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POISSON KERNEL FOR A PARABOLIC PROBLEM

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In this paper we obtain a representation for the solution of a parabolic mixed problem.

Introduction.

In this work we obtain a representation for the Poisson kernel $\Phi(\tau, z, z'; \rho)$ of the following mixed problem, with real parameter ρ:

$$\text{(I)} \qquad \begin{cases} \partial_\tau u = (\partial_z^2 - z^2)u & (\tau, z) \in R_\tau^+ \times R_z^+ \\ (\partial_z - 2\rho)u(\tau, 0) = 0 & \tau \in R_\tau^+ \\ u(0, z) = g(z) & z \in R_z^+. \end{cases}$$

with $g(z) \in L^2(R_z^+)$.
We construct $\Phi(\tau, z, z'; \rho)$ using the parabolic cilinder functions $D_\upsilon(z)$, $z \in R$. The problem (I) arises when we apply the process used in [2] to an oblique derivative problem for the operator $L = \partial_t - \partial_y^2 - y^2\partial_x^2$ where $(t, y, x) \in]0, +\infty[\times]0, +\infty[\times R$.
The main result can be stated as follows:

Theorem 1. *For every $\rho \in R$ there is a sequence of real numbers $\{\nu_k(\rho)\}_{k\in N_0}$, positively diverging such that $\{D_{\nu_k(\rho)}(z)\}_{k\in N_0}$ is orthogonal system in $L^2(R_z^+)$.*

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Moreover, put

$$\text{(II)}\qquad \varphi_k(z;\rho) = \left[D_\nu(\sqrt{2z}) / \| D_\nu(\sqrt{2z}) \|_{L^2(R_z^+)} \right]_{\nu=\nu_k(\rho)} \qquad z \geq 0,\ k \in N_0$$

$$\text{(III)}\qquad \Phi(\tau, z, z'; \rho) = \sum_{k=0}^{+\infty} e^{-(2\nu_k(\rho)+1)\tau} \varphi_k(z;\rho)\varphi_k(z';\rho)$$

$$\tau > 0,\ z, z' \in [0, +\infty[,$$

it results $\Phi(\tau, z, z'; \rho) \in C^\infty(R_\tau^+ \times \overline{R}_z^+ \times \overline{R}_{z'}^+) \frown C^0(\overline{R}_\tau^+ \times \overline{R}_z^+, D'(R_{z'}^+))$*, and*

$$\text{(IV)}\qquad \begin{cases} (\partial_\tau - \partial_z^2 + z^2)\Phi(\tau, z, z'; \rho) = 0 & (\tau, z, z') \in R_\tau^+ \times R_z^+ \times R_{z'}^+ \\ (\partial_z - 2\rho)\Phi(\tau, 0, z'; \rho) = 0 & (\tau, z') \in R_\tau^+ \times R_{z'}^+ \\ \lim\limits_{\tau \to 0} \Phi(\tau, z, z'; \rho) = \delta(z' - z) & (\tau, z) \in R_\tau^+ \times R_z^+. \end{cases}$$

The elements of the sequence $\{\nu_k(\rho)\}_{k \in N_0}$ are zeros for the function $\rho + \frac{\Gamma((1-\nu)/2)}{\Gamma(-\nu/2)}$, where Γ is the Eulero function; it follows that $\nu_k(0)$, $k \in N_0$, are even natural numbers and $\varphi_k(z; 0)$, $k \in N_0$, are the Hermite functions $\varphi_{2k}(z)$. Then, the result in [2] is obtained for $\rho \to \infty$.

This paper is organized as follows: in section 1 we study the sequence $\{\nu_k(\rho)\}_{k \in N_0}$; in section 2 we establish estimates for the functions $D_\nu(z)$, using well-known integral and asymptotic representations (see. [6], [3]); in section 3 we construct and study the resolvent set of the operator $A = \partial_z^2 - z^2$, with domain $D(A, \rho) \subseteq H^2(R_z^+)$ formed by the functions $\omega(z)$ such that $\omega'(0) = 2\rho\omega(0)$. In section 4 we prove that A is the generator of a holomorphic semigroup, so we obtain, among other things, the completeness of the system $\{\varphi_k(z;\rho)\}_{k \in N_0}$ in $L^2(R_z^+)$. Finally in Section 5 we introduce the Poisson kernel $\Phi(\tau, z, z'; \rho)$ and we complete the proof of Theorem 1.

1. Auxiliary functions.

If $\Gamma(\nu)$, $\nu \in C$, is the Eulero function, we consider the following analytic function

$$\text{(1.1)}\qquad \alpha(\nu) = \frac{\Gamma((1-\nu)/2)}{\Gamma(-\nu/2)} \qquad \nu \in C.$$

For this function the values $\nu = 2k$, $k \in N_0$, are zeros of order 1, while the values $\nu = 2k + 1$, $k \in N_0$, are poles of order 1. Moreover, if $\nu \in R$ we have:

$$\nu < 0 \Rightarrow \alpha(\nu) > 0, \tag{1.2}$$

$$\nu \in]2k, 2k + 1[, \ k \in N_0 \Rightarrow \alpha(\nu) < 0, \tag{1.3}$$

$$\nu \in]2k - 1, 2k[, \ k \in N \Rightarrow \alpha(\nu) > 0. \tag{1.4}$$

Proposition 1.1. *If $\nu \in R$ and $\nu \neq 2k + 1$, $k \in N_0$, then results:*

$$\alpha'(\nu) < 0. \tag{1.5}$$

Proof. Since the values $\nu = 2k$, $k \in N_0$, are simple zeros for $\alpha(\nu)$, we have $\alpha'(2k) \neq 0$, $\forall k \in N_0$. Then it is sufficient to prove (1.5) for $\nu \notin N_0$. That being stated we have

$$\alpha'(\nu) = -\frac{1}{2}\frac{\Gamma((1-\nu)/2)}{\Gamma(-\nu/2)}\left[\frac{\Gamma'((1-\nu)/2)}{\Gamma((1-\nu)/2)} - \frac{\Gamma'(-\nu/2)}{\Gamma(-\nu/2)}\right] = \tag{1.6}$$

$$= -\frac{1}{2}\,\alpha(\nu)\beta(\nu).$$

So we must prove that $\forall \nu \in R - N_0$, $\beta(\nu)$ is not zero and it has the sign of $\alpha(\nu)$. From the well known representation of the logarithmic derivatives of $\Gamma(\nu)$ (see [6]) we obtain:

$$\beta(\nu) = 2\sum_{n=0}^{+\infty} \frac{1}{(\nu - 2n)(\nu - (2n + 1))} \tag{1.7}$$

Now we observe that if $\nu \in]2k - 1, 2k[$, $k \in N$, or $\nu < 0$, the terms of series in (1.7) are all positive; it follows that $\beta(\nu) > 0$ and so for (1.2), (1.4) and (1.6) we have (1.5) for these values of ν.

