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SINGULAR BIELLIPTIC CURVES AND WEIERSTRASS POINTS

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Here we study the Weierstrass points of singular bielliptic curves in characteristic 0. Most of our results are existence results of the type "there exists a bielliptic curve Y with certain singular points and with a weierstrass point $P \in Y_{reg}$ with a prescribed gap sequence". Another main result is a smoothness one for the set of all genus g bielliptic curves with prescribed singularities.

0. Introduction.

In this paper we study the Weierstrass points of singular bielliptic curves. Unless otherwise stated, we work over an algebraically closed field $\mathbf{K}$ with char($\mathbf{K}$) $= 0$. For the case char($\mathbf{K}$) > 0, but char($\mathbf{K}$) $\neq 2$, see Remark 1.12. Let Y be an integral projective bielliptic curve with $g := p_a(Y) \geq 6$. Hence there exists a double covering $f : Y \to C$ with $p_a(C) = 1$. We assume that C is smooth. For a remark in the case in which C is singular, i. e. in which C is a rational curve with a unique ordinary node or a unique ordinary cusp as singularities, see Remark 1.11. We want to study simultaneously the smooth points of Y which are Weierstrass points and all the singular points. We list all possible singularities and the possible types of singular points of Y as Weierstrass points (see 1.11). Most of our results are existence results of the

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type "there exists a bielliptic curve Y with certain singular points and with a Weierstrass point $P \in Y_{reg}$ with a prescribed gap sequence" (see Theorems 0.1, 1.9 and 1.10). Another main result is a smoothness one for the set of all genus g bielliptic curves with prescribed singularities (see Theorem 0.2). To get the flavour of our results we state the two main ones.

Theorem 0.1. *Fix positive integers g, k, z and $\delta_1, \dots, \delta_k$. For every integer i with $1 \leq i \leq k$ fix a label: "cusp with invariant δ_i" or "tacnode with invariant δ_i". Let γ be the number of labels "cusp!". Assume $z \geq g \geq 6$ and $z + \sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 2$. Fix an elliptic curve C and $M \in Pic^{(g-1)}(C)$. Consider the associated surface cone $T \subset \mathbf{P}^{g-1}$ induced by M and vertex $\nu \notin H$. Fix $P \in T \setminus \{\nu\}$ and then take k general points $Q_1, \dots, Q_k$ of $T \setminus \{\nu\}$. Then there exists an integral genus g bielliptic curve Y whose canonical model is embedded in $T \setminus \{\nu\}$ and contains $\{P, Q_1, \dots, Q_k\}$, with $P \in Y_{reg}$, P a Weierstrass point of Y of any prescribed in advance type according to the rules of Lemma 1.7 with respect to the integer z and such that each Q_I is a singular point of Y whose type is exactly the one prescribed by its label.*

Theorem 0.2. *Let $Y \subset \mathbf{P}^{g-1}$ be a canonically embedded integral bielliptic curve and $T \subset \mathbf{P}^{g-1}$ the associated elliptic cone with $Y \subset T \setminus \{\nu\}$. Let $Q_1, \dots, Q_k$ be the singular points of Y; call δ_i the invariant associated to Q_i and assume that Q_i is a cusp if $0 \leq i \leq \gamma$ and a tacnode if $\gamma + 1 \leq i \leq k$. Let $\mathbf{B}(Y)$ the subset of $Hilb(T)$ parametrizing integral bielliptic curves with the same type of singularities as Y. Assume $2g - 2 > \sum_{1 \leq i \leq k} 2\delta_i + \gamma + k$. Then $\mathbf{B}(Y)$ is smooth at Y with the expected dimension*

$$3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i).$$

As the reader has certainly noticed, to make sense of the statements of 0.1 and 0.2 we need to introduce several definitions and a few notation. This will be done (together with their proof and a few related remarks) in the only section of this paper. The main tool will be the study of the subset $\mathbf{B}(Y)$ of the Hilbert scheme $Hilb(T)$ parametrizing the equisingular deformations of Y. For the case in which Y is smooth, see [4]. To study $\mathbf{B}(Y)$ near Y when Y is singular we use [7] and [8].

