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ON THE EIGENVALUES OF A KERNEL CONSIDERED BY A.M. OSTROWSKI

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Dedicated to Mario R. Occorsio on his 65th birthday.

By using the inverse iteration method we improve approximation of the eigenvalues of a kernel connected with a problem considered by A.M. Ostrowski.

1. Introduction.

A.M. Ostrowski [10] considered the problem of finding the maximum value M of the functional, $\forall \phi \in L^2(0, 1)$:

$$(1) \qquad I(\phi) := \left[\int_0^1 \phi^2(t)\, dt \right]^{-1} \int_0^1 \int_0^1 \left[\frac{1}{x-y} \int_y^x \phi(t)\, dt \right]^2 dx dy.$$

As a consequence of a theorem by A. Garsia, cited in the same article of A.M. Ostrowski, the preceding problem can be reduced to the computation of the

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greatest eigenvalue μ_1 of the positive compact operator, defined in $L^2(0, 1)$:

$$\mathcal{K}\phi = \int_2^1 K(x, y)\phi(y)\, dy, \tag{2}$$

where

$$K(x, y) = \begin{cases} 2\log\dfrac{x(1-y)}{x-y}, & \text{if } 0 \le y < x \le 1 \\ 2\log\dfrac{y(1-x)}{y-x}, & \text{if } 0 \le x < y \le 1. \end{cases} \tag{3}$$

This problem was solved by G. Fichera - M.A. Sneider in a paper [7] dedicated to Mauro Picone on the occasion of his 90th birthday. By a very accurate computation they have found the inequalities

$$1.202931525711 < M = \mu_1 < 1.202931525733.$$

They used the classical Rayleigh-Ritz method for the lower bounds, and the orthogonal invariants method (a method developed by G. Fichera, see e.g. [5]) for computing the upper bounds.

The computation of lower bounds for M was considerably simplified by A. Ghizzetti in [8] by using a basis of modified Legendre polynomials, instead of powers, in the application of the Rayleigh-Ritz method.

In the above mentioned paper [7] G. Fichera - M.A. Sneider dealing with a more general framework with respect to the problem introduced by A.M. Ostrowski, considered also the problem of approximating the first few eigenvalues of the same operator (2) with kernel (3).

They have found these following results for the first five eigenvalues μ_k ($K = 1, 2, 3, 4, 5$):

$1.202931525711 < \mu_1 < 1.202931525733$
$0.729012156602 < \mu_2 < 0.740766$
$0.52709392618 < \mu_3 < 0.54902$
$0.4135343103 < \mu_4 < 0.44775$
$0.3407196076 < \mu_5 < 0.38842$

Table I

In a recent paper [4], by using a refinement of the orthogonal invariants technique, B. Firmani and M.L. Leuzzi have obtained a sensible improvement for lower and upper bounds of the eigenvalue μ_2 : $0.73041 \leq \mu_2 \leq 0.73130$. Also, they have provided better estimates for the eigenvalues $\mu_3 (\leq 0.54483)$ and $\mu_4 (\leq 0.44141)$.

In the sequel we will apply a method proposed in [1] in order to improve bounds for the eigenvalues of Table I. This can be done by using the inverse iteration method described in [1], which will be summarized in Section 2, anyway we want to point out that our method cannot ensure a monotone convergence of the iterations, while in the method of Fichera a monotone but cumbersome approach to the approximation of eigenvalues is given.

2. The inverse iteration method.

Let $\mathcal{K} : L^2(0, 1) \to L^2(0, 1)$ be a compact operator defined by

$$\mathcal{K}\phi = \int_2^1 K(x, y)\phi(y)\, dy, \tag{4}$$

denote by $\mathcal{I}$ the identity operator, and consider the second kind homogeneous Fredholm integral equation

$$(\mathcal{I} - \lambda\mathcal{K})\phi = 0, \tag{5}$$

where $\lambda \in \mathbf{C}$ is a complex parameter.

By Fredholm theorems, it is well known that eq. (5) admits at most a denumerable set of non vanishing characteristic values which does not accumulate to finite points.

Writing (5) in the form

$$(\mathcal{K} - \mu\mathcal{I})\phi = 0\,, \qquad (\mu = \lambda^{-1}) \tag{6}$$

i.e.

$$\mathcal{K}\phi = \mu\phi, \tag{7}$$

the eigenvalues of the operator $\mathcal{K}$ can be ordered with respect to their modulus in a decreasing sequence:

$$0 < \ldots \leq |\mu_3| \leq |\mu_2| \leq |\mu_1|.$$

In the particular case of symmetric or hermitian positive $K(x, y) = K(y, x)$ or $K(x, y) = \overline{K(y, x)}$, $(\mathcal{K}\phi, \phi) > 0$ if $\phi \neq 0 \in L^2(0, 1)$, in the last formula the modulus symbol can be avoided, since the eigenvalues are real and positive:

$$0 < \ldots \leq \mu_3 \leq \mu_2 \leq \mu_1. \tag{8}$$

We will limit ourselves to this last case, since this is the most important case in physical applications.