If $\nu \in]2k, 2k+1[$, $k \in N_0$, in (1.7) the only negative term has index k, moreover we have

$$\frac{1}{(\nu - 2k)(\nu - (2k + 1))} \leq -4$$

$$\frac{1}{(\nu-2n)(\nu-(2n+1))} < \frac{1}{4(n-k-\frac{1}{2})^2} \qquad k < n,\ n \in N$$

$$\frac{1}{(\nu-2n)(\nu-(2n+1))} < \frac{1}{4(k-n-\frac{1}{2})^2} \qquad n < k,\ n \in N_0,$$

and then

$$\beta(\nu) \le 2\Big[-4+2\sum_{h=1}^{+\infty}\frac{1}{h^2}\Big] = 2\Big[-4+\frac{\pi^2}{6}\Big] < 0; \tag{1.8}$$

from (1.3) the thesis follows. □

Let $\rho \in R$, we consider the equation

$$\alpha(\nu)+\rho = 0 \qquad \nu \in R. \tag{1.9}$$

From (1.2), (1.3), (1.4) and from Proposition 1.1, as well as from Dini Theorem, we have the following

Proposition 1.2. *The solutions of the equation* (1.9) *form a sequence* $\{\nu_k(\rho)\}_{k\in N_0}$ *positively diverging such that* $\nu_k(\rho) \in C^\infty(R)$ $\forall k \in N_0$, *and* $\{\nu_k(0)\}_{k\in N_0} = \{2k\}_{k\in N_0}$*; moreover*

$$\rho > 0 \Rightarrow \nu_k(\rho) \in]2k, 2k+1[\quad \forall k \in N_0,$$

$$\rho < 0 \Rightarrow \nu_0(\rho) < 0, \quad \nu_k(\rho) \in]2k-1, 2k[\quad \forall k \in N,$$

$$\lim_{\rho\to+\infty} \nu_k(\rho) = 2k+1 \quad \forall k \in N_0\ , \qquad \lim_{\rho\to-\infty} \nu_k(\rho) = 2k-1 \quad \forall k \in N,$$

$$\lim_{\rho\to-\infty} \nu_0(\rho) = -\infty, \qquad -\nu_0(\rho) < 6\rho^2 \ \ if \ \ \rho < 0.$$

We observe that we obtain the estimate of $-\nu_0(\rho)$ for $\rho < 0$ by (1.9) and (1.1) using the asyimptotic expansion (see [4]):

$$\Gamma(x) = e^{-x}\, x^{x-1/2}\, (2\pi)^{1/2}\, e^{\theta/12x} \qquad 0 < \theta < 1 \ \ x \in R^+.$$

Afterwards it will be usefull

Proposition 1.3. *For every* $\rho \in R$ $\exists\delta > 0$ *depending of* ρ *such that, if* $Re\,\nu = 2n+1$, $n \in N_0$ *and* $|Im\,\nu| < \delta$ *results:*

$$\left|\frac{\Gamma(-\nu)}{\alpha(\nu)+\rho}\right| \le \frac{c}{(2n+1)!} \tag{1.10}$$

where c *is an absolute constant.*

Proof. Put $\nu = 2n + 1 + iy$, using recurrence formulae of Γ, we have

$$|\Gamma(-\nu)| \le \frac{|\Gamma(1-iy)|}{(2n+1)!|y|} \le \frac{1}{(2n+1)!|y|}, \tag{1.11}$$

and for $|y| \le 1$

$$|\alpha(\nu)| \ge \left|\frac{\Gamma(1-iy/2)}{\Gamma((1-iy)/2)}\right|\left|\frac{1+iy}{y}\right| > \frac{|\Gamma(1-iy/2)|}{\sqrt{\pi}|y|} \ge \frac{2}{c|y|}, \tag{1.12}$$

where $\dfrac{2}{c} = \min\limits_{|y|\le 1} \dfrac{|\Gamma(1-iy/2)|}{\sqrt{\pi}}$.

Putting $\delta = \min\{1, 1/c|\rho|\}$ for $\rho \ne 0$ and $\delta = 1$ for $\rho = 0$, the thesis follows from (1.11) and (1.12). $\square$

2. Estimates for the functions $D_\nu(z)$.

We remember that the parabolic cilinder function $D_\nu(z)$, $\nu \in C$, is solution of (see [6], [4], [3]):

$$\frac{d^2}{dz^2} D_\nu(z) - \frac{1}{4} z^2 D_\nu(z) = -\left(\nu + \frac{1}{2}\right) D_\nu(z) \quad z \in C \tag{2.1}$$

moreover

$$D_\nu(0) = \frac{\sqrt{\pi}\, 2^{\nu/2}}{\Gamma((1-\nu)/2)} \qquad D'_\nu(0) = -\frac{\sqrt{2\pi}\, 2^{\nu/2}}{\Gamma(-\nu/2)}. \tag{2.2}$$

The functions $D_\nu(z)$ are susceptible of many formulas of representation, we will use the following:

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0 \qquad \forall\ \nu,\ z \in C, \tag{2.3}$$

$$D_\nu(z) = \frac{2^{-1-\nu/2}}{\Gamma(-\nu)} e^{-z^2/4} \sum_{k=0}^{+\infty} \frac{(-\sqrt{2}\, z)^k}{k!} \Gamma((k-\nu)/2) \quad \forall\ \nu, z \in C. \tag{2.4}$$

By (1.7) (see [3]) we have:

$$\int_0^{+\infty} |D_\nu(z)|^2\, dz = \sqrt{\pi}\, 2^{-3/2} \beta(\nu)/\Gamma(-\nu) \quad \nu \in C,\ \nu \notin N_0, \tag{2.5}$$