1. Proofs and related remarks.

We use the notation introduced at the beginning of section 0. Let $\pi : X \to Y$ be the normalization map. Since $g \geq 6$ the bielliptic structure of Y is unique ([2], Remark 2.4). For a refined study of case Y smooth and $3 \leq g \leq 5$, see [3], sec. 5. Since C is assumed to be smooth, Y is Gorenstein (see e.g. [5], Ch. 0, sec. 1). Since Y is Gorenstein and not hyperelliptic, the canonical map of Y is an embedding ([10], Th. 15) and we will always see Y canonically embedded in $\mathbf{P}^{g-1}$ as a linearly normal curve of degree $2g-2$. Y is contained in a cone $T \subset \mathbf{P}^{g-1}$ with vertex $\nu \notin Y$ and with as base a degree $g-1$ elliptic curve, E, embedded in a hyperplane H of $\mathbf{P}^{g-1}$ as a linearly normal curve ([2], Prop. 4.2). The restriction to Y of the projection of $T \setminus \{\boldsymbol{\nu}\} \to E$ from the vertex ν induces the double covering f. In particular there is an isomorphism $\mathbf{j} : C \to E$ and we will omit it identifying C and E when there is no danger of misunderstanding. Since char($\mathbf{K}$) $\neq 2$ and C is smooth, the double covering f is associated to a unique $M \in Pic^{(g-1)}(C)$. The linearly normal embedding $\mathbf{j}$ is associated to M. Since char($\mathbf{K}$) $= 0$, there are exactly $(g-1)^2$ points $P_i \in E \cong C$, $1 \leq i \leq (g-1)^2$, such that $\mathbf{O}_C((g-1)P_i \cong M$. Let $u : S \to T$ be the blowing-up of T at $\boldsymbol{\nu}$. We have $S \cong \mathbf{P}(\mathbf{O}_C \oplus M)$ and this isomorphism is compatible with the projections $\alpha : \mathbf{P}(\mathbf{O}_C \oplus M) \to C$ and $T \setminus \{\boldsymbol{\nu}\} \to E \cong C$. $Pic(S) \cong \mathbf{Z}[\mathbf{h}] \oplus \alpha^*(Pic(C))$, where $\mathbf{h} := u^{-1}(\boldsymbol{\nu}) \cong C$. The conormal bundle of $\mathbf{h}$ in S is isomorphic to M and hence for all $L, L' \in \alpha^*(Pic(C))$ we have $L \cdot L' = 0$, $L \cdot \mathbf{h} = deg(L)$ and $\mathbf{h}^2 = 1 - g$. by the adjunction formula we obtain $\omega_S \cong \mathbf{O}_S(-2\mathbf{h} - \alpha^*(M))$. Fix $Q \in Sing(Y)$ and let $D \subset \mathbf{P}^{g-1}$ be the line $(Q, \boldsymbol{\nu})$ spanned by Q and $\boldsymbol{\nu}$. Since the projection of Y from $\boldsymbol{\nu}$ as degree 2, we see that the scheme $D \cap Y$ has length 2 and $(D \cap Y)_{reg} = \{Q\}$. In particular Q is a double point and if it is not unibranch it has exactly two branches, both of them smooth, and with D transversal to the two branches, while if Y is unibranch at Q, the line D is not in the tangent cone of Y at Q. Let $\delta(Q, Y)$ be the codimension as $\mathbf{K}$-vector space of $\mathbf{O}_{Y,Q}$ in its normalization. Hence $0 < \delta(Q, Y) \leq g$ and $\delta(Q, Y) = g$ if and only if $X \cong \mathbf{P}^1$ and $\{Q\} = Sing(Y)$. If Y is unibranch at Q with invariant $\delta(Q, Y)$, then it is a cusp formally equivalent to the plane singularity $y^2 = x^{2k+1}$, $k := \delta(Q, Y)$; blowing-up the cusp singularity with invariant k we obtain a smooth germ of plane curve if $k = 1$ and a cusp singularity with invariant $k-1$ if $k \geq 2$. If has two branches, then it is a tacnode formally equivalent to the plane singularity $y^2 = x^{2k}$, $k = \delta(Q, Y)$ (see e.g. [11], bottom of p. 100); blowing-up the tacnode singularity with invariant k we obtain a smooth germ of plane curve if $k = 1$ and a tacnode singularity with invariant $k-1$ if $k \geq 2$. Now we fix an integer δ with $0 < \delta \leq g$ and $Q \in T \setminus \{\mathbf{v}\} = S \setminus \{\mathbf{h}\}$. Let $\Delta(Q, \delta, 1)$

