $Q$ has weight $(g^2 - 5g + 10)/2$. By [6], Prop. 5.5, $P$ has extraweight $E(P) = (g^2 - 5g + 10)/2$. In particular the extraweight does not depend from $\delta$. Following the terminology of [4] for smooth ramification points, we will say that $P$ is a cusp of type $\Pi$ of $Y$.

An easy modification proof of Theorem 0.1 gives the following existence theorem for bielliptic curves with prescribed singularities; instead of specializing each point $Q_i$, $1 \leq i \leq k$, to $P$, loose directly $\delta_i$ (or $\delta_i + 1$ for a cusp) conditions to handle the postulation of the scheme $\Delta(Q_i, \delta_i, 2)$ (resp. $\Delta(Q_i, \delta_i, 1)$); for the value of the dimension, see Theorem 0.2 and its proof.
Theorem 1.9. Fix positive integers $g, k, \gamma, \delta_1, \ldots, \delta_k$ with $g \geq 6, 0 \leq \gamma \leq k$, and $\sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 3$. Fix a smooth elliptic curve $C$ and $M \in \text{Pic}^{(g-1)}(C)$. Use $M$ to obtain a linearly normal embedding of $C$ into a hyperplane, $H$, of $\mathbb{P}^{g-1}$. Fix $v \notin H$ and call $T \subset \mathbb{P}^{g-1}$ the associated elliptic cone with vertex $v$. Fix $k$ general points of $Q_1, \ldots, Q_k$ of $T$. For each integer $i$ with $1 \leq i \leq k$ fix one of the following four labels: “tacnode of type I with invariant $\delta_i$”, “tacnode of type II with invariant $\delta_i$”, “cusp of type I with invariant $\delta_i$” or “cusp of type II with invariant $\delta_i$”. Assume that exactly $\gamma$ of the labels say “cusp!”. Then there exists a canonically embedded integral bielliptic curve $Y \subset T \setminus \{v\}$ with $\text{Sing}(Y) = (Q_1, \ldots, Q_k)$ and such that $Y$ has at each $Q_i$ the singularity prescribed by the corresponding label. Furthermore, there exists such curve $Y$ with the property that the subset of the Hilbert scheme $\text{Hilb}(T)$ of $T$ parametrizing such curves is, near $Y$, a smooth variety of dimension $3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i)$.

Taking the union for all possible $C, M$ and $v$ from Theorem 1.9 we obtain the following result; just note that since $k \leq g$ the union of $k$ general points of $\mathbb{P}^{g-1}$ is contained in an elliptic degree $g - 1$ two-dimensional cone; here we use that for every bielliptic curve $Y$ there is a non-empty finite set of elliptic cones $T \subset \mathbb{P}^{g-1}$ containing the canonical model of $Y$, that any elliptic cone, $T$, with vertex $v \notin H \cong \mathbb{P}^{g-2}$ is uniquely determined by $H \cap T$, that $\dim(\mathbb{P}^{g-1})$ (i.e. the possible vertices, $v$, are $\infty^{g-1}$) and that the subset of $\text{Hilb}(H)$ parametrizing the linearly normal smooth elliptic curves of degree $g - 1$ is a smooth variety of dimension $(g - 1)^2$.

Theorem 1.10. Fix positive integers $g, k, \gamma, \delta_1, \ldots, \delta_k$ with $g \geq 6, 0 \leq k$, and $\sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 3$. Fix $k$ general points of $Q_1, \ldots, Q_k$ of $\mathbb{P}^{g-1}$. For each integer $i$ with $1 \leq i \leq k$ fix one of the following four labels: “tacnode of type I with invariant $\delta_i$”, “tacnode of type II with invariant $\delta_i$”, “cusp of type I with invariant $\delta_i$”, or “cusp of type II with invariant $\delta_i$”. Assume that exactly $\gamma$ of the labels say “cusp!”. Then there exists a canonically embedded integral bielliptic curve $Y \subset \mathbb{P}^{g-1}$ with $\text{Sing}(Y) = (Q_1, \ldots, Q_k)$ and such that $Y$ has at each $Q_i$ the singularity prescribed by the corresponding label. Furthermore, there exists such curve $Y$ with the property that the subset of the Hilbert scheme $\text{Hilb}(\mathbb{P}^{g-1})$ of $\mathbb{P}^{g-1}$ parametrizing such curves is, near $Y$, a smooth variety of dimension $3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i) + g(g - 1)$.

Now we may prove Theorem 0.2.

Proof of Theorem 0.2. By remark 1.3 the smoothness criterion [7], Th. 3.5 and
Remark 3.8, part 3, is satisfied.

**Remark 1.11.** Let \( f : Y \rightarrow C \) is a double covering with \( p_a(C) = 1 \) and \( C \) singular, i.e. with \( C \) rational and with a unique singular point, \( Q \), which is either an ordinary node or an ordinary cusp; The canonical model of \( Y \) is again contained in a cone with vertex \( v \notin Y \) and, as base, a degree \( g - 1 \) curve isomorphic to \( C \) and embedded into a hyperplane, \( H \), of \( \mathbb{P}^{g-1} \) ([2], Prop. 4.2). Since \( Pic^0(C) \cong K^* \), outside the singular point such curve \( C \subset H \) has exactly \( g - 1 \) asculating points, say \( P_i \), \( 1 \leq i \leq g - 1 \). For each \( P \in Sing(Y) \) with \( f(P) \neq Q \), the classification of all possible singularities not mapped into \( Q \) and their division into types works verbatim, taking the points \( P_i \), \( 1 \leq i \leq g - 1 \), instead of the points \( P_i \), \( 1 \leq i \leq (g - 1)^2 \).

**Remark 1.12.** Here we assume \( char(K) \not\equiv 0 \), but \( char(K) \neq 2 \). Let \( f : Y \rightarrow C \) is a double covering with \( p_a(C) = 1 \). The canonical model of \( Y \) is again contained in a cone with vertex \( v \notin Y \) and, as base a degree \( g - 1 \) curve isomorphic to \( C \) and embedded into a hyperplane, \( H \), of \( \mathbb{P}^{g-1} \) ([2], Prop. 4.2). The classification of singular points, \( P \), of \( Y \) with \( f(P) \notin Sing(C) \) as cusps or tacnodes works even in this case (see [11], pp. 100–101). Obviously here \( C \subset H \) may have a smaller number of osculating points if \( p \leq g - 1 \), but the only difference is that their weight is bigger than 1. From now on we assume \( p > 2g - 2 \). Under this assumption we are sure that the Hermite invariants of the linear system, \( V \), induced by \( \pi : X \rightarrow Y \subset \mathbb{P}^{g-1} \) at a generic point of \( X \) are the classical ones ([9], Th. 15). Furthermore, only if we have such a restrictive assumption on \( char(K) \) we are sure that the weight of a Weierstrass point \( Q \in X \) of \( V \) is computed using the gap sequence of \( V \) at \( Q \) ([9], Th. 15, part (iii)). With this very restrictive assumption on \( char(K) \) we may copy [6], Prop. 5.5, and extend (1.8). For the case \( Y \) smooth, see [1].

REFERENCES


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