(2.6) $$\int_0^{+\infty} |D_n(z)|^2\, dz = \sqrt{2\pi}\; n! \quad n \in N_0.$$

We recall that for $n \in N_0$ it results

(2.7) $$D_n(z) = e^{-z^2/4} H_n(z),$$

where $H_n(z)$ is the Hermite polynomial of degree n.
Now we prove the following

Proposition 2.1. *If $\nu \in R - N_0$, it results*

(2.8) $$\sup_{z \in \overline{R}_z^+} |D_\nu(z)| = O(|\nu|^{1/4}) \|D_\nu\|,$$

where $\|\cdot\|$ denotes the usual norm in $L^2(R_z^+)$.
Proof. Multiplying (2.1) by $\overline{D}_\nu(z)$ and integrating on R_z^+, we obtain

$$D_\nu(0)D_\nu(0) + \|D_\nu(z)\|^2 + \frac{1}{4}\|zD_\nu(z)\|^2 = (\nu + \frac{1}{2})\|D_\nu(z)\|^2;$$

then from (2.2) and by duplication formula of Legendre (see [4]) we get

$$\|D'_\nu(z)\|^2 + \frac{1}{4}\|zD_\nu(z)\|^2 = (\nu + \frac{1}{2})\|D_\nu(z)\|^2 + \sqrt{\frac{\pi}{2}}(\Gamma(-\nu))^{-1}.$$

From (2.5), for $\nu \in R - N_0$, we deduce

$$\|D'_\nu\|^2 \le (|\nu| + 2^{-1} + 2|\beta(\nu)|^{-1})\|D_\nu\|^2,$$

from which

(2.9) $$\|D'_\nu\|^2 = O(|\nu|)\|D_\nu\|^2.$$

because the function $|\beta(\nu)|^{-1}$ is bounded (see Prop. 1.1).
Being $D_\nu(z) \in \mathcal{S}(\overline{R}_z^+)$, it results

$$|D_\nu(z)|^2 \le 2\|D_\nu\|\;\|D'_\nu\| \qquad \forall z \in \overline{R}_z^+$$

so, by (2.9), we have the thesis. □

Remark. If $n \in N_0$, from (2.7) and from well known properties of the Hermite function, follows

(2.10) $$\sup_{z \in \overline{R}_z^+} |D_n(z)| = O(1)\sqrt{n!} = O(1)\|D_n\|.$$

Now we need some estimates for functions $D_\nu(z)$ when $\nu \in C$ and $Re\,\nu = 2n-1$, $n \in N$.

Proposition 2.2. - *There is a real and continuous function $C(z)$ such that if $Re\,\nu = n \in N$ and $|Im\,\nu| \le 1$:*

$$|D_\nu(z)| \le C(z)(1+|z|)^n 2^n \Gamma(n/2) \qquad \forall\, z \in C. \tag{2.11}$$

Proof. First of all we prove (2.11) when $n = 1$. Using (2.4) we have $\forall z \in C$:

$$|D_{-1+iy}(z)| \le \frac{e^{-Rez^2/4}}{|\Gamma(1-iy)|} \sum_{k=0}^{+\infty} \frac{|\sqrt{2}z|^k}{k!}\, \Gamma((k+1)/2) \tag{2.12}$$

$$|D_{iy}(z)| \le \frac{e^{-Rez^2/4}}{|\Gamma(-iy)|} \left(|\Gamma(-iy/2)| + \sum_{k=1}^{+\infty} \frac{|\sqrt{2}z|^k}{k!}\, \Gamma(k/2) \right). \tag{2.13}$$

By Legendre duplication formula it results

$$|D_{iy}(z)| \le e^{-Rez^2/4} \left(\frac{2\sqrt{\pi}}{|\Gamma((1-iy)/2)|} + \right. \tag{2.14}$$

$$\left. + \frac{1}{|\Gamma(-iy)|} \sum_{k=1}^{+\infty} \frac{|\sqrt{2}z|^k}{k!}\, \Gamma(k/2) \right).$$

Series in (2.12), (2.13) and (2.14) have radius of convergence infinite; so put

$$M = \max_{|y|\le 1} \left\{ \frac{1}{|\Gamma(1-iy)|},\ \frac{2\sqrt{\pi}}{\Gamma((1-iy)/2)|},\ \frac{1}{|\Gamma(-iy)|} \right\}$$

and

$$C_0(z) = M\, e^{-Re\,z^2/4} \left(1 + \Gamma(1/2) + \sum_{k=1}^{+\infty} \frac{|\sqrt{2}z|^k}{k!}\, (\Gamma(k/2) + \Gamma((k+1)/2)) \right)$$

we have

$$|D_{-1+iy}(z)| \le C_0(z)\ , \quad |D_{iy}(z)| \le C_0(z) \qquad \forall\, y \in [-1, 1]. \tag{2.15}$$

Utilizing (2.3) with $\nu = 1 + iy$, we find

$$|D_{1+iy}(z)| \le |z|\,|D_{iy}(z)| + |iy|\,|D_{-1+iy}(z)|$$

and so, by (2.15), we have the assert in the case $n = 1$, with $C(z) = C_0(z)$.
If $n = 2$, reasoning as above with $\nu = 2 + iy$, $|y| \le 1$, and using (2.11) for $n = 1$, we obtain

$$|D_{2+iy}(z)| \le 2C_0(z)(1 + |z|)^2\Gamma(1/2). \tag{2.16}$$

From (2.16) it is clear that the proposition is true if $n = 2$, with $C(z) = 2C_0(z)$.
By (2.16) and (2.11) with $n = 1$, by similar arguments one can prove that the proposition is true even if $n = 3$, with $C(z) = 2C_0(z)$.
Now we suppose that (2.11) holds untill $n = Re\,\nu > 3$. Then from (2.3) it follows

$$\begin{aligned}|D_{\nu+1}(z)| \le C(z)\Big\{&|z|(1 + |z|)^n 2^n\Gamma(n/2) +\\ &+ |n + iy|(1 + |z|)^{n-1}2^{n-1}\Gamma((n-1)/2)\Big\} \qquad \forall z \in C.\end{aligned} \tag{2.17}$$