be the following zero-dimensional scheme with $\Delta(Q, \delta 1)_{red} = \{Q\}$; we fix a germ at Q, Y'', of a cusp singularity with invariant δ and with tangent cone not containig the line $(\{Q, \mathbf{v}\})$, i.e. the vertical fiber of S through Q; let $\Delta(Q, \delta, 1)$ be the generalized singularity scheme associated to Y'' in the sense of [8], Def. 2.3; by [8], Lemma 2.6, we have length $(\Delta(Q, \delta, 1)) = 3\delta + 1$; if $\mathbf{m}$ is the maximal ideal of the local ring $\mathbf{O}_{S,Q,}$, then the ideal sheaf of $\Delta(Q, \delta, 1)$ contains $\mathbf{m}^{2k+1} + \mathbf{I}_{Y'',Q}$ and it is contained in $\mathbf{m}^{2k} + \mathbf{I}_{Y'',Q}$. Let $\Delta(Q, \delta, 2)$ be the following zero-dimensional scheme with $\Delta(Q, \delta, 1)_{red} = \{Q\}$; we fix a germ at Q, Y'', of a tacnode singularity with invariant δ and with tangent cone not containing the line $(\{Q, \mathbf{v}\})$; let $\Delta(Q, \delta, 2)$ be the generalized singularity scheme associated to Y'' in the sense of [8], Def. 2.3; by [8], Lemma 2.6, we have length $(\Delta(Q, \delta, 1)) = 3\delta$; the difference with the cuspidal case is that after δ blowing-ups the strict transform of Y'' is transversal to the tree of exceptional divisors; if $\mathbf{m}$ is the maximal ideal of the local ring $\mathbf{O}_{S,Q,}$ then the ideal sheaf of $\Delta(Q, \delta, 2)$ is $\mathbf{m}^{2k} + \mathbf{I}_{Y'',Q}$. We will see in (1.8) that $\Delta(Q, \delta, 1)$ (resp. $\Delta(Q, \delta, 2)$) is related to bielliptic curves, Y, with $Q \in Sing(Y)$ and having a cusp (resp. a tacnode) with $\delta(Q, Y) = \delta$.

Remark 1.1. Fix $Q \in S \setminus \mathbf{h} \cong T \setminus \{\mathbf{v}\}$ and an integer $\delta > 0$. Call L the line $(\{Q, \mathbf{v}\})$. The residual scheme $Res_L(\Delta(Q, \delta, 1))$ of the scheme $\Delta(Q, \delta, 1)$ with respect to the Cartier divisor L of $S \setminus \mathbf{h}$ is just Q with its reduced structure if $\delta = 1$, while $Res_L(\Delta(Q, \delta, 1)) = \Delta(Q, \delta, 1, 1)$ if $\delta \geq 2$. We have $Res_L(\Delta(Q, 1, 2)) = \phi$ and $Res_L(\Delta(Q, \delta, 2)) = \Delta(Q, \delta - 1, 2)$ if $\delta \geq 2$.

Remark 1.2. We have $h^0(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2})) = h^0(C, M^{\otimes 2}) + h^0(C, M) + h^0(C, \mathbf{O}_C) = 3g - 2$ and $h^1(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2})) = 1$.

Remark 1.3. Fix a curve $A \in |2\mathbf{h}+M^{\otimes 2}|$ on S. Since $\mathbf{h}\cdot M = -\mathbf{h}^2 = 1-g$, we see that if A has a vertical fiber as component, then A has $\mathbf{h}$ as a component. Every curve $B \in |\mathbf{h} + \alpha^*(R)|$, $R \in Pic(C)$, is the union of a smooth curve isomorphic to C and possibly some vertical fibers. Hence we easily see that if $Y \in |2\mathbf{h} + M^{\otimes 2}|$ has sufficiently many tacnodes or cusps, then it must be irreducible. Fix positive integers $g, k, \delta_1, \dots, \delta_k$ with $g \geq 6$. Fix k general distinct points of $Q_1, \dots, Q_k$ of $S \setminus \mathbf{h}$, no two of them contained in the same fiber of the ruling of S. For each integer i with $1 \leq i \leq k$ fix one of the following two labels: "tacnode with invariant δ_i" or "cusp with invariant δ_i". Fix an integral curve $Y \in |2\mathbf{h} + M^{\otimes 2}|$ with $\{Q_1, \dots, Q_k\} \subseteq Sing(Y)$ and such that Q_i is a singularity of Y with the formal isomorphic type prescribed by its label. Both tacnodes and cusps are rational double points and for these type of singularities the invariants considered in [7] are known; the Tyurina and the Milnor numbers of a cusp (resp. tacnode) with invariant δ_i are $2\delta_i$ (resp.

