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SYMMETRIZATION RESULTS FOR A MULTI-EXPONENT, DEGENERATE AND ANISOTROPIC ELECTROSTATIC PROBLEM

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In this paper, we give some isoperimetric inequalities for the capacity c_p of an anisotropic configuration where each connected component has the form $\Omega_i = \omega_i \setminus \overline{\omega}'_i$, $i \in \{1, \dots, n\}$, ω_i and ω'_i are regular bounded open sets in $\mathbb{R}^{N_i}$, $(N_i \geq 1)$. The anisotropy of Ω_i is described by a Finsler metric (or gauge function) $\phi_i(\xi)$, $\xi \in \mathbb{R}^{N_i}$ and the growth exponent p may vary with i. Using the convex symmetrization, we prove in particular that $c_p \geq \tilde{c}_p$, where $\tilde{c}_p$ is the capacity of a suitable symmetrized anisotropic configuration.

1. Statement of the problem.

Let $\Omega_i (i = 1, \dots, n)$ be open sets of the form $\Omega_i = \omega_i \setminus \overline{\omega}'_i$, where ω_i and ω'_i are regular bounded open sets in $\mathbb{R}^{N_i} (N_i \geq 1)$ such that $\overline{\omega}'_i \subset \omega_i$. Let $\gamma_i = \partial\omega_i$ and $\gamma'_i = \partial\omega'_i$ be the respective boundaries of ω_i and ω'_i.

Let $r = (r_i)$, $p = (p_i)$, $q = (q_i)$, $i = 1 \dots, n$ be multi-exponents such that

$$(1.1) \qquad 1 \leq r_i \leq \infty,\ 1 + \frac{1}{r_i} < p_i < \infty,\ q_i = \begin{cases} p_i & \text{if} \quad r_i = \infty \\ \dfrac{r_i}{1 + r_i} p_i & \text{if} \quad r_i < \infty \end{cases}$$

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(hence $1 < q_i \le p_i$) and let $a_i : \Omega_i \to \mathbb{R}$ be a (a.e.) positive function such that

$$a_i \in L^1(\Omega_i), \quad a_i^{-1} = \frac{1}{a_i} \in L^{r_i}(\Omega_i) \tag{1.2}$$

where $L^1(\Omega_i)$ and $L^{r_i}(\Omega_i)$ are classical Lebesgue spaces. Let

$$L^{p_i}_{a_i}(\Omega_i) = \Big\{ v : \Omega_i \to \mathbb{R}, \quad \int_{\Omega_i} a_i |v|^{p_i}\, dx < +\infty \Big\}$$

be the weighted Lebesgue space equipped with the norm

$$\|v\|_{L^{p_i}_{a_i}(\Omega_i)} = \Big(\int_{\Omega_i} a_i |v|^{p_i}\, dx \Big)^{1/p_i}$$

and let us introduce the spaces

$$\mathbb{L}^q = \{ v = (v_1, \ldots, v_n), \quad \forall i = 1, \ldots, n,\, v_i \in L^{q_i}(\Omega_i) \},$$

$$\mathbb{L}^p_a = \{ v = (v_1, \ldots, v_n), \quad \forall i = 1, \ldots, n,\, v_i \in L^{p_i}_{a_i}(\Omega_i) \}.$$

We equip them with the respective norms

$$\|v\|_{\mathbb{L}^q} = \sum_{i=1}^n \|v_i\|_{L^{q_i}(\Omega_i)}, \qquad \|v\|_{\mathbb{L}^p_a} = \sum_{i=1}^n \|v_i\|_{L^{p_i}_{a_i}(\Omega_i)}.$$

By Hölder's inequality, (1.1) and (1.2), it is easy to check that

$$\|v\|_{\mathbb{L}^q} \le \max_{i \in \{1, \ldots, n\}} \Big\{ \|a_i^{-1}\|^{1/p_i}_{L^{r}_{i}(\Omega_i)} \Big\} \|v\|_{\mathbb{L}^p_a} \tag{1.3}$$

and it follows that $\mathbb{L}^p_a \hookrightarrow \mathbb{L}^q$ with continuous imbedding. Moreover, let us set

$$\mathbb{W}^{1,q} = \Big\{ v = (v_1, \ldots, v_n),\ \forall i = 1, \ldots, n,\ v_i \in W^{1,q_i}(\Omega_i) \Big\},$$
$$\mathbb{W}_a = \Big\{ v = (v_1, \ldots, v_n),\ \forall i = 1, \ldots, n,\ v_i \in L^{q_i}(\Omega_i),\, \nabla v_i \in L^{p_i}_{a_i}(\Omega_i)^{N_i} \Big\},$$

where for simplicity ∇ denotes the gradient (in the sense of distributions) in any dimension.

By the previous remark, $\mathbb{W}_a \hookrightarrow \mathbb{W}^{1,q}$ with continuous imbedding. In particular if $v \in \mathbb{W}_a$ then $v_{|\gamma_i}$ and $v_{|\gamma_i'}$ are well defined and belong respectively to $L^{q_i}(\gamma_i)$ and $L^{q_i}(\gamma_i')$. Hence, we can define

$$\mathbb{H} = \Big\{ v \in \mathbb{W}_a,\, v_1 = 1 \text{ on } \gamma_1',\ v_n = 0 \text{ on } \gamma_n \text{ and } v_{i|\gamma_i} = v_{i+1_{|\gamma_{i+1}'}} = k_i$$

(undetermined constant) for $i = 1, \dots, n-1\big\}$.

Let $\phi_i : \mathbb{R}^{N_i} \to [0, +\infty[(i = 1, \dots, n)$, be non negative strictly convex functions, differentiable off the origin, homogeneous in the sense

$$\forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^{N_i}, \phi_i(t\xi) = |t|\phi_i(\xi) \tag{1.4}$$

and with linear growth

$$\exists \delta > 0,\ \forall \xi \in \mathbb{R}^{N_i}, |\xi| \le \phi_i(\xi) \le \delta|\xi|, \tag{1.5}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^{N_i}$.

Let $G_i : \Omega_i \times \mathbb{R}^{N_i} \to G_i(x, \xi) \in \mathbb{R}(i = 1, \dots, n)$, be Carathéodory functions (i.e. measurable with respect to x and continuous with respect to ξ) such that

- for almost every $x \in \Omega_i$, $G_i(x, .)$ is strictly convex, homogeneous of degree p_i in the sense

  $$\forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^{N_i}, G_i(x, t\xi) = |t|^{p_i} G_i(x, \xi)$$

  and it admits a gradient $g_i(x, .)$,
- there exists $c \ge 1$ such that for almost every $x \in \Omega_i$ and for every $\xi \in \mathbb{R}^{N_i}$

$$a_i(x)\phi_i(\xi)^{p_i} \le G_i(x, \xi) \le c a_i(x)|\xi|^{p_i}. \tag{1.6}$$

We consider the following problem

$$\inf\Big\{J(v) = \sum_{i=1}^{n} \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla v_i)\, dx,\ v \in \mathbb{H}\Big\}, \tag{1.7}$$

the integral being finite thanks to (1.6).