Since

$$\Gamma((n-1)/2) = 2\Gamma((n+1)/2)/(n-1)\,, \quad \Gamma(n/2) < \Gamma((n+1)/2) \quad n \ge 3$$

we have for $|y| \le 1$

$$\begin{aligned}|D_{\nu+1}(z)| \le C(z)\Big\{&|z|(1 + |z|)^n 2^n\Gamma((n+1)/2) +\\ &+ (1 + |z|)^{n-1}2^n\Gamma((n+1)/2)(n+1)/(n-1)\Big\}\end{aligned}$$

so the thesis follows. □

We observe that if $Re\,\nu = 2n + 1$, $n \in N$ and $|Im\,\nu| \le 1$, then (2.11) entails

$$|D_\nu(z)| \le C(z)(1 + |z|)^{2n}2^{2n}n! \qquad \forall\ z \in C. \tag{2.18}$$

Now we conclude with the following

Proposition 2.3. - *If $Re\,\nu = 2n+1$, $n \in N$ and $|Im\,\nu| \leq 1$, there is a positive constant c independent of n such that*

$$\left|\Gamma(-\nu)e^{z^2/4}\big[D_\nu(z)+D_\nu(-z)\big]\right| \leq c\frac{2^{-n}}{n!}\left((1+|z|^2)^n e^{|z|^2}\right) \quad \forall z\in C. \tag{2.19}$$

Proof. Put $\nu = 2n+1+iy$, by (2.4) we deduce

$$\left|\Gamma(-\nu)e^{z^2/4}\big[D_\nu(z)+D_\nu(-z)\big]\right| \leq \tag{2.20}$$

$$\leq 2^{-n}\sum_{h=0}^{+\infty}\frac{\left|\sqrt{2}\,z\right|^{2h}}{(2h)!}|\Gamma(h-n-(1+iy)/2)|.$$

Now, for $h \leq n$, by recurrence formulae of Γ, being $(2h)! \geq 2^h(h!)^2$ we have

$$\left|\frac{\Gamma(-(n-h)-(1+iy)/2)}{(2h)!}\right| \leq \frac{c}{(n-h)!(2h)!} \leq \frac{c}{n!}\,2^{-h}\binom{n}{h}$$

where $c = \max\limits_{|y|\leq 1}|2\Gamma((1-iy)/2)/(1+iy)|$; whereas, for $h \geq n+1$,

$$\left|\frac{\Gamma((h-n)-(1+iy)/2)}{(2h)!}\right| \leq \sqrt{\pi}\,\frac{\Gamma(h-n+1)}{(2h)!} = \sqrt{\pi}\,\frac{(h-n)!}{(2h)!} \leq \frac{\sqrt{\pi}}{2^h n! h!}$$

Because of these inequalities we can increase the sum of series in (2.20) as follows:

$$\frac{c}{n!}\left[\sum_{h=0}^{n}\binom{n}{h}|z|^{2h} + \sum_{h=n+1}^{+\infty}\frac{|z|^{2h}}{h!}\right] \leq \frac{c}{n!}\left[(1+|z|^2)^n + e^{|z|^2}\right].$$

This completes the proof. □

3. A differential problem with parameter.

Let $f \in \mathcal{S}(\overline{R_z^+})$, $\lambda \in C$, $\rho \in R$, and we consider the following problem:

$$\begin{cases}\lambda\,\omega - \omega'' + z^2\omega = f\\ \omega'(0) = 2\rho\,\omega(0)\end{cases} \tag{3.1}$$

Our goal is to construct a solution belonging to $\mathcal{S}(\overline{R}^+_z)$.
Putting $\omega(z) = w(\sqrt{2}\, z)$ in (3.1), by sostitutions $z \to \sqrt{2}\, z$ and $\nu + 1/2 = -\lambda/2$, we have

$$\begin{cases} w'' + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) w = -\frac{1}{2} f(z/\sqrt{2}) \\ w'(0) = \sqrt{2}\,\rho\, w(0) \quad \rho \in R. \end{cases} \tag{3.2}$$

The homogeneous equation associated to (3.2) coincides with (2.1), so, for $\nu \notin N_0$, it admits two independent solutions $D_\nu(z)$ and $D_\nu(-z)$, and their Wronskiano is

$$W(\nu) = -2 D_\nu(0) D'_\nu(0);$$

consequently, for (2.2) and Legendre duplication formula, we have:

$$W(\nu) = \frac{\sqrt{2\pi}}{\Gamma(-\nu)}. \tag{3.3}$$

That being stated, the general solution of equation in (3.2), always for $\nu \notin N_0$, is

$$w(z) = c_1(\nu) D_\nu(z) + c_2(\nu) D_\nu(-z) + \tag{3.4}$$

$$- \frac{1}{2W(\nu)} \int_0^z \left[D_\nu(s) D_\nu(-z) - D_\nu(-s) D_\nu(z)\right] f(s/\sqrt{2})\, ds.$$

Computing $w(0)$ and $w'(0)$, and imposing that the function (3.4) is solution of (3.2), we obtain

$$c_1(\nu)\left(\frac{D'_\nu(0)}{D_\nu(0)} - \sqrt{2}\,\rho\right) = c_2(\nu)\left(\frac{D'_\nu(0)}{D_\nu(0)} + \sqrt{2}\,\rho\right). \tag{3.5}$$

From (2.2), by (1.1), we have

$$\frac{D'_\nu(0)}{D_\nu(0)} = -\sqrt{2}\,\alpha(\nu)$$

and so

$$c_1(\nu) = \frac{\alpha(\nu) - \rho}{\alpha(\nu) + \rho}\, c_2(\nu). \tag{3.6}$$

On the other hand, imposing that function w is rapidly decreasing in $\overline{R}_z^+$, we have

$$c_2(\nu) = \frac{1}{2W(\nu)} \int_0^{+\infty} D_\nu(s) f(s/\sqrt{2})\, ds. \tag{3.7}$$

so, if $\alpha(\nu) + \rho \neq 0$ and $w(z) \in \mathcal{S}(\overline{R}_z^+)$ is solution of (3.2), it results

$$w(z) = w_\nu(z; \rho, f) = \frac{\Gamma(-\nu)}{2\sqrt{2\pi}} \Bigg\{ \int_0^{+\infty} D_\nu(s) \left[\frac{\alpha(\nu) - \rho}{\alpha(\nu) + \rho} D_\nu(z) + D_\nu(-z) \right] \tag{3.8}$$

$$f(s/\sqrt{2})\, ds - \int_0^z \Big[D_\nu(s) D_\nu(-z) - D_\nu(-s) D_\nu(z) \Big] f(s/\sqrt{2})\, ds \Bigg\}.$$