$2\delta_i - 1$); the local isomorphism defect (in the sense of [7], 3.4) of the germ of the normal sheaf of Y at Q_i is ([7], Ex. 4.5). Assume $2(\sum_{1\le i\le k} \delta_i) + \gamma \le 2g-3$, where γ is the number of cusp among the labels. Since $\omega_S \cong \mathbf{0}_S(-2\mathbf{h}-\alpha^*(M))$, we have $-\omega_S \cdot Y = 2g-2$ and $Y \cdot Y = 4g-4$. By [7], Remark 3.8, part 3, we obtain that the subset of the Hilbert scheme Hilb(S) of S parametrizing curves near Y which are equisingular to Y at each point Q_i is smooth of the expected dimension $3g-3+k-\gamma-2(\sum_{1\le i\le k} \delta_i)$ (see [7], 3.14, for the case $Sing(Y) \neq \{Q_1, \dots, Q_k\}$).

The following result is just [2], Prop. 2.3. For reader's sake we reproduce its proof

Lemma 1.4. *Assume C smooth and $g \ge 3$. Fix $L \in Pic^d(X)$ with $0 < d \le g-2$ and L spanned. Then d is even and there exists a unique $R \in Pic^{d/2}(C)$ with $L \cong f^*(R)$ and $h^0(Y, L) = h^0(C, R)$.*

Proof. The uniqueness of R follows from [2], Lemma 2.2. Fix a general linear suspace V of $H^0(Y, L)$. Since L is a spanned line bundle, V spans L. Hence V induces a morphism $\nu : Y \to \mathbf{P}^1$ with $L \cong \nu^*(\mathbf{0}^1_\mathbf{P}(1))$ and $V = \nu^*(H^0(\mathbf{P}^1, \mathbf{O}^1_\mathbf{P}(1)))$. If the morphism ν factors through f, we obtain d even and the existence of $R \in Pic^{d/2}(C)$ with $L \cong f^*(R)$. By the uniqueness of R we obtain that the image of the injective linear map $\gamma : H^0(C, R) \to H^0(Y, L)$ contains a general two-dimensional subspace of $H^0(Y, L)$. Hence γ is an isomorphism. Hence we may assume that ν does not factor through f, i.e. that the induced morphism $h = (f, \nu) : Y \to C \times \mathbf{P}^1$ is birational. Thus $p_a(h(Y)) \ge g$. Since $h(Y)$ is a divisor of $C \times \mathbf{P}^1$ of bidegree $(2, d)$, we conclude using the adjunction formula on the smooth surface $C \times \mathbf{P}^1$, exactly as in the classical case with Y smooth.

The following result was checked in [4] (see [4], Lemma 0.2) if Y is smooth. The proof in the general case is the same quoting Lemma 1.4 as a reference for Castelnuovo - Severi inequality.

Lemma 1.5. *Let $P \in Y_{reg}$ be a Weierstrass point which is not a ramification point of f.*

Then one of the following 3 cases occurs:

Type (a): the sequence of non gaps of P is $g-1$ and $g+2+j$ for all $j \ge 0$; P has weight $w(P) = 2$;

Type (b): there is an integer k with $1 \le k \le g-2$, $k \neq g-3$, such that the sequence of non gaps of P is $g-1$, $g+j$ for all j with $1 \le j \le k$ and $g+k+2+t$ for all $t \ge 0$; P has weight $w(P) = k+2$;

Type (c): there is an integer k with $0 \leq k \leq g-2$ such that the sequence of non gaps of P is $g+j$ for all j with $0 \leq j \leq k$ and $g+k+2+t$ for all integers $t \geq 0$; P has weight $w(P) = k+1$.