For $N_i = N, \phi_i(\xi) = |\xi|$ and $p_i = p$ for any $i \in \{1, \dots, n\}$, similar problems have been considered by V. Ferone and L. Boukrim. In an interesting paper [9], V. Ferone has given an isoperimetric inequality for the p-capacity c_p of a configuration $\Omega = (G \setminus E)\setminus(\cup_i H_i)$, where Ω represents a nonhomogeneous isotropic medium, ∂G and ∂E have given potentials respectively equal to 0 and 1, and the H_i have constant unknown potentials K_i. He has shown that $c_p \ge c_p^*$ where c_p^* is the p-capacity of a symmetrical configuration which has no interior conductor such as H_i. In his thesis [6] (see also the short note [5]), L. Boukrim has extended and completed Ferone's result when Ω is multiconnected and when the H_i separate the different connected components of Ω. He proved that $c_p \geq \overline{c}_p \geq c_p^*$, where $\overline{c}_p$ is the p-capacity of a symmetrized isotropic configuration (having inner conductors) and gave isoperimetric estimates for the unknown potentials K_i.

In this paper the anisotropy function ϕ_i, as well the growth exponent p_i, may be different when i varies. Our purpose is to show that the generalized p-capacity of the collection of $\Omega_i (i = 1, \ldots, n)$, denoted c_p (see section 2 below) is not smaller than the p-capacity $\tilde{c}_p$ of a symmetrized anisotropic configuration and to give isoperimetric estimates for the unknown potentials K_i. The proof, inspired by the work of L. Boukrim, uses the notion of relative rearrangement introduced by J. Mossino and R. Temam [12] and developed in [13, 14]. But the anisotropy of Ω_i requires other arguments related to the new notion of convex symmetrization introduced in [1].

2. Study of the problem.

In this section we study the existence, uniqueness and characterization of solution of problem (1.7).

Theorem 1. *Problem* (1.7) *admits a solution and only one.*

Proof. The proof is not quite standard in this context of degenerate problems in several domains in different dimensions and with different exponents. Let u^m be a minimizing sequence: $u^m \in \mathbb{H}$ and $J(u^m) \to I$, where I denotes the infimum in (1.7). We have, due to the coerciveness condition in (1.6) together with (1.5),

$$\sum_{i=1}^{n} \int_{\Omega_i} a_i(x) |\nabla u_i^m|^{p_i} dx \leq J(u^m) \leq c$$

and hence $\|\nabla u_i^m\|_{L_{a_i}^{p_i}(\Omega_i)^{N_i}} \leq c$ where here (and in the following) we denote by c any constant.

In particular ∇u_n^m is bounded in $L^{q_n}(\Omega_n)^{N_n}$. As $u_n^m = 0$ on γ_n, u_n^m is bounded in $W^{1,q_n}(\Omega_n)$ by Poincaré inequality. By continuity of the trace mapping (i.e. $W^{1,q_n}(\Omega_n) \to L^{q_n}(\gamma_n')$), $k_{n-1}^m = u^m_{n|\gamma_n'}$ is bounded in $\mathbb{R}$.

Now ∇u_{n-1}^m is bounded in $L^{q_{n-1}}(\Omega_{n-1})^{N_{n-1}}$ and $k_{n-1}^m = u^m_{n-1|_{\gamma_{n-1}}}$ is bounded in $\mathbb{R}$. It follows from Poincaré inequality that u_{n-1}^m is bounded in $W^{1,q_{n-1}}(\Omega_{n-1})$ and, just as above $k_{n-2}^m = u^m_{n-1|_{\gamma'_{n-1}}}$ is bounded in $\mathbb{R}$, so that by induction u_i^m is bounded in $W^{1,q_i}(\Omega_i)$ (for any $i = 1,\dots,n$) and $k_i^m = u^m_{i|\gamma_i} = u^m_{i+1|\gamma'_{i+1}}$ is bounded in $\mathbb{R}$ (for any $i = 1,\dots,n-1$).

Up to an extraction of a subsequence we may suppose that for any $i = 1,\dots,n$

$$\begin{aligned}
u_i^m &\rightharpoonup u_i \text{ weakly in } W^{1,q_i}(\Omega_i),\\
u_i^m &\to u_i \text{ strongly in } L^{q_i}(\Omega_i) \text{ (by compactness)},\\
u^m_{i|\gamma_i} \text{ (resp. } u^m_{i|\gamma'_i}) &\to u_{i|\gamma_i} \text{ ((resp. } u_{i|\gamma'_i}) \text{ strongly in } L^{q_i}(\Gamma_i) \text{ (resp. } L^{q_i}(\gamma'_i)),\\
\nabla u_i^m &\rightharpoonup \zeta_i \text{ weakly in } L^{p_i}_{a_i}(\Omega_i)^{N_i},\\
k_i^m &\to k_i \text{ in } \mathbb{R}.
\end{aligned}$$

As $u^m \in \mathbb{H}$, we get $u_1 = 1$ on γ'_1, $u_n = 0$ on γ_n, $u_{i|\gamma_i} = u_{i+1|\gamma'_{i+1}} = k_i$ ($i = 1,\dots,n-1$). As $\nabla u_i^m \rightharpoonup \zeta_i$ weakly in $L^{p_i}_{a_i}(\Omega_i)^{N_i}$, we get $\nabla u_i^m \rightharpoonup \zeta_i$ weakly in $L^{q_i}(\Omega_i)^{N_i}$ by using the continuity of the imbedding $L^{p_i}_{a_i}(\Omega_i) \hookrightarrow L^{q_i}(\Omega_i)$. Since $u_i^m \to u_i$ in $L^{q_i}(\Omega_i)$, it follows that $\zeta_i = \nabla u_i \in L^{p_i}_{a_i}(\Omega_i)^{N_i}$ and $u \in \mathbb{H}$.

It remains to prove that u solves (1.7). We note that $(x,\xi) \in \Omega_i \times \mathbb{R}^{N_i} \to G_i(x,\xi) \in \mathbb{R}$ is a Carathéodory function such that by (1.5) and (1.6)

$$a_i(x)|\xi|^{p_i} \le G_i(x,\xi) \le c a_i(x)|\xi|^{p_i}.$$

Hence the mapping $r \to G_i(x,r)$ is continuous from $L^{p_i}_{a_i}(\Omega_i)^{N_i}$ into $L^1(\Omega_i)$ and the mapping $r \to \int_{\Omega_i} G_i(x,r)\,dx$ is continuous from $L^{p_i}_{a_i}(\Omega_i)^{N_i}$ into $\mathbb{R}$. It is also convex, so that it is lower semicontinuous for the weak topology of $L^{p_i}_{a_i}(\Omega_i)^{N_i}$ and as $\nabla u_i^m \rightharpoonup \nabla u_i$ in $L^{p_i}_{a_i}(\Omega_i)^{N_i}$,

$$\begin{aligned}
I = \liminf \sum_{i=1}^n \frac{1}{p_i}\int_{\Omega_i} G_i(x,\nabla u_i^m)\,dx &\ge \sum_{i=1}^n \frac{1}{p_i} \liminf \int_{\Omega_i} G_i(x,\nabla u_i^m)\,dx\\
&\ge \sum_{i=1}^n \frac{1}{p_i}\int_{\Omega_i} G_i(x,\nabla u_i)\,dx
\end{aligned}$$

which proves that u solves (1.7). By the strict convexity, the gradient is the same in each Ω_i for all solutions of (1.7) and it follows from the boundary conditions in $\mathbb{H}$ that the solution of (1.7) is unique (and then the above convergences hold for the whole sequence u^m). This finishes the proof of Theorem 1. □