The function $w_\nu(z; \rho, f)$, defined in (3.8), is an analytic function of ν for every $z \in \overline{R}_z^+$ and $\rho \in R$. Its singular points are the only zeros of $\alpha(\nu) + \rho$ (see section 1). The integer and not negative values of ν seem zeros when $\rho \neq 0$. Since these zeros are eigenvalues for the adjoint problem (3.2), such zeros are all real and so they form the sequence $\{\nu_k(\rho)\}_{k \in N_0}$ (see section 1).
The following proposition holds

Proposition 3.1. *Let $X \subset R_z^+$ a compact set and $f \in C_0^\infty(\overline{R}_z^+)$. Then there is a positive number μ, depending of ρ, X and supp f, and there is a positive number δ, such that if $Re\,\nu = 2n + 1$ and $|\,Im\,\nu| < \delta$, results*

$$|\, w_\nu(z; \rho, f)| \leq \mu^n \max_{\overline{R}_z^+} |f| \qquad \forall\, z \in X. \tag{3.9}$$

Proof. We rewrite (3.8) as follows

$$w_\nu(z; \rho, f) = \frac{1}{2\sqrt{2\pi}} \Bigg\{ - D_\nu(-z) \int_0^z \Gamma(-\nu) \big[D_\nu(s) + D_\nu(-s) \big] f(s/\sqrt{2})\, ds +$$

$$+ \Gamma(-\nu) \big[D_\nu(z) + D_\nu(-z) \big] \int_0^z D_\nu(-s) f(s/\sqrt{2})\, ds - \frac{2\rho}{\alpha(\nu) + \rho} \Gamma(-\nu) D_\nu(z)$$

$$\left.\int_0^{+\infty} D_\nu(s) f(s/\sqrt{2}) ds + \Gamma(-\nu)\big[D_\nu(z) + D_\nu(-z)\big] \int_0^{+\infty} D_\nu(s) f(s/\sqrt{2}) ds \right\} =$$

$$= \frac{1}{2\sqrt{2\pi}} \big\{A_1(z) + A_2(z) + A_3(z) + A_4(z)\big\}.$$

Put $X(f) = supp\ f$ and $\alpha = 2\Big(1 + \max_{X \cup X(f)} |z|\Big)$, from Propositions 2.2 and 2.3 we draw that there is $\delta > 0$ such that if $Re\,\nu = 2n + 1$ and $|Im\,\nu| < \delta$

$$|A_i(z)| \leq c\,\alpha^{4n} \max |f| \qquad \forall i \in \big\{1, 2, 4\big\}$$

uniformly with respect to $Im\,\nu \in] - \delta, \delta[$ and to $z \in X$, with c constant independent of n.
Likewise, from Propositions 1.3 and 2.2 we draw

$$|A_3(z)| \leq c|\rho|\alpha^{4n} \max |f|,$$

hence the thesis. □

Now we return to problem (3.1). From arguments above follows that eigenvalues of this problem are:

(3.10) $$\lambda_k(\rho) = -(2\nu_k(\rho) + 1) \qquad k \in N_0$$

and respective normalized eigensolutions belonging to $\mathcal{S}(\overline{R}_z^+)$ are

(3.11) $$\varphi_k(z; \rho) = \left[D_\nu(\sqrt{2}z) \Big/ \big\| D_\nu(\sqrt{2}z)\big\|_{L^2(R_z^+)} \right]_{\nu=\nu_k(\rho)} \qquad k \in N_0.$$

If λ is not an eigenvalue for the problem (3.1), the function

(3.12) $$\omega_\lambda(z; \rho, f) = w_{-\frac{\lambda+1}{2}}(\sqrt{2}z; \rho, f)$$

is solution of (3.1); moreover if this problem admits a solution in $\mathcal{S}(\overline{R}_z^+)$, it is $\omega_\lambda(z; \rho, f)$.
From (3.8) and (3.12) one deduces that the elements of sequence $\big\{\lambda_k(\rho)\big\}_{k\in N_0}$ are the only singular points of $\omega_\lambda(z; \rho, f)$ and they are poles of first order.

Called $R_k(z;\rho)$ the residue of $\omega_\lambda(z;\rho,f)$ in $\lambda_k(\rho)$, by simple calculations one proves

$$R_k(z;\rho) = \varphi_k(z;\rho)\int_0^{+\infty} \varphi_k(s;\rho) f(s)\,ds \qquad k \in N_0. \tag{3.13}$$

It is easy to prove that from (3.8) and (3.12) follows that if $f \in C_0^\infty(\overline{R}_z^+)$ the function $\omega_\lambda(z;\rho,f)$ belongs to $\mathcal{S}(\overline{R}_z^+)$ for every $\lambda \in C$ such that λ is not an eigenvalue for (3.1).

Now fixed $\rho \in R$ and put

$$m_\rho = \begin{cases} 0 & \text{if } \rho \ \geq \ 0 \\ 16\rho^2 & \text{if } \rho \ < \ 0 \end{cases} \tag{3.14}$$

we prove the following

Theorem 3.2. - *Let* $f \in C_0^\infty(\overline{R}_z^+)$. *If*

$$Re\,\lambda \ > \ m_\rho \tag{3.15}$$

we have

$$\|\omega_\lambda(z;\rho,f)\| \leq \ 2\sqrt{2}\frac{\|f\|}{|\lambda|}. \tag{3.16}$$

Proof. If $\lambda \in C$ verifies (3.15), the function $\omega_\lambda(z;\rho,f)$ is well defined because by Proposition 1.2 we have $\lambda_0(\rho) < \ 0$ if $\rho \ \geq \ 0$ and $\lambda_0(\rho) < \ 12\rho^2$ if $\rho < \ 0$. That being said, because $\omega_\lambda(z;\rho,f) \in \mathcal{S}(\overline{R}_z^+)$ is solution of the problem (3.1), we obtain

$$\lambda\|\omega_\lambda\|^2 + 2\rho|\omega_\lambda(0)|^2 + \|\omega_\lambda'\|^2 + \|z\omega_\lambda\|^2 = \int_0^{+\infty} f(z)\overline{\omega}_\lambda(z)\,dz \tag{3.17}$$