Suppose we know an initial approximation $\tilde{\mu}$ of the searched eigenvalue μ_j, such that

$$|\tilde{\mu} - \mu_j| < \frac{1}{2} \min_{\substack{\mu_k \neq \mu_j \\ k=1,2,\ldots,\nu}} |\mu_k - \mu_j|$$

for a suitable choice of the integer ν. In practice in this condition the eigenvalues will be replaced by their Raileigh-Ritz approximations, for sufficiently large ν:

$$|\tilde{\mu} - \mu_j^{(\nu)}| < \frac{1}{2} \min_{\substack{\mu_k^{(\nu)} \neq \mu_j^{(\nu)} \\ k=1,2,\ldots,\nu}} |\mu_k^{(\nu)} - \mu_j^{(\nu)}|\,. \tag{9}$$

From (7) we get:

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\phi = (\mu - \tilde{\mu})\phi. \tag{10}$$

Consequently, if μ_j is an eigenvalue of $\mathcal{K}$ with eigenfunction ϕ_j, then $\mu_j - \tilde{\mu}$ is an eigenvalue of $\mathcal{K} - \tilde{\mu}\mathcal{I}$ with eigenfunction ϕ_j. By writing (10) in the form

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\phi = (\mu - \tilde{\mu})^{-1}\phi \tag{11}$$

it follows that $(\mu_j - \tilde{\mu})^{-1}$ is an eigenvalue of $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$ with the same eigenfunction ϕ_j.

By using condition (9), for ν sufficiently large, the eigenvalue $(\mu_j - \tilde{\mu})^{-1}$ becomes the (unique) eigenvalue of maximum modulus for the operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$. This leads to the possibility to apply the Kellog method (see [9]) in order to approximate $(\mu_j - \tilde{\mu})^{-1}$, and a corresponding eigenfunction. This can be done in the usual way, starting from an arbitrary function ω_0 (which theoretically should not be orthogonal to the eigenspace associated with $(\mu_j - \tilde{\mu})^{-1}$), and defining the sequence

$$\omega_{n+1} := (\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\omega_n, \qquad (n = 0, 1, 2, \ldots).$$

Then (see [9]):

$$\lim_{n\to\infty} \frac{\|\omega_{n+1}\|_2}{\|\omega_n\|_2} = (\mu_j - \tilde{\mu})^{-1}, \tag{12}$$

$$\lim_{n\to\infty} \frac{\omega_{2n}}{\|\omega_{2n}\|_2} = \pm\phi_j. \tag{13}$$

After computing with prescribed accuracy (see Section 3) the eigenvalue

$$\xi_j := (\mu_j - \tilde{\mu})^{-1},$$

one finds

$$\mu_j = \frac{1}{\xi_j} + \tilde{\mu},$$

so that, by recalling $\mu = \lambda^{-1}$ ($\tilde{\mu} =: \tilde{\lambda}^{-1}$), we obtain for the characteristic values of the kernel the expressions

$$\lambda_j = \frac{\tilde{\lambda}\xi_j}{\tilde{\lambda} + \xi_j}.$$

It is important to note that (as in the finite dimensional case) we can avoid the determination of the inverse operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$, since the equation

$$\omega_{n+1} = (\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\omega_n$$

is equivalent to

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\omega_{n+1} = \omega_n. \tag{14}$$

However, this leads to the necessity to solve numerically, at each step, a Fredholm integral equation of the first kind.

This can be done by using different methods (see [2]–[3]), namely we could use, e.g., the Fast Galerkin method, or the Nyström method. The latter method was used, since it turned out to be very simple and efficient both with respect to time and number of iterations. In applying the Nyström method we have used the modified Gauss-Legendre quadrature formula with 60 nodes.

3. Error estimate and prescribed accuracy.

The rate of convergence of the method is given by the formula:

$$\frac{\|\omega_n\|_2}{\|\omega_0\|_2} = \mathcal{O}[(\mu'(\mu_j - \tilde{\mu}))^n],$$

where $\mu' \neq (\mu_j - \tilde{\mu})^{-1}$ denotes a suitable eigenvalue of $(\mathcal{K} - \tilde{\mu}\,\mathcal{I})^{-1}$ (see [12]).

As a matter of fact, by the numerical point of view, the use of Nyström method in the solution of equation (14) is substantially equivalent to the substitution of the original kernel $K(x, y)$ by an approximating kernel $\tilde{K}(x, y)$ given by a suitably defined two-dimensional step function (i.e. instead of the original operator, we consider an approximating finite dimensional operator given by a suitable matrix). In order to define this finite dimensional operator, and to discuss the accuracy of our approximation we introduce some notations.

Let n be the number of nodes in the application of the Nyström method, and denote by $x_1, x_2, \ldots, x_n$ (or $y_1, Y_2, \ldots, y_n$) the knots of the modified Gauss-Legendre quadrature formula on the x (or y) axis, and by $w_1, w_2, \ldots, w_n$ the corresponding Christoffel constants.