Let u be the solution of (1.7). It is classical that u is characterized by the variational formulation: $u \in \mathbb{H}$ and

$$0 = \sum_{i=1}^n \frac{1}{p_i}\int_{\Omega_i} g_i(x,\nabla u_i).\,\nabla v_i\,dx, \quad \forall v \in \mathbb{H}_0, \tag{2.1}$$

with

$$\mathbb{H}_0 = \Big\{ v \in \mathbb{W}_a,\, v_1 = 0 \text{ on } \gamma_1',\, v_n = 0 \text{ on } \gamma_n \text{ and } v_{i|\gamma_i} = v_{i+1|\gamma_{i+1}'} = k_i \\ \text{(undetermined constant) for } \quad i = 1, \ldots, n-1 \Big\}.$$

It follows that u satisfies

$$\begin{cases} \mathcal{A}_i u_i = 0 & \text{in } \Omega_i \text{ (in the sense of distributions)} \\ u_1 = 1 & \text{on } \gamma_1', \\ u_n = 0 & \text{on } \gamma_n, \\ u_{i|\gamma_i} = u_{i+1|_{\gamma_{i+1}'}} = k_i & \text{(unprescribed constant) for } i = 1, \ldots, n-1, \end{cases}$$

where

$$\mathcal{A}_i\, u_i = -\frac{1}{p_i} \operatorname{div}\, (g_i(x, \nabla u_i))$$

and for simplicity div (resp. ∇) denotes the divergence (resp. gradient in any dimension N_i.

Let $v_i (i = 1, \ldots, n)$ be the unique solution of

$$\text{(2.2)} \quad \inf \left\{ \frac{1}{p_i} \int_{\Omega_i} G_i(x, \nabla w)\, dx,\, w \in W_{a_i}(\Omega_i),\, w = 1 \text{ on } \gamma_i',\, w = 0 \text{ on } \gamma_i \right\},$$

where

$$W_{a_i}(\Omega_i) = \{ v \in L^{q_i}(\Omega_i),\, \nabla v \in L^{p_i}_{a_i}(\Omega_i)^{N_i} \}.$$

Then v_i is characterized by $v_i \in W_{a_i}(\Omega_i)$, $v_i = 1$ on γ_i', $v_i = 0$ on γ_i and

$$\text{(2.3)} \qquad \int_{\Omega_i} g_i(x, \nabla v_i).\, \nabla \varphi\, dx = 0,\ \forall \varphi \in W_{a_i}(\Omega_i),\, \varphi = 0 \text{ on } \gamma_i' \cup \gamma_i,$$

and it follows that

$$\text{(2.4)} \qquad \begin{cases} \mathcal{A}_i v_i = 0 & \text{in } \Omega_i \text{ (in the sense of distributions),} \\ v_i = 1 & \text{on } \gamma_i', \\ v_i = 0 & \text{on } \gamma_i. \end{cases}$$

Next, we prove that the solution u of (1.7) is explicit in terms of the solutions v_i $(i = 1, \ldots, n)$ of (2.2).

Theorem 2. *Let u be the solution of* (1.7), $k_i = u_{i+1|_{\gamma'_{i+1}}}$ *and let v_i be the solution of* (2.2). *Let*

$$c_p = \sum_{i=1}^{n} \int_{\Omega_i} G_i(x, \nabla u_i)\, dx$$

be a generalized p-capacity of the collection of $\Omega_i (i = 1, \ldots, n)$. We have

(a) $c_p > 0,$

(b) $c_p = \dfrac{1}{p_i} \displaystyle\int_{\Omega_i} g_i(x, \nabla u_i). \nabla v_i \, dx,$ *for* $i = 1, 2, \ldots, n,$

(c) $\displaystyle\int_{\Omega_i} G_i(x, \nabla u_i)\, dx > 0,$

(d) $k_i \neq k_{i-1},$

(e) $u_i = (k_{i-1} - k_i)v_i + k_i,$

(f) $\displaystyle\int_{\Omega_i} G_i(x, \nabla u_i)\, dx = (k_{i-1} - k_i)c_p,$

(g) $0 = k_n < k_{n-1} < \ldots < k_{i+1} < k_i < \ldots < k_i < k_0 = 1,$

(h) $\displaystyle\int_{\Omega_i} G_i(x, \nabla v_i)\, dx = \frac{c_p}{(k_{i-1} - k_i)^{p_{i-1}}},$

(i) $0 < v_i < 1, \;\; k_i < u_i < k_{i-1}.$

Proof. **(a)** If $c_p = 0$ then we get from (1.5) and (1.6) that u_i is constant in each connected component Ω_i. Using the transmission conditions (because $u \in \mathbb{H}$), we obtain a contradiction.

(b) Let $\tilde{v}^i = (\tilde{v}^i_1, \ldots, \tilde{v}^i_n)$ be the function defined by

$$\tilde{v}^i_i = v_i, \qquad \tilde{v}^i_j = \begin{cases} 1 & \text{if} \quad j < i \\ 0 & \text{if} \quad j > i. \end{cases}$$

As $\tilde{v}^i - u \in \mathbb{H}_0$, we get, using the variational formulation of u,

$$0 = \sum_j \frac{1}{p_j} \int_{\Omega_j} g_j(x, \nabla u_j). \nabla(\tilde{v}^i_j - u_j)\, dx$$

which is equivalent to

$$c_p = \sum_j \int_{\Omega_j} G_j(x, \nabla u_j)\, dx = \frac{1}{p_i} \int_{\Omega_i} g_i(x, \nabla u_i). \nabla v_i \, dx.$$

(c) If $\int_{\Omega_i} G_i(x, \nabla u_i)\, dx = 0$ then from (1.5) and (1.6), we have $\nabla u_i = 0$ and hence $c_p = 0$ using **(b)**; but this contradicts **(a)**.

(d) If $k_i = k_{i-1}$, then we can define $m^i = (m_1^i, \ldots, m_n^i)$ by

$$m_j^i = \begin{cases} k_i & \text{for} \quad j = i \\ u_i & \text{otherwise} \end{cases}$$

and m^i belongs to $\mathbb{H}$. It follows from **(c)** that

$$\sum_j \int_{\Omega_j} \frac{1}{p_j} G_j(x, \nabla m_j^i)\, dx < \sum_j \int_{\Omega_j} \frac{1}{p_j} G_j(x, \nabla u_j)\, dx$$

which contradicts the minimality property of u.