(1.6). Fix an integral curve $Y \subset T$, with $\nu \notin Y$ and such that the projection from ν makes Y a double covering of $E \cong C$, say $f : Y \to C$. Since $\nu \notin Y$, we have $u^{-1}(Y) \cong Y$. Assume that the corresponding double covering $u^{-1}(Y) \to C$ is induced by $M \in Pic^{(g-1)}(C)$. Since $u^*(\mathbf{O}_T(1)) \cong \mathbf{O}_S(\mathbf{h}+M)$ (Using additive notation in $Pic(S)$), $deg(Y) = 2g-2$, $p_a(Y) = g$ and $\omega_C \cong \mathbf{O}_C$, the adjunction formula implies that $u^{-1}(Y) \in |2\mathbf{h} + M^{\otimes 2}|$. Call Q[H] the set of points of Y which are mapped onto one of the points P_i, $1 \leq i \leq (g-1)^2$. Fix $P \in Y_{reg}$ and an integer z with $g \leq z \leq 2g-3$; if $P \in Q[H]$, assume $z \geq 2g-4$. Fix a general hyperplane H' of $\mathbf{P}^{g-1}$ with $P \in H'$ and set $C' := T \cap H$. Hence $C' \cong C$. For every integer $w > 0$, let $\{wP\}$ be the zero-dimensional subscheme of C' of degree w supported by P. Assume that Y contains $\{zP\}$ but not $\{(z+1)P\}$. Then the proof of [4], Lemma 1.1, (in which it was assumed Y smooth instead of just assuming $P \in Y_{reg}$) works verbatim and gives the following result.

Lemma 1.7. *Assume that P is not a ramification point of f. Then we have:*

(1) P is a Weierstrass point of Y;

(2) if $P \in Q[H]$, then P is a Weierstrass point of type (a) or of type (b) of Y;

(3) if $P \notin Q[H]$, then P is a Weierstrass point of type (c) of Y with associated integer $k = z-g$;

(4) if $P \in Q[H]$ and $z \geq g+1$, then P is a Weierstrass point of type (b) of Y with associated integer $k = z-g$;

(5) if $P \in Q[H]$ and $z = g$, then P is a Weierstrass point of type (a) of Y.

Proof of Theorema 0.1. We use the notation introduced for the statement of Lemma 1.7. Let Γ be the union of the points whose label says “cusp !”. Let Δ be the union of the schemes $\Delta(Q_i, \delta_i, 1)$ if $Q_i \in \Gamma$ and $\Delta(Q_i, \delta_i, 2)$ if $\Delta_i \notin \Gamma$. Set $W := \mathbf{P}(H^0(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2}) \otimes \mathbf{I}_{\{zP\}\cup\Delta}))$. Let Y be a general element of $\mathbf{P}(H^0(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2}) \otimes \mathbf{I}_{\{zP\}\cup\Delta}))$.

First Claim: We have $h^0(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2}) \otimes \mathbf{I}_{\{zP\}} \cup \Delta) = h^0(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2})) - \text{length}(\{zP\} \cup \Delta) = 3g-2-z-\sum_{1\leq i\leq k} 3\delta_i - \gamma$ and $h^1(S, \mathbf{O}_S(\mathbf{h}+M^{\otimes 2}(-\sum_{1\leq i\leq k}\delta_i Q_i)) \otimes \mathbf{I}_{\{zP\}\cup\Gamma}) = 1$.