Divide the square $Q := [0, 1]x[0, 1]$ into the sub-squares $Q_{i,j}$ defined by

$$Q_{i,j} := \Big\{(x, y) | \sum_{l=1}^{i-1} w_l < x_i < \sum_{l=1}^{i} w_l; \sum_{k=1}^{j-1} w_k < y_j < \sum_{k=1}^{j} w_k\Big\},$$

assuming $\sum_{s=1}^{i-1} w_s := 0$, if $i = 1$, and recalling that obviously $\sum_{s=1}^{n} w_s = 1$.

Then define

$$\tilde{K}(x, y) = \begin{cases} K(x_i, y_j), & \text{if } i \neq j, \text{ and } (x, y) \in Q_{i,j} \\ M_i, & \text{if } i = j, \end{cases} \tag{15}$$

where M_i are such constants that

$$\|K(x, y) - \tilde{K}(x, y)\|_{L^2(\cup_{i \neq j} Q_{i,j})} < eps,$$

where *eps* denotes the smallest positive number used by the computer (i.e. the *machine epsilon*). This condition can always be satisfied provided that n is sufficiently large.

Then the numerical computation by using the inverse power method yields to approximate the exact eigenvalues $\tilde{\mu}_j, (j = 1, 2, \ldots, n)$ of the kernel $\tilde{K}(x, y)$. Anyway, by using the well known Aronszajn Theorem (see e.g. [6]),

it is possible to find an upper bound for the absolute error $|\mu_j - \tilde{\mu}_j|$, which is given simply, for every j, by the estimate

$$|\mu_j - \tilde{\mu}_j| \leq \|K(x, y) - \tilde{K}(x, y)\|_{L^2(Q)}.$$

Then, in order to find an approximation $\tilde{\mu}_j$ which is exact, with respect to the corresponding μ_j, up to the p-th digit, it is sufficient to increase n (and eventually to use adaptive composite quadrature formulas, increasing the number of knots close to the singularities) in such a way that further inequality $\|K(x, y) - \tilde{K}(x, y)\|_{L^2(Q)} < .5x10^{-p}$ holds true.

This can always be done, and permits to control the error of our approximation, independently by the use of the orthogonal invariants method.

4. Approximation of the first five eigenvalues of the kernel considered by A.M. Ostrowski.

The numerical results for the first five eigenvalue of the kernel (3) considered by A.M. Ostrowski have been obtained by using a Turbo C^{++} program written by S. Delle Monache.

The computer algebra system MATHEMATICA© has been used in order to compute the above mentioned a priori estimate of the absolute error.

Starting from the secular equation considered by A. Ghizzetti in Section 3 of paper [8], we have found (for $n = 4$) the fifth order equation:

$$\mu^5 - 2.7\overline{6}\mu^4 + 2.689\overline{629}\mu^3 - 1.1314120370374\mu^2 +$$

$$+ 0.20130529835381\mu - 0.01241169410151 = 0$$

whose roots are given by:

$$\tilde{\mu}_1 = 1.20292106807967$$

$$\tilde{\mu}_2 = 0.71630799473620$$

$$\tilde{\mu}_3 = 0.50460544336472$$

$$\tilde{\mu}_4 = 0.20035867193046$$

$$\tilde{\mu}_5 = 0.14247348855561.$$

By considering $n = 19$, and the corresponding equation of order 20, we have found the first Rayleigh-Ritz approximation values:

$$\tilde{\mu}_1 = 1.20293152571347$$

$$\tilde{\mu}_2 = 0.72901215660725$$

$$\tilde{\mu}_3 = 0.52709392620653$$

$$\tilde{\mu}_4 = 0.41353431036983$$

$$\tilde{\mu}_5 = 0.34071960754549,$$

which improve the Rayleigh-Ritz (lower) approximations of Table I.

These roots have been considered as initial approximations $\tilde{\mu}$ for the inverse iteration method.

Our results are contained in the following Table II.

$1.20293152571347 < \mu_1 < 1.209$
$0.72901215660725 < \mu_2 < 0.739$
$0.52709392620653 < \mu_3 < 0.530$
$0.41353431036983 < \mu_4 < 0.414$
$0.34071960754549 < \mu_5 < 0.345$

Table II

Since we have used a simple personal computer, we have chosen $p = 2$, and consequently we can find only the first two correct decimal figures of the eigenvalues $\mu_k (k = 1, 2, 3, 4, 5)$ of the given kernel. We want to point out that the very close approximation for the first eigenvalue which appears in Table I was obtained by G. Fichera and M.A. Sneider by using an IBM mainframe, and setting a 100 digits multiple precision during computations. Anyway by using the orthogonal invariants method they was able to find only the first correct decimal figure for the subsequent four eigenvalues of the operator $\mathcal{K}$.

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