If $|\mathrm{Arg}\,\lambda| \geq \pi/4$, equalizing the imaginary parts in (3.17) we have (3.16).
Now we suppose $0 \leq |\mathrm{Arg}\,\lambda| \leq \pi/4$. If $\rho \geq \ 0$, we obtain (3.16) equalizing the real parts in (3.17). If $\rho < 0$, being

$$|\omega_\lambda(0)|^2 \leq 2\|\omega_\lambda\|\,\|\omega_\lambda'\| \leq \ 4|\rho|\,\|\omega_\lambda\|^2 + \frac{1}{4|\rho|}\|\omega_\lambda'\|^2 \tag{3.18}$$

from (3.17) we have

$$(3.19)\qquad (Re\,\lambda - 8\rho^2)\|\omega_\lambda\|^2 + \frac{1}{2}\|\omega'_\lambda\|^2 + \|z\omega_\lambda\|^2 \le \|f\|\,\|\omega_\lambda\|.$$

By (3.14) and (3.15) we have

$$Re\,\lambda - 8\rho^2 \ge Re\,\lambda - \frac{m_\rho}{2} \ge \frac{1}{2}Re\,\lambda \ge \frac{1}{2\sqrt{2}}|\lambda|$$

and so (3.16). □

Now we put

$$(3.20)\qquad |||w||| = \|w\| + \|zw\| + \|w'\| \qquad \forall\, w \in \mathcal{S}(\overline{R}^+_z)$$

and prove the following

Theorem 3.3. *If* $\lambda \in C-]-\infty, m_\rho]$ *there is a constant* $c = c(\lambda, \rho)$ *such that*

$$(3.21)\qquad |||\omega_\lambda(z;\rho, f)||| \le c\|f\| \qquad \forall\, f \in C_0^\infty(\overline{R}^+_z).$$

Proof. If $\lambda \in C-]-\infty, m_\rho]$ with $Im\,\lambda \ne 0$ there is $\theta_0 \in]0, \pi/2[$ such that $\theta_0 \le |\mathrm{Arg}\,\lambda| \le \pi - \theta_0$.
Equalizing the imaginary parts in (3.17) we have

$$(3.22)\qquad \|\omega_\lambda\| \le \frac{\|f\|}{|\lambda|\sin\theta_0} \qquad f \in C_0^\infty(\overline{R}^+_z),$$

and by (3.18) one obtains

$$|\omega_\lambda(0)|^2 \le \frac{4|\rho|}{|\lambda|^2\sin^2\theta_0}\|f\|^2 + \frac{1}{4|\rho|}\|\omega'_\lambda\|^2$$

so (3.21) follows from (3.17) in the case $Im\,\lambda \ne 0$.
If $\lambda = Re\,\lambda$, being $\lambda > m_\rho$, (3.21) follows directly from (3.19). □

From (3.17) by similar calculations we deduce

Proposition 3.4. *For every* $\delta > 0$, *if* $|Im\,\lambda| > \delta$, *then*

$$\left|\omega_\lambda(z;\rho, f)\right| \le c(|\lambda| + m_\rho)^{1/2}\|f\| \qquad \forall\, z \in \overline{R}^+_z, \quad \forall\, f \in C_0^\infty(\overline{R}^+_z),$$

where c is a constant depending only of δ.

4. A holomorphic semigroup.

We have recourse to Semigroups Theory to solve the problem (I). In order to make more simple the reading of this section, we report a theorem which descends from the mutually equivalent of three conditions proved by Yosida (see [7]).

Theorem 4.1. *- Let* $\{T_\tau;\, \tau \geq 0\}$ *be an equi-continuous semigroup of class* (C_0) *and let A be its infinitesimal generator. If*

i) a positive constant C_1 *exists such that the family of operators*

$$\left\{\left(C_1 \lambda R(\lambda;\, A)\right)^n\right\}$$

is equi-continuous with respect to $n \geq 0$ *and to* λ *with* $Re\,\lambda \geq 1 + \varepsilon$, $\varepsilon > 0$,

then we have also:

1) there exists an angle $\theta_0 \in\,]0, \pi/2[$ *such that the resolvent set of A,* $\rho(A)$, *includes the set*

$$\sum_{m,\theta_0} = \left\{\lambda \in C : |\lambda| \geq m \;\; and \;\; |\mathrm{Arg}\,\lambda| \leq \pi - \theta_0\right\}$$

with m suitably large;

2) for every $x \in X$ *and* $\tau > 0$ *we have*

$$T_\tau x = \frac{1}{2\pi i} \int_{C_2} e^{\lambda\tau} R(\lambda;\, A) x\, d\,\lambda$$

where the path of integration $C_2 = \lambda(\sigma)$, $-\infty < \sigma < +\infty$, *is such that* $\lim_{|\sigma| \to +\infty} |\lambda(\sigma)| = +\infty$ *and for some* $\varepsilon > 0$,

$$\pi/2 + \varepsilon \leq \mathrm{Arg}\,\lambda(\sigma) \leq \pi - \theta_0 \quad and \quad -(\pi - \theta_0) \leq \mathrm{Arg}\,\lambda(\sigma) \leq -(\pi/2 + \varepsilon)$$

when $\sigma \to +\infty$ *and* $\sigma \to -\infty$ *respectively;*

3) exists $\theta_1 \in\,]0, \pi/2[$ *such that* T_τ *admits a weakly holomorphic extension* T_λ *for* $|\mathrm{Arg}\,\lambda| \leq \theta_1$, *that is* T_τ *is a holomorphic semigroup.*

Now we introduce the following operator

$$A : \omega \in \mathcal{S}(\overline{R}_z^+) \to A\omega = \left(\frac{d^2}{dz^2} - z^2\right)\omega.$$

By Theorems of section 3 we can be able to extend A to more general spaces. Let $H(R_z^+)$ be the subspace of $H^1(R_z^+)$ formed by functions ω with $|||\omega||| < +\infty$ for which there is a sequence $\{\omega_n\} \subset \mathcal{S}(\overline{R}_z^+)$ such that:

$$\lim_n |||\omega_n - \omega||| = 0, \tag{4.1}$$

$$\{A\omega_n\}_{n\in N_0} \text{ is convergent to an element of } L^2(R_z^+). \tag{4.2}$$

If $\omega \in H(R_z^+)$ then $\lim_n A\omega_n$ is independent of $\{\omega_n\}$, so it is right to put

$$A\omega = \lim_n A\omega_n \qquad \text{in } L^2(R_z^+).$$

We equip $H(R_z^+)$ by the norm:

$$\|\omega\|_H = |||\omega||| + \|A\omega\|, \tag{4.3}$$

so $H(R_z^+)$ is a complete Banach space.