Proof of the First Claim. Since $h^1(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2}) = 1$ (Remark 1.2) the last equality is true if and only if the first equality is true, i.e. if the zero-dimensional scheme $\{zP\} \cup \Delta$ imposes independent conditions to the linear system $|2\mathbf{h}+M^{\otimes 2}|$. By semicontinuity it is sufficient to prove the result for some special configuration of points $Q_1, \dots, Q_k$. Let F be vertical fiber containing P. We specialize Q_1 to a general point of F. Let Δ' the residual scheme $Res_F(\Delta)$ of Δ with respect to F; Δ' and Δ have outside Q_1 the same connected components; length(Δ') $-$ length(Δ) $-$ 2 and the connected component of Δ' is empty if Q_1 is labelled an ordinary node, $\{Q_1\}$ if Q_1 is labelled as an ordinary cusp and $\Delta(Q_1, \delta_1 - 1, i)$ ($i = 1$ or 2 according to the label of Q_1) if $\delta_1 \geq 2$. We have $Res_F(\{zP\} \cup \Delta) = \{(z-1)P\} \cup \Delta'$ (Remark 1.1 and 1.3). Since the scheme $F \cap (\{zP\} \cup \Delta)$ has lenght 3, we have $h^0(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2}) \otimes \mathbf{I}_{\{zP\}\cup\Delta}) = h^0(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2}(-Q_1 \otimes \mathbf{I}_{\{(z-1)P\}\cup\Delta'})$. If $\delta_1 \geq 2$ we have $lenght(F \cap (\{(z-1)P\} \cup \Delta')) = 3$ and hence we continue $\delta_1 - 1$ times obtaining $h^0(S, \mathbf{0}_S(2\mathbf{h}+M^{\otimes 2}) \otimes \mathbf{I}_{\{zP\}\cup\Delta}) = h^0(S, \mathbf{O}_S(2\mathbf{h}+M^{\otimes 2}(-\delta_1 P) \otimes \mathbf{I}_{\{(z-\delta_1)P\}\cup\Delta''\cup *})$, where Δ'' is the union of the connected components of Δ not supported by Q_1 and $*$ is the empty set if Q_1 has label "tacnode", while $* = \{Q_1\}$ if Q_1 has "cusp !" as label). If Q_1 has "cusp !" as label, i.e. $* \neq \phi$, we just prove the weaker statement $h_1(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2}(-(\delta_1 + 1)P) \otimes \mathbf{I}_{\{(z-\delta_1 P\}\cup\Delta''}) = 0$. Now, and only now, we specialize Q_2 to a general point of F. At the end it is sufficient to check that $h^1(S, \mathbf{0}_S(\mathbf{h} + M^{\otimes 2}(-(\sum_{1 \leq i \leq k} \delta_i + \gamma)P) \otimes \mathbf{I}_{\{(z - \sum_{1\leq i\leq k} \delta_k)P\}}) = 0$.

Second Claim: Y is integral, $Sing(Y) = \{Q_1, \dots, Q_k\}$, and Y has at each Q_i the singularity prescribed by the label of Q_i and with the invariant δ_i.
Proof of the Second Claim. We claim that the assertions on $Sing(Y)$ follow from the last part of the First Claim, the definition of singularity scheme given in [8], sec. 2, and its use made in [8] to construct plane curves with prescribed singularities. To check the claim see in particular [8], Lemma 2.4, and the fact that the bijectivity of a map $H_1(S, A \otimes \mathbf{J}) \to H^1(S, A \otimes \mathbf{J})$, $A \in Pic(S)$, $\mathbf{J}$ ideal of a zero-dimensional scheme, is what is needed to obtain that $H^0(S, A \otimes \mathbf{J})$ spans $\mathbf{J}$; remember that $h^1(S, \mathbf{O}_S(2\mathbf{h} + M^{\otimes 2})) = 1$ (Remark 1.2). The first assertion follows from the first part of the First Claim and the proof of [4], Th 0.3.

By Lemma 1.7 P is a Weierstrass point of Y with the type we want. Hence we conclude the proof of Theorem 0.1.

(1.8). Fix $P \in Sing(Y)$ and set $\delta := \delta(P, Y) > 0$. By [6], Prop. 3.5, P is a Weierstrass point of Y with weight $w(P) \geq g(g-1)\delta$. The non-negative integer $E(P) := w(P) - g(g-1)\delta$ was called the extraweight of P and it is the real measure of how much P is a Weierstrass point of Y, not just how singular is Y at P. By [6], Prop. 5.5, it is possible to compute $E(P)$ looking at the gap sequences of all points of $\pi^{-1}(P)$ with respect to a suitable linear system, V, on X with $V \cong \mathbf{P}(\pi^*(H^0(Y, \omega_Y)))$. We distinguish four cases.

(1.8.1). Here we assume that P has two branches and that $f(P)$ is not one of the points P_i, $1 \leq i \leq (g-1)^2$. Set $\{P', P''\} := \pi^{-1}(P)$. Since the line $(\{P, \nu\})$ is transversal to each of the two branches of Y at P and $f(P)$ is not a osculating point of E, we see that P' and P'' are not Weierstrass point of the linear system V on X. Thus $E(P) = 0$ ([6], Prop. 5.5). Following the terminology of [4] for smooth ramification points, we will say that P is a tacnode of type I of Y.