(e) Following **(d)**, one can define $w_i = \frac{u_i - k_i}{k_{i-1} - k_i}$. It is easy to check (from $\mathcal{A}_i u_i = 0$), that $\mathcal{A}_i w_i = 0$, $w_i = 1$ on γ_i', $w_i = 0$ on γ_i. The functions v_i and w_i satisfy the same equation which has a unique solution. It follows that $w_i = v_i$.

(f) It is sufficient to replace v_i by $\frac{u_i - k_i}{k_{i-1} - k_i}$ in **(b)**.

(g) Clear from **(a)**, **(c)** and **(f)**.

(h) Replace u_i by $(k_{i-1} - k_i)v_i + k_i$ in **(b)**.

(i) Using convenient test functions in (2.3), it is easy to prove that $0 < v_i < 1$ and then **(e)** gives $k_i < u_i < k_{i-1}$. □

Remark 1. *From* **(h)** *of Theorem 2,* $\sum_{i=1}^{n}(k_{i-1} - k_i) = 1$ *and* $k_i = 1 - \sum_{j=1}^{i}(k_{j-1} - k_j)$, *we get*

$$1 = \sum_{i=1}^{n} \left(\frac{c_p}{\int_{\Omega_i} G_i(x, \nabla v_i)\, dx} \right)^{\frac{1}{p_i - 1}}$$

and

$$k_i = 1 - \sum_{j=1}^{i} \left(\frac{c_p}{\int_{\Omega_j} G_j(x, \nabla v_j)\, dx} \right)^{\frac{1}{p_j - 1}}$$

Remark 2. *If Green's formula is valid, then we have from* $\mathcal{A}_i u_i = 0$ *in* Ω_i *and from* **(b)** *of Theorem 2 that for all* $i = 1, \ldots, n$

$$c_p = -\int_{\gamma_i'} \frac{\partial u_i}{\partial \nu^{\mathcal{A}_i}} d\gamma = -\int_{\gamma_i} \frac{\partial u_i}{\partial \nu^{\mathcal{A}_i}} d\gamma$$

where

$$\frac{\partial u_i}{\partial v^{\mathcal{A}_i}} = \frac{1}{p_i} g_i(x, \nabla u_i).\, v$$

and for simplicity v *denotes the outer normal to* Ω_i *on* γ_i *as well as the inner normal to* Ω_i *on* γ_i'.

3. Main inequalities.

Let us recall some notions of (unidimensional and relative) rearrangement (see for example [3], [8], [11], [12], [13], [14]). In this paper, we use only the Lebesgue measure on $\mathbb{R}^N$ (for different values of N). For a measurable set E in $\mathbb{R}^N$, let $|E|$ be its measure. Let u be a measurable function from E into $\mathbb{R}$. The (unidimensional) decreasing rearrangement u_* of u is defined on $\overline{E}^* = [0, |E|]$ by $u_*(|E|) = ess_E \inf u$ and for $s < |E|$, $u_*(s) = \inf\{\theta \in \mathbb{R}, |u > \theta| \leq s\}$ where $|u > \theta| = |\{x \in E : u(x) > \theta\}|$; the increasing rearrangement of u, denoted u^*, is then $u^*(s) = u_*(|E| - s)$. The functions u, u_* and u^* satisfy $|u > \theta| = |u_* > \theta| = |u^* > \theta|$.

For $v \in L^1(E)$ and $u : E \to \mathbb{R}$ measurable, we define the function $\mathcal{W}$ on $\overline{E^*}$ by

$$\mathcal{W}(s) = \begin{cases} \displaystyle\int_{u>u_*(s)} v(x)\,dx & \text{if } |u = u_*(s)| = 0, \\ \displaystyle\int_{u>u_*(s)} v(x)\,dx + \int_0^{s-|u>u_*(s)|} (v|_{P(s)})_*(\sigma)\,d\sigma & \text{otherwise,} \end{cases}$$

where $(v|_{P(s)})_*$ is the decreasing rearrangement of v restricted to $P(s) = \{x \in E : u(x) = u_*(s)\}$. The integrable function $\frac{d\mathcal{W}}{ds}$ is called (according to [12], [13], [14]) the relative rearrangement of v with respect to u and is denoted v_{*_u}.

We recall also some facts about the function ϕ_i defined in section 1. As it has been said earlier, the function $\phi_i : \mathbb{R}^{N_i} \to [0, +\infty[$ is strictly convex, homogeneous of degree one, with linear growth and differentiable off the origin.

Let

$$B_{\phi_i} = \{\xi \in \mathbb{R}^{N_i}; \phi_i(\xi) \leq 1\}$$

be the unit ball of $\mathbb{R}^{N_i}$ relative to ϕ_i. It follows from the definition of ϕ_i that the ball B_{ϕ_i} (the so-called Wulff shape relative to ϕ_i) is bounded, convex and symmetric with respect to the origin.

We denote by $\phi_i^0 : \mathbb{R}^{N_i} \to [0, +\infty[$ the dual function of ϕ_i defined by

$$\phi_i^0(\xi^*) = \sup\{\xi^*.\,\xi;\, \xi \in B_{\phi_i}\} = \sup_{\xi\neq 0} \frac{\xi^*.\,\xi}{\phi_i(\xi)}, \qquad \forall \xi^* \in \mathbb{R}^{N_i}.$$

One can check that ϕ_i^0 is also a convex function and satisfies the properties (1.4) and $\frac{1}{\delta}|\xi^*| \le \phi_i^0(\xi^*) \le |\xi^*|$ (see for example [15]). In the sequel, we assume that the dual function ϕ_i^0 is strictly convex and differentiable everywhere but in the origin. The corresponding unit ball $B_{\phi_i^0}$ is known as Frank diagram. One can also prove from (1.4) the following useful properties of the functions ϕ_i and ϕ_i^0 (see for example [4]). Let $\xi \in \mathbb{R}^{N_i} \setminus \{0\}$ and let $t \neq 0$, then

$$\nabla\phi_i(t\xi) = \frac{t}{|t|}\nabla\phi_i(\xi), \qquad \nabla\phi_i^0(t\xi) = \frac{t}{|t|}\nabla\phi_i^0(\xi) \tag{3.1}$$

$$\phi_i(\xi) = \nabla\phi_i(\xi).\,\xi, \qquad \phi_i^0(\xi) = \nabla\phi_i^0(\xi).\,\xi \tag{3.2}$$

$$1 = \phi_i(\nabla\phi_i^0(\xi)) = \phi_i^0(\nabla\phi_i(\xi)) \tag{3.3}$$

$$\xi = \phi_i^0(\xi)\nabla\phi_i(\nabla\phi_i^0(\xi)) = \phi_i(\xi)\nabla\phi_i^0(\nabla\phi_i(\xi)). \tag{3.4}$$

All the isoperimetric inequalities of this section are consequences of the following theorem.