For every $\rho \in R$, let $D(A, \rho)$ be a subspace of $H(R_z^+)$. We say that the function ω belongs to $D(A, \rho)$ if

$$\int_0^{+\infty} (A\omega)\nu \, dz = \int_0^{+\infty} \omega(A\nu)\, dz \qquad \forall \nu \in C_0^\infty(\overline{R}_z^+) : \quad \nu'(0) = 2\rho\nu(0). \tag{4.4}$$

It is clear that $D(A, \rho)$ is a close subspace of $H(R_z^+)$, dense in $L^2(R_z^+)$, moreover if $\omega \in H^2(R_z^+) \cap D(A, \rho)$, we have $\omega'(0) = 2\rho\omega(0)$. It is easy to prove that the operator A, with domain $D(A, \rho)$ and range in $L^2(R_z^+)$, is a close operator.

Let $\omega_\lambda(z) = \omega_\lambda(z; \rho, f)$ the function defined by (3.12) and (3.8). We prove:

Proposition 4.2. *For every $\rho \in R$, let $\lambda \in C -]-\infty, m_\rho]$. Then, for every $f \in L^2(R_z^+)$, the function ω_λ belongs to $D(A, \rho)$.*

Proof. Let $f \in L^2(R_z^+)$ and $f_n \in C_0^\infty(\overline{R}_z^+)$, $n \in N$, such that

$$f_n \to f \qquad \text{in } L^2(R_z^+). \tag{4.5}$$

If $(\omega_\lambda)_n = \omega_\lambda(z; \rho, f_n)$, $\forall n \in N$, we have that $(\omega_\lambda)_n$ is classical solution of problem (3.1), therefore

$$(\omega_\lambda)_n \in D(A, \rho) \cap \mathcal{S}(\overline{R}_z^+). \tag{4.6}$$

By Theorem 3.3 and by (4.5) we have that the sequence $(\omega_\lambda)_n$ is foundamental so it is convergent in $H(R_z^+)$. By (3.8) and (3.12) we have

$$(\omega_\lambda)_n(z) \to \omega_\lambda(z) \qquad \forall z \in \overline{R}_z^+.$$

So $\omega_\lambda \in H(R_z^+)$ and

$$\left\|(\omega_\lambda)_n - \omega_\lambda\right\|_H \to 0.$$

Since $D(A, \rho)$ is a closed subset of $H(R_z^+)$ we have the assert. □

We consider the operator

$$A_{m_\rho} : \omega \in \mathcal{S}(\overline{R}_z^+) \to (A - m_\rho I)\omega$$

where I is the identity in $\mathcal{S}(\overline{R}_z^+)$.

By Proposition 4.2 we obtain that $C -]-\infty, 0]$ is included in the resolvent set of A_{m_ρ} in $D(A, \rho)$ and results

$$R(\lambda; A_{m_\rho})f = \omega_{\lambda+m_\rho}(f) \qquad \forall f \in L^2(R_z^+). \tag{4.7}$$

Now we prove the following

Theorem 4.3. *For every $\rho \in R$ the operator A_{m_ρ}, with domain $D(A, \rho)$, is the infinitesimal generator of a contraction semigroup of class (C_0) in $L^2(R_z^+)$. This semigroup is also holomorphic.*

Proof. We prove that for every $\rho \in R$, A_{m_ρ} and $D(A, \rho)$ satisfy hypotheses of Philips and Lumer Theorem (see [7]).
It is obvious that the range of A_{m_ρ} and $D(A, \rho)$ are subspaces of $L^2(R_z^+)$. The density of $D(A, \rho)$ in $L^2(R_z^+)$ has been observed. By Proposition 4.2 and by (4.7) we have that the range of $I - A_{m_\rho}$ is $L^2(R_z^+)$. We must only verify that A_{m_ρ} is dissipative. In order to do this, let $\omega \in D(A, \rho)$ and let $\{\omega_n\}$ be a sequence of $\mathcal{S}(\overline{R}_z^+)$ corvenging to ω in $H(R_z^+)$.
Since

$$\int_0^{+\infty} A_{m_\rho}(\omega_n)\overline{\omega}_n \, dz = -m_\rho \|\omega_n\|^2 - 2\rho|\omega_n(0)|^2 - \|\omega_n'\|^2 - \|z\omega_n\|^2 \tag{4.8}$$

we have

$$Re \int_0^{+\infty} A_{m_\rho}(\omega_n)\overline{\omega}_n \, dz \le 0,$$

(this is obvious if $\rho \geq 0$, if $\rho < 0$ it follows from (3.18) and (3.16)). Passing to limit in (4.8), we have that the operator A_{m_ρ} is dissipative.
We denote by $\left\{\overline{T}_{\tau,\rho},\ \tau \geq\ 0\right\}$ the contraction semigroup of class (C_0) generated by A_{m_ρ}.
By Theorem 3.2 we have that the condition i) of Theorem 4.1 is verified, so $\left\{\overline{T}_{\tau,\rho},\ \tau \geq 0\right\}$ is a holomorphic semigroup.
The theorem is proved. □

Now we are able to prove

Theorem 4.4. *For every $\rho \in R$ the operator A, with domain $D(A, \rho)$, is the infinitesimal generator of a holomorphic semigroup $\left\{T_{\tau,\rho},\ \tau \geq 0\right\}$ and results*

$$T_{\tau,\rho} = e^{m_\rho \tau}\overline{T}_{\tau,\rho}. \tag{4.9}$$

Proof. Since the operator $m_\rho I$ is bounded by Theorem 4.3 we have that $A = A_{m_\rho} + m_\rho I$ is the infinitesimal generator of a holomorphic semigroup $\left\{T_{\tau,\rho},\ \tau \geq 0\right\}$ (see [5], chapter 3, section 3.2, Corollary 2.2), and its expression is given by (4.9). □

Theorem 4.5. *For every $f \in L^2(R_z^+)$ and for every $\rho \in R$, results*

$$T_{\tau,\rho} f(z) = \sum_{k=0}^{+\infty} e^{\lambda_k(\rho)\tau}\varphi_k(z;\rho)\int_0^{+\infty} f(z')\varphi_k(z';\rho)\,dz' \qquad \forall\tau > 0,\ \ \forall z \in \overline{R_z^+}. \tag{4.10}$$