(1.8.2). Here we assume that P has two branches and that $f(P)$ is one of the points P_i, $1 \leq i \leq (g-1)^2$. Set $\{P', P''\} := \pi^{-1}(P)$. Since the line $(\{P, \nu\})$ is transversal to each of the two branches of Y at P and $f(P)$ is a osculating point of E with weight 1, we see that the Hermite invariants, $\{h_i\}_{0 \leq i \leq g-1}$ of P' and P'' with respect to V are the same and $h_i = i$ for $i \leq g-2$, $h_{g-1} = g$. Thus P' and P'' are Weierstrass points with weight 1 for the linear system V on X (see [9], Th. 15). Thus $E(P) = 2$ ([6], Prop. 5.5). Following the terminology of [4] for smooth ramification points, we will say that P is a tacnode of type Π of Y.

(1.8.3). Here we assume that P has one branch and that $f(P)$ is not one of the points P_i, $1 \leq i \leq (g-1)^2$. Set $\{Q\} := \pi^{-1}(P)$. Since the line $(\{P, \nu\})$ is not in the tangent cone of Y at P, we see that the sequence of non gaps of Q for the linear system V on X is given by the integers $2t$ $(2 \leq t \leq g-1)$, $2g-1, 2g, \ldots$. Hence Q has weight $(g^2-5g+6)/2$. By [6], Prop. 5.5, P has extraweight $E(P) = (g^2-5g+6)/2$. In particular the extraweight does not depend from δ. Following the terminology of [4] for smooth ramification points, we will say that P is a cusp of type I of Y.

(1.8.4). Here we assume that P has one branch and that $f(P)$ is one of the points P_i, $1 \leq i \leq (g-1)^2$. Set $\{Q\} := \pi^{-1}(P)$. Since the line $(\{P, \nu\})$ is not in the tangent cone of Y at P, we see that the sequence of non gaps of Q for the linear system V on X is given by the integers $2t$ $(2 \leq t \leq g-2)$, $2g-3, 2g-2, 2g, \ldots$. Hence Q has weight $(g^2-5g+10)/2$. By [6], Prop. 5.5, P has extraweight $E(P) = (g^2-5g+10)/2$. In particular the extraweight does not depend from δ. Following the terminology of [4] for smooth ramification points, we will say that P is a cusp of type Π of Y.

An easy modification proof of Theorem 0.1 gives the following existence theorem for bielliptic curves with prescribed singularities; instead of specializing each point Q_i, $1 \leq i \leq k$, to P, loose directly δ_i (or $\delta_i + 1$ for a cusp) conditions to handle the postulation of the scheme $\Delta(Q_i, \delta_i, 2)$ (resp. $\Delta(Q_i, \delta_i, 1)$); for the value of the dimension, see Theorem 0.2 and its proof.

Theorem 1.9. *Fix positive integers* $g, k, \gamma, \delta_1, \dots, \delta_k$ *with* $g \geq 6$, $0 \leq \gamma \leq k$, *and* $\sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 3$. *Fix a smooth elliptic curve* C *and* $M \in Pic^{(g-1)}(C)$. *Use* M *to obtain a linearly normal embedding of* C *into a hyperplane,* H, *of* $\mathbf{P}^{g-1}$. *Fix* $\nu \notin H$ *and call* $T \subset \mathbf{P}^{g-1}$ *the associated elliptic cone with vertex* ν. *Fix* k *general points of* $Q_1, \dots, Q_k$ *of* T. *For each integer* i *with* $1 \leq i \leq k$ *fix one of the following four labels: "tacnode of type I with invariant* δ_i*", "tacnode of type II with invariant* δ_i*", "cusp of type I with invariant* δ_i*" or "cusp of type II with invariant* δ_i*". Assume that exactly* γ *of the label says "cusp !". Then there exists a canonically embedded integral bielliptic curve* $Y \subset T \setminus \{\nu\}$ *with* $Sing(Y) = (Q_1, \dots, Q_k)$ *and such that* Y *has at each* Q_i *the singularity prescribed by the corresponding label. Furthermore, there exists such curve* Y *with the property that the subset of the Hilbert scheme* $Hilb(T)$ *of* T *parametrizing such curves is, near* Y, *a smooth variety of dimension* $3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i)$.