Theorem 3. *Let $i \in \{1, \dots, n\}$. Let α_i be the Lebesgue measure of the unit ball (i.e. Frank diagram) $B_{\phi_i^0} = \{\xi \in \mathbb{R}^{N_i};\, \phi_i^0(\xi) \le 1\}$ in $\mathbb{R}^{N_i}$. Let p_i' be such that $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ and let v_i be the unique solution of* (2.2)*. Then for all t, t' such that $0 \le t \le t' \le 1$, we have*

$$t' - t \le N_i^{-p_i'}\alpha_i^{-p_i'/N_i}\Big(\int_{\Omega_i} G_i(x, \nabla v_i)\,dx\Big)^{p_i'/p_i}.$$
$$\cdot \int_{|v_i>t'|}^{|v_i>t|} (|\omega_i'| + \sigma)^{\frac{p_i'}{N_i} - p_i'}(a_i^*)^{-\frac{p_i'}{p_i}}(\sigma - |v_i > t'|)\,d\sigma.$$

Proof. For $\theta \in]0, 1[$, let us set

$$z_i = \theta - (v_i - \theta)_- = \begin{cases} v_i & \text{if} \quad v_i \le \theta \\ \theta & \text{if} \quad v_i > \theta. \end{cases}$$

Then the function $\varphi = z_i - \theta v_i$ satisfies the conditions $\varphi \in W_{a_i}(\Omega_i)$, $\varphi = 0$ on $\gamma_i' \cup \gamma_{i+1}$. In consequence, we have using (2.3)

$$0 = \int_{\Omega_i} g_i(x, \nabla v_i). \nabla(z_i - \theta v_i)\, dx.$$

Hence

$$\int_{v_i \leq \theta} G_i(x, \nabla v_i)\, dx = \theta \int_{\Omega_i} G_i(x, \nabla v_i)\, dx$$

and then

$$\frac{d}{d\theta} \int_{v_i > \theta} G_i(x, \nabla v_i)\, dx = - \int_{\Omega_i} G_i(x, \nabla v_i)\, dx. \tag{3.5}$$

Moreover, by using (1.6), (1.2) and Hölder's inequality, we have for $h > 0$,

$$\frac{1}{h} \int_{\theta < v_i \leq \theta + h} \phi_i(\nabla v_i)\, dx \leq \left(\frac{1}{h} \int_{\theta < v_i \leq \theta + h} a_i^{-p_i'/p_i}\, dx \right)^{1/p_i'} \cdot \left(\frac{1}{h} \int_{\theta < v_i \leq \theta + h} G_i(x, \nabla v_i)\, dx \right)^{1/p_i}$$

and letting h tend to 0, we get (thanks to (3.5))

$$-\frac{d}{d\theta} \int_{v_i > \theta} \phi_i(\nabla v_i)\, dx \leq \left(-\frac{d}{d\theta} \int_{v_i > \theta} a_i^{-p_i'/p_i}\, dx \right)^{1/p_i'} \cdot \left(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \right)^{1/p_i}$$

By using the following formula of derivation (see [14])

$$\frac{d}{d\theta} \int_{v_i > \theta} a_i^{-p_i'/p_i}\, dx = \mathcal{W}'(v_i(\theta)) v_i'(\theta)$$

where $v_i(\theta) = |v_i > \theta|$ and $\mathcal{W}' = \left(a_i^{-p_i'/p_i}\right)_{*v_i}$ is the relative rearrangement of $a_i^{-p_i'/p_i}$ with respect to v_i it comes

$$-\frac{d}{d\theta} \int_{v_i > \theta} \phi_i(\nabla v_i)\, dx \leq \left(-\mathcal{W}'(v_i(\theta)) v_i'(\theta) \right)^{1/p_i'} \left(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \right)^{1/p_i} \tag{3.6}$$

Let $P_{\phi_i,\Omega_i}(\{v_i > \theta\})$ be the generalized perimeter relative to ϕ_i and Ω_i of the set $\{x \in \Omega_i,\, v_i(x) > \theta\}$ defined in [2] by

$$P_{\phi_i,\Omega_i}(\{v_i > \theta\}) = \sup\left\{ \int_{v_i>\theta} \operatorname{div}(\sigma)\, dx;\ \sigma \in C_0^1(\Omega_i, \mathbb{R}^{N_i}),\ \phi_i^0(\sigma) \le 1 \right\}.$$

The following two results hold (see [1]):

$$-\frac{d}{d\theta}\int_{v_i>\theta} \phi_i(\nabla v_i)\, dx = P_{\phi_i,\Omega_i}(\{v_i > \theta\}), \tag{3.7}$$

$$P_{\phi_i,\Omega_i}(\{v_i > \theta\}) \le N_i \alpha_i^{1/N_i}(|\omega_i'| + v_i(\theta))^{1-\frac{1}{N_i}} \tag{3.8}$$

Let's note that for $\phi_i(\xi) = |\xi|$, the result (3.7) is nothing else the Fleming-Rishel formula (see [10]) and the corresponding inequality (3.8) is known as the isoperimetric inequality for the perimeter of De Giorgi (see [7]).

Now, using (3.6), (3.7) and (3.8), we get

$$1 \le N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \left(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \right)^{p_i'/p_i} \cdot$$
$$\cdot (|\omega_i'| + v_i(\theta))^{\frac{p_i'}{N_i} - p_i'} \mathcal{W}'(v_i(\theta))(-v_i'(\theta)).$$

By integrating between t and t',

$$t' - t \le N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \Big(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \Big)^{p_i'/p_i} \cdot$$
$$\cdot \int_0^{|\Omega_i|} \chi[v_i(t'), (t)](\sigma)(|\omega_i'| + \sigma)^{\frac{p_i'}{N_i} - p_i'} \Big(a_i^{-p_i'/p_i} \Big)_{*v_i}(\sigma)\, d\sigma$$
$$\le N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \Big(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \Big)^{p_i'/p_i} \cdot$$
$$\cdot \int_0^{|\Omega_i|} \Big(\chi[v_i(t'), v_i(t)](.)(|\omega_i'| + .)^{\frac{p_i'}{N_i} - p_i'} \Big)_*(\sigma) \big(a_i^{-p_i'/p_i} \big)_*(\sigma)\, d\sigma$$

(for this latest inequality, see Theorem 3 in [13])

$$= N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \Big(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \Big)^{p_i'/p_i} \cdot$$

$$\cdot \int_0^{|\Omega_i|} \chi[0, v_i(t) - v_i(t')](\sigma)(|\omega_i'| + v_i(t') + \sigma)^{\frac{p_i'}{N_i} - p_i'} (a_i^*)^{-\frac{p_i'}{p_i}}(\sigma)\, d\sigma$$

(using the properties of the (unidimensional) decreasing rearrangement)

$$= N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \Big(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \Big)^{p_i'/p_i} \cdot$$

$$\cdot \int_{v_i(t')}^{v_i(t)} (|\omega_i'| + \sigma)^{\frac{p_i'}{N_I} - p_i'} (a_i^*)^{-\frac{p_i'}{p_i}} (\sigma - v_i(t'))\, d\sigma.$$