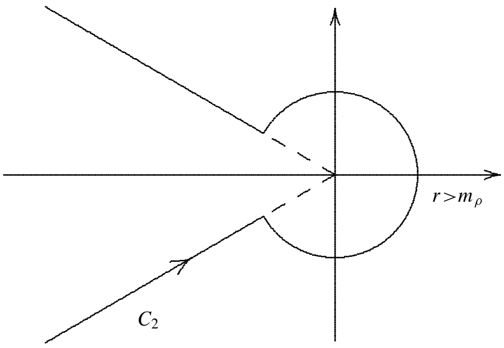
Proof. By 2) of Theorem 4.1 and by (4.7) we have

$$\overline{T}_{\tau,\rho} f(z) = \frac{1}{2\pi i}\int_{C_2} e^{\lambda\tau}\omega_{\lambda+m_\rho}(z)\,d\lambda. \tag{4.11}$$

By sostitution $\lambda + m_\rho \to \lambda$, from (4.9) we have

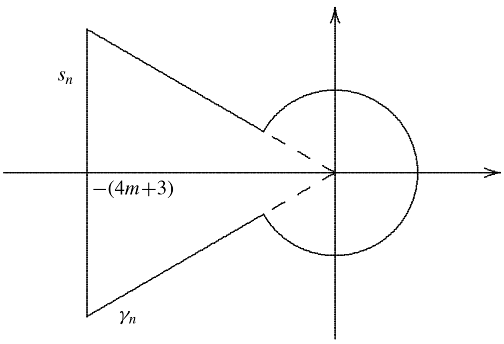
$$T_{\tau,\rho} f(z) = \frac{1}{2\pi i}\int_{C_2} e^{\lambda\tau}\omega_\lambda(z)\,d\lambda \tag{4.12}$$

here C_2 is the path included in $\sum\limits_{m,\theta_0}$ of the type showed in Picture 1



Picture 1

We fix now $z \in \overline{R}_z^+$. If τ is large enough and $f \in C_0^\infty(\overline{R}_z^+)$, it is possible to evaluate integral (4.12) by residues method. Fixed $n \in N$, let γ_n be the path in Picture 2.



Picture 2

By (3.13) we have

$$\frac{1}{2\pi i} \lim_{n \to +\infty} \oint_{\gamma_n} e^{\lambda\tau} \omega_\lambda(z)\, d\lambda = \sum_{k=0}^{+\infty} e^{\lambda_k(\rho)\tau} R_k[z; \rho] = \tag{4.13}$$

$$= \sum_{k=0}^{+\infty} e^{\lambda_k(\rho)\tau}\varphi_k(z;\rho)\int_0^{+\infty} f(z')\varphi_k(z';\rho)\,dz'.$$

Let s_n be the segment of γ_n belonging to the straight line of equation $Re\,\lambda = -(4n+3)$; let δ be a positive number for which the Proposition 3.1 holds. We have:

$$\left|(s_n)\int_{|Im\,\lambda|<\delta} e^{\lambda\tau}\omega_\lambda(z)\,d\lambda\right| \le \delta\mu^n e^{-(4n+3)\tau}\max|f|$$

so if τ such that

$$e^{4\tau} > \mu \tag{4.14}$$

results

$$\lim_{n\to+\infty}(s_n)\int_{|Im\lambda|<\delta} e^{\lambda\tau}\omega_\lambda(z)\,d\lambda = 0.$$

Then by Proposition 3.4 we have

$$\left|(s_n)\int_{|Im\lambda|>\delta} e^{\lambda\tau}\omega_\lambda(z)\,d\lambda\right| \le e^{-(4n+3)\tau}O(n^{\frac{3}{2}})\|f\|$$

and so

$$\lim_{n\to+\infty}(s_n)\int_{|Im\,\lambda|>\delta} e^{\lambda\tau}\omega_\lambda(z)\,d\lambda = 0.$$

Then (4.10) follows from (4.13) for $f \in C_0^\infty(\overline{R}_z^+)$ and τ satisfying (4.14). Moreover the series

$$\sum_{k=0}^{+\infty} e^{\lambda_k(\rho)\lambda}\varphi_k(z;\rho)\int_0^{+\infty} f(z')\varphi_k(z';\rho)\,dz' \quad \lambda \in C$$

for $Re\,\lambda \ge a > 0$ is increased in absolute value by series $M\|f\|\sum_{k=0}^{+\infty} e^{-(4k-3)a}k^{\frac{1}{4}}$, for (2.8) and (2.10), so it is a holomorphic function with respect to λ in the halfplane $Re\,\lambda > 0$. By 3) of Theorem 4.1 and by Principle of identity for holomorphic functions, we have (4.10) $\forall\tau > 0$. Since $C_0^\infty(\overline{R}_z^+)$ is dense in $L^2(R_z^+)$ we have the thesis. □

From this theorem we get

Theorem 4.6. *For every $\rho \in R$, the system $\{\varphi_k(z;\rho)\}_{k\in N_0}$ is complete in $L^2(R_z^+)$.*

We conclude this section with a theorem of representation

Theorem 4.7. *Let $\rho \in R$ and $\lambda \in C$. If $\lambda \neq \lambda_k(\rho)$, $\forall k \in N_0$, and $f \in L^2(R_z^+)$, the function $\omega_\lambda(z;\rho, f)$ belongs to $D(A,\rho)$ and it has the following representation:*

$$\omega_\lambda(z;\rho, f) = \sum_{k=0}^{+\infty} \frac{\varphi_k(z;\rho)}{\lambda - \lambda_k(\rho)} \int_0^{+\infty} f(z')\varphi_k(z';\rho)\,dz'. \tag{4.15}$$

Proof. We fix $\lambda \neq \lambda_k(\rho)$, $\forall k \in N_0$, and $f \in C_0^\infty(R_z^+)$. In this case $\omega_\lambda(z;\rho, f) \in \mathcal{S}(\overline{R}_z^+)$ and we find that its Fourier expansion in terms of the system $\{\varphi_k(z;\rho)\}$ is the series in (4.15).
If $\lambda \notin \left]-\infty, m_\rho\right]$ results $\omega_\lambda(z;\rho, f) \in D(A,\rho)$ for every $f \in L^2(R_z^+)$ (see Prop. 4.2), and by an approximation argument we arrive at (4.15) $\forall f \in L^2(R_z^+)$. Finally, being $\omega_\lambda(z;\rho, f)$ weakly holomorphic in $C - \bigcup_