Taking the union for all possible C, M and ν from Theorem 1.9 we obtain the following result; just note that since $k \leq g$ the union of k general points of $\mathbf{P}^{g-1}$ is contained in an elliptic degree $g-1$ two-dimensional cone; here we use that for every bielliptic curve Y there is a non-empty finite set of elliptic cones $T \subset \mathbf{P}^{g-1}$ containing the canonical model of Y, that any elliptic cone, T, with vertex $\nu \notin H \cong \mathbf{P}^{g-2}$ is uniquely determined by $H \cap T$, that dim($\mathbf{P}^{g-1}$ (i.e. the possible vertices, ν, are ∞^{g-1}) and that the subset of $Hilb(H)$ parametrizing the linearly normal smooth elliptic curves of degree $g - 1$ is a smooth variety of dimension $(g-1)^2$.

Theorem 1.10. *Fix positive integers* $g, k, \gamma, \delta_1, \dots, \delta_k$ *with* $g \geq 6$, $0 \leq k$, *and* $\sum_{1 \leq i \leq k} 3\delta_i + \gamma \leq 2g - 3$. *Fix* k *general points of* $Q_1, \dots, Q_k$ *of* $\mathbf{P}^{g-1}$. *For each integer* i *with* $1 \leq i \leq k$ *fix one of the following four labels: "tacnode of type I with invariant* δ_i*", "tacnode of type II with invariant* δ_i*", "cusp of type I with invariant* δ_i*", or "cusp of type II with invariant* δ_i*". Assume that exactly* γ *of the labels say "cusp !". Then there exists a canonically embedded integral bielliptic curve* $Y \subset \mathbf{P}^{g-1}$ *with* $Sing(Y) = (Q_1, \dots, Q_k)$ *and such that* Y *has at each* Q_i *the singularity prescribed by the corresponding label. Furthermore, there exists such curve* Y *with the property that the subset of the Hilbert scheme* $Hilb(\mathbf{P}^{g-1})$ *of* $\mathbf{P}^{g-1}$ *parametrizing such curves is, near* Y, *a smooth variety of dimension* $3g - 3 + k - \gamma - 2(\sum_{1 \leq i \leq k} \delta_i) + g(g-1)$.

Now we may prove Theorem 0.2.

Proof of Theorem 0.2. By remark 1.3 the smoothness criterion [7], Th. 3.5 and

Remark 3.8, part 3, is satisfied.

Remark 1.11. Let $f : Y \to C$ is a double covering with $p_a(C) = 1$ and C singular, i.e. with C rational and with a unique singular point, Q, which is either an ordinary node or an ordinary cusp; The canonical model of Y is again contained in a cone with vertex $\nu \notin Y$ and, as base, a degree $g - 1$ curve isomorphic to C and embedded into a hyperplane, H, of $\mathbf{P}^{g-1}$ ([2], Prop. 4.2). Since $Pic^0(C) \cong \mathbf{K}^*$, outside the singular point such curve $C \subset H$ has exactly $g - 1$ asculating points, say P_i, $1 \le i \le g - 1$. For each $P \in Sing(Y)$ with $f(P) \neq Q$, the classification of all possibile singularities not mapped into Q and their division into types works verbatim, taking the points P_i, $1 \le i \le g-1$, instead of the points P_i, $1 \le i \le (g - 1)^2$.

Remark 1.12. Here we assume char(**K**) ¿ 0, but char(**K**) $\neq 2$. Let $f : Y \to C$ is a double covering with $p_a(C) = 1$. The canonical model of Y is again contained in a cone with vertex $\nu \notin Y$ and, as base a degree $g - 1$ curve isomorphic to C and embedded into a hyperplane, H, of $\mathbf{P}^{g-1}$ ([2], Prop. 4.2). The classification of singular points, P, of Y with $f(P) \notin Sing(C)$ as cusps or tacnodes works even in this case (see [11], pp. 100–101). Obviously here $C \subset H$ may have a smaller number of osculating points if $p \le g - 1$, but the only difference is that their weight is bigger than 1. From now on we assume $p > 2g - 2$. Under this assumption we are sure that the Hermite invariants of the linear system, V, induced by $\pi : X \to Y \subset \mathbf{P}^{g-1}$ at a generic point of X are the classical ones ([9], Th. 15). Furthermore, only if we have such a restrictive assumption on char(**K**) we are sure that the weigth of a Weierstrass point $Q \in X$ of V is computed using the gap sequence of V at Q ([9], Th. 15, part (iii)). With this very restrictive assumption on $char(\mathbf{K})$ we may copy [6], Prop. 5.5, and extend (1.8). For the case Y smooth, see [1].

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