Therefore

$$t' - t \le N_i^{-p_i'} \alpha_i^{-p_i'/N_i} \Big(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \Big)^{p_i'/p_i} \cdot$$

$$\cdot \int_{|v_i > t'|}^{|v_i > t|} (|\omega_i'| + \sigma)^{\frac{p_i'}{N_i} - p_i'} (a_i^*)^{-\frac{p_i'}{p_i}} (\sigma - |v_i > t'|)\, d\sigma$$

for all t, t' such that $0 \le t \le t' \le 1$. □

Making $t = 0$ and $t' = 1$ in Theorem 3, we obtain

Corollary 1. *Let $i \in \{1, \dots, n\}$, $f_i(\sigma) = (|\omega_i'| + \sigma)^{\frac{p_i'}{N_i} - p_i'} (a_i^*)^{-\frac{p_i'}{p_i}}(\sigma)$ for $\sigma \in [0, |\Omega_i|]$ and $I_i = \int_0^{|\Omega_i|} f_i(\sigma) d\sigma$. We have*

$$\left(\int_{\Omega_i} G_i(x, \nabla v_i)\, dx \right)^{p_i'/p_i} \ge \frac{N_i^{p_i'} \alpha_i^{\frac{p_i'}{N_i}}}{I_i}.$$

Now we are able to state our main results of this section.

Theorem 4. *Let S be the unique positive solution of $\sum_{i=1}^n \frac{I_i S^{p_i'-1}}{N_i^{p_i'} \alpha_i^{p_i'/N_i}} = 1$. We have $c_p \ge S$. Moreover for any $s \in [0, |\Omega_i|]$,*

$$(c_p)^{-\frac{p_i'}{p_i}} (k_{i-1} - u_{i*}(s)) \le N_i^{-p_i'} \alpha_i^{-\frac{p_i'}{N_i}} \int_0^s f_i(\sigma) d\sigma, \tag{3.9}$$

which gives for $s = |\Omega_i|$

$$(c_p)^{-\frac{p_i'}{p_i}} (k_{i-1} - k_i) \le N_i^{-p_i'} \alpha_i^{-\frac{p_i'}{N_i}} I_i. \tag{3.10}$$

Proof. **(a)** From Remark 1 and Corollary 1, we have

$$1 = \sum_{i=1}^{n} \left(\frac{c_p}{\int_{\Omega_i} G_i(x, \nabla v_i)\, dx} \right)^{\frac{1}{p_i - 1}} \leq \sum_{i=1}^{n} \frac{I_i c_p^{p_i' - 1}}{N_i^{p_i'} \alpha_i^{p_i'/N_i}}.$$

As the last expression is (strictly) increasing in c_p, we get $c_p \geq S$.

(b) From Theorem 2 and Theorem 3, we deduce that for all t such that $k_i \leq t \leq k_{i-1}$

$$k_{i-1} - t \leq N_i^{-p_i'} \alpha_i^{-\frac{p_i'}{N_i}} (c_p)^{\frac{p_i'}{p_i}} \int_0^{|u_i > t|} f_i(\sigma) d\sigma.$$

Making $t = u_{i*}(s)$, s in $[0, |\Omega_i|]$ and noticing that $|u_i > u_{i*}(s)| \leq s$, we obtain (3.9).

4. Symmetrized problem and isoperimetric inequalities.

We begin by recalling the notion of convex symmetrization introduced in the paper of A. Alvino, V. Ferone, P. L. Lions and Trombetti [1].

For $i = 1, \ldots, n$, let $\phi_i : \mathbb{R}^{N_i} \to [0, +\infty[$ be a strictly convex function, differen differentiable off the origin, satisfying (1.4) and (1.5). Let ϕ_i^0 be its dual and $B_{\phi_i^0} = \{\xi \in \mathbb{R}^{N_i}; \phi_i^0(\xi) \leq 1\}$ be the unit ball of $\mathbb{R}^{N_i}$ relative to ϕ_i^0 (i.e. the Frank diagram relative to ϕ_i) with Lebesgue measure α_i. Moreover, we assume that the dual function ϕ_i^0 is strictly convex and differentiable everywhere but in the origin.

Let E be a measurable set in $\mathbb{R}^{N_i}$ and let u be a measurable function from E into $\mathbb{R}$. Let $\widetilde{E}_i$ be the set homothetic to the Frank diagram $B_{\phi_i^0}$ such that $|\widetilde{E}_i| = |E|$. Note that both E and $\widetilde{E}_i$ are subsets of $\mathbb{R}^{N_i}$.

The convex symmetrization (or convex symmetric decreasing rearrangement) relative to ϕ_i^0 of u, denoted by u_i^c is defined on $\widetilde{E}_i$ by

$$u_i^c(x) = u_*(\alpha_i(\phi_i^0(x))^{N_i}); \quad x \in \widetilde{E}_i.$$

The function u and u_i^c are equimeasurable. The level sets of u_i^c, $\{x \in \widetilde{E}_i; u_i^c(x) > t\}$, are homothetic to $B_{\phi_i^0}$ and have the same measure as $\{x \in E; u(x) > t\}$. Indeed, the convex symmetrization coincides with the Schwarz symmetrization (or spherically symmetric increasing rearrangement) when $\phi_i(\xi) = |\xi|$.

Now let $\tilde{\omega}_i$ (resp. $\tilde{\omega}'_i$) be the set of $\mathbb{R}^{N_i}$, homothetic to the ball $B_{\phi_i^0}$ such that $|\tilde{\omega}_i| = |\omega_i|$ (resp. $|\tilde{\omega}'_i| = |\omega'_i|$). The sets $\tilde{\omega}_i$ and $\tilde{\omega}'_i$ are bounded, convex, symmetric with respect to the origin and homothetic. Moreover $\overline{\tilde{\omega}'_i} \subset \tilde{\omega}_i$. Let $A_i = \tilde{\omega}_i \setminus \overline{\tilde{\omega}'_i}$, $\tilde{\gamma}_i = \partial\tilde{\omega}_i$, $\tilde{\gamma}'_i = \partial\tilde{\omega}'_i$. Let μ be the normal to $\tilde{\gamma}_i$ pointing outside A_i or the normal to $\tilde{\gamma}'_i$ pointing inside A_i.

Let $\tilde{a}_i : A_i \to \mathbb{R}$ be the function defined by

$$\tilde{a}_i(x) = a_i^*(\alpha_i(\phi_i^0(x))^{N_i} - |\tilde{\omega}'_i|)$$

where a_i^* is the increasing rearrangement of a_i. As the function a_i, the function $\tilde{a}_i$ also satisfies (1.2) (with A_i instead of Ω_i).

We begin by the explicit resolution of the symmetrized problem corresponding to (2.4) in A_i.

Proposition 1. *For $i \in \{1, \ldots, n\}$, let $\mathcal{B}_i$ and $\frac{\partial}{\partial_\mu \mathcal{B}_i}$ be the operators defined by*

$$\mathcal{B}_i V = -\mathrm{div}[\tilde{a}_i \phi_i(\nabla V)^{p_i-1}\nabla\phi_i(\nabla V)],$$
$$\frac{\partial V}{\partial \mu^{\mathcal{B}_i}} = \tilde{a}_i \phi_i(\nabla V)^{p_i-1}\nabla\phi_i(\nabla V).\,\mu$$

and let V_i be the solution of the following problem

$$\begin{cases} \mathcal{B}_i V_i = 0 & in \quad A_i, \\ V_i = 0 & on \quad \tilde{\gamma}_i, \\ V_i = 1 & on \quad \tilde{\gamma}'_i. \end{cases} \tag{4.1}$$

We have, with f_i and I_i defined in Corollary 1,

(a) $V_i(x) = \dfrac{1}{I_i}\displaystyle\int_{\alpha_i\phi_i^0(x)^{N_i} - |\omega'_i|}^{|\Omega_i|} f_i(\sigma)d\sigma,$

(b) $-\displaystyle\int_{\tilde{\gamma}'_i} \frac{\partial V_i}{\partial\mu^{\mathcal{B}_i}} d\gamma = N_i^{p_i}\alpha_i^{p_i/N_i} I_i^{1-p_i}.$

Proof. **(a)** With $x \in A_i$ and $r = \phi_i^0(x)$, we obtain $\nabla V_i = \frac{dV_i}{dr}\nabla\phi_i^0$. Using the properties of ϕ_i and ϕ_i^0, we get for $x \in A_i$,

$$\phi_i(\nabla V_i(x)) \quad = \quad \phi_i\Big(\frac{dV_i}{dr}\nabla\phi_i^0(x)\Big) = \left|\frac{dV_i}{dr}\right| \quad \text{by (1.4) and (3.3),}$$

$$\nabla\phi_i(\nabla V_i(x)) = \nabla\phi_i\Big(\frac{dV_i}{dr}\nabla\phi_i^0(x)\Big) = \frac{dV_i}{dr}\left|\frac{dV_i}{dr}\right|^{-1}\frac{x}{\phi_i^0(x)} \text{ by (3.1) and (3.4).}$$

Therefore the operator $\mathcal{B}_i$ can be rewritten as

$$\begin{aligned} -\mathcal{B}_i V_i &= a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\left|\frac{dV_i}{dr}\right|^{p_i-2}\frac{dV_i}{dr}\frac{1}{r}\Big(N_i - \frac{1}{r}\nabla\phi_i^0(x).\,x\Big) \\ &+ (p_i-1)a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\left|\frac{dV_i}{dr}\right|^{p_i-2}\frac{d^2V_i}{dr^2}\frac{1}{r}\nabla\phi_i^0(x).\,x \\ &+ \frac{d}{dr}a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\left|\frac{dV_i}{dr}\right|^{p_i-2}\frac{dV_i}{dr}\frac{1}{r}\nabla\phi_i^0(x).\,x. \end{aligned}$$

We see by (3.2) that

$$\nabla\phi_i^0(x).\,x = \phi_i^0(x) = r$$

and finally

$$\begin{aligned} -\mathcal{B}_i V_i &= \left|\frac{dV_i}{dr}\right|^{p_i-2}\Big[\frac{d}{dr}a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\frac{dV_i}{dr} + \\ &+ a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\Big((p_i-1)\frac{d^2V_i}{dr^2} + \frac{N_i-1}{r}\frac{dV_i}{dr}\Big)\Big]. \end{aligned}$$

Therefore $\mathcal{B}_i V_i = 0$ is equivalent to

$$\frac{dV_i}{dr} = kr^{\frac{N_i-1}{1-p_i}}(a_i^*)^{-\frac{p_i'}{p_i}}(\alpha_i r^{N_i} - |\omega_i'|)$$

where k is a constant. Hence, since $V_i = 0$ on $\tilde{\gamma}_i$ and $V_i = 1$ on $\tilde{\gamma}_i'$, we deduce that

$$V_i(x) = \frac{1}{I_i}\int_{\alpha_i\phi_i^0(x)^{N_i} - |\omega_i'|}^{|\Omega_i|} f_i(\sigma)\,d\sigma$$

for all $x \in A_i$.

(b) We have, from earlier computations, for x in $\tilde{\gamma}_i'$,

$$\frac{\partial V_i}{\partial\mu^{\mathcal{B}_i}} = a_i^*(\alpha_i r^{N_i} - |\omega_i'|)\left|\frac{dV_i}{dr}\right|^{p_i-2}\frac{dV_i}{dr}\frac{x.\,\mu}{\phi_i^0(x)} =$$

$$= - N_i^{p_i-1}\alpha_i^{\frac{p_i}{N_i}-1} r^{-N_i}(I_i)^{1-p_i} x.\mu$$

and then,

$$-\int_{\tilde{\gamma}_i'} \frac{\partial V_i}{\partial \mu^{\mathscr{B}_i}} d\gamma = N_i^{p_i-1}\alpha_i^{\frac{p_i}{N_i}-1} r^{-N_i}(I_i)^{1-p_i} \int_{\tilde{\gamma}_i'} x.\mu d\gamma = N_i^{p_i}\alpha_i^{p_i/N_i} I_i^{1-p_i}$$

because $\int_{\tilde{\gamma}_i'} x.\mu\, d\gamma = N_i|\tilde{\omega}_i'| = N_i|\omega_i'|$ and if $x \in \tilde{\gamma}_i'$ we have $\alpha_i\phi_i^0(x)^{N_i} = |\tilde{\omega}_i'| = |\omega_i'|$.

This ends the proof of Proposition 1. □

Remark 3. *Similar computations show that one has also*

$$\int_{\tilde{\gamma}_i} \frac{\partial V_i}{\partial \mu^{\mathscr{B}_i}} d\gamma = \int_{\tilde{\gamma}_i'} \frac{\partial V_i}{\partial \mu^{\mathscr{B}_i}} d\gamma,$$

$$\int_{A_i} \tilde{a}_i\phi_i(\nabla V_i)^{p_i}\, dx = N_i^{p_i}\alpha_i^{p_i/N_i} I_i^{1-p_i} = -\int_{\tilde{\gamma}_i'} \frac{\partial V_i}{\partial \mu^{\mathscr{B}_i}} d\gamma$$

that is Green's formula is valid.

We consider the symmetrized problem defined as follows

$$\inf\left\{ \sum_{i=1}^{n} \frac{1}{p_i} \int_{A_i} Q_i(x, \nabla V_i(x))\, dx, \quad V \in \tilde{\mathbb{H}} \right\} \tag{4.2}$$

where

$$Q_i(x,\xi) = \tilde{a}_i(x)\phi_i(\xi)^{p_i}$$

and

$$\tilde{\mathbb{H}} = \{V \in \mathbb{W}_{\tilde{a}},\ V_1 = 1 \text{ on } \tilde{\gamma}_i',\ V_n = 0 \text{ on } \tilde{\gamma}_n \text{ and } V_{i|\tilde{\gamma}_i} = V_{i+1|\tilde{\gamma}_{i+1}'} = K_i$$
$$\text{(undetermined constant) for} \quad i = 1, \dots, n-1\}.$$

Remark 4. *It follows from Theorem 1 that the symmetrized problem* (4.2) *admits too one solution and only one.*

Let us denote by U the solution of the symmetrized problem (4.2). Let K_i be the common value of U_i on $\tilde{\gamma}_i$ and U_{i+1} on $\tilde{\gamma}'_{i+1}(i = 1, \ldots, n-1)$. Let $\tilde{c}_p = \sum_1^n \int_{A_i} Q_i(x, \nabla U_i(x))\, dx$ be the generalized p-capacity of the collection of $A_i(i = 1, \ldots, n)$. It follows from Theorem 2 applied with $\tilde{c}_p, U_i, K_i, V_i$ instead of c_p, u_i, k_i, v_i and Remark 3 that Green's formula is also valid for U_i, so that $U_1, \ldots, U_n$ satisfy:

$$(4.3)\quad \begin{cases} \mathcal{B}_i U_i = 0 \quad \text{in} \quad A_i, \\ U_i = 1 \quad \text{on} \quad \tilde{\gamma}'_1, \\ U_n = 0 \quad \text{on} \quad \tilde{\gamma}_n, \\ U_{i|\tilde{\gamma}_i} = U_{i+1|\tilde{\gamma}'_{i+1}} = K_i \quad \text{(unprescribed constant)} \\ \qquad\qquad \text{for} \quad (i = 1, \ldots, n-1), \\ \displaystyle\int_{\tilde{\gamma}_i} \frac{\partial U_i}{\partial \mu^{\mathcal{B}_i}}\, d\gamma = \int_{\tilde{\gamma}'_i} \frac{\partial U_i}{\partial \mu^{\mathcal{B}_i}}\, d\gamma \quad \text{is independent of} \quad i = 1, \ldots, n. \end{cases}$$

The symmetrized problem can be solved explicitly:

Theorem 5. (explicit resolution of the symmetrized problem). *Let* U *be the solution of* (4.2), $K_i = U_{i|\tilde{\gamma}_i} = U_{i+1|\tilde{\gamma}'_{i+1}}(i = 1, \ldots, n-1)$. *Let* $\tilde{c}_p = \sum_{i=1}^{n} \int_{A_i} Q_i(x, \nabla U_i(x))\, dx$. *Then the values* $\tilde{c}_p$, K_i *and* U_i *are given respectively by*

(1) $1 = \displaystyle\sum_{i=1}^{n} \frac{I_i(\tilde{c}_p)^{p'_i - 1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}}$

(2) $K_i = 1 - \displaystyle\sum_{j=1}^{i} \frac{I_j(\tilde{c}_p)^{p'_j - 1}}{N_j^{p'_j} \alpha_j^{p'_j/N_j}}$

and for $i \in \{1, \ldots, n\}$, $x \in A_i$

(3) $U_i(x) = K_{i-1} - N_i^{-p'_i} \alpha_i^{-p'_i/N_i} (\tilde{c}_p)^{\frac{p'_i}{p_i}} \displaystyle\int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma)\, d\sigma.$

(As already mentioned there exists a unique $\tilde{c}_p > 0$ solution of **(1)**.)

Proof. Using Theorem 2, Proposition 1 and Remarks 1 and 3, we have

$$1 = \sum_{i=1}^{n} \left(\frac{\tilde{c}_p}{\int_{A_i} \tilde{a}_i \phi_i(\nabla V_i)^{p_i}\, dx} \right)^{\frac{1}{p_i - 1}} = \sum_{i=1}^{n} \frac{I_i(\tilde{c}_p)^{p'_i - 1}}{N_i^{p'_i} \alpha_i^{p'_i/N_i}},$$

$$1 - K_i = \sum_{j=1}^{i} \left(\frac{\tilde{c}_p}{\int_{A_j} \tilde{a}_j \phi_j(\nabla V_j)^{p_j}\, dx} \right)^{\frac{1}{p_j-1}} = \sum_{j=1}^{i} \frac{I_j(\tilde{c}_p)^{p'_j-1}}{N_j^{p'_j} \alpha_j^{p'_j/N_j}},$$

and finally for $x \in A_i (i = 1, \dots, n)$,

$$\begin{aligned} U_i(x) &= (K_{i-1} - K_i)V_i(x) + K_i = K_{i-1} - (K_{i-1} - K_i)(1 - V_i(x)) \\ &= K_{i-1} - \frac{I_i(\tilde{c}_p)^{p'_i-1}}{N_i^{P'_i} \alpha_i^{p'_i/N_i}} \left(1 - \frac{1}{I_i} \int_{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|}^{|\Omega_i|} f_i(\sigma)\, d\sigma \right) \\ &= K_{i-1} - N_i^{-p'_i} \alpha_i^{-p'_i/N_i} (\tilde{c}_p)^{\frac{p'_i}{p_i}} \int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma)\, d\sigma. \end{aligned}$$

□

Remark 5. *For the symmetrized problem, it follows from Theorem 5 that* (3.9) *becomes an equality. Actually for* $x \in \overline{A}_i$, $s = \alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|$ *belongs to* $[0, |\Omega_i|]$ *and* $U_{i_*}(s) = U_i(x)$.

To summarize, the following theorem says that the inequalities in Theorem 4 are all isoperimetric.

Theorem 6. (isoperimetric inequalities).

a) $c_p \geq \tilde{c}_p$,

b) $(c_p)^{-\frac{p'_i}{p_i}}(k_{i-1} - u_{i_*}(\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|)) \leq (\tilde{c}_p)^{-\frac{p'_i}{p_i}}(K_{i-1} - U_i(x))$ *for* $x \in \overline{A}_i$,

c) $(c_p)^{-\frac{p'_i}{p_i}}(k_{i-1} - k_i) \leq (\tilde{c}_p)^{-\frac{p'_i}{p_i}}(K_{i-1} - K_i)$.

Proof. **a)** is already proved (see Theorem 4 and **(1)** in Theorem 5).

b) Let $i \in \{1, \dots, n\}$. For $x \in \overline{A}_i$ and $s = \alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|$, we have by (3.9) of Theorem 4 and **(3)** of Theorem 5,

$$\begin{aligned} (c_p)^{-\frac{p'_i}{p_i}}(k_{i-1} - u_{i_*}(s)) &\leq N_i^{-p'_i} \alpha_i^{-\frac{p'_i}{N_i}} \int_0^{\alpha_i \phi_i^0(x)^{N_i} - |\omega'_i|} f_i(\sigma)\, d\sigma \\ &= (\tilde{c}_p)^{-\frac{p'_i}{p_i}}(K_{i-1} - U_i(x)). \end{aligned}$$

Finally, **(c)** is a particular case of **(b)** with $x \in \tilde{\gamma}_i$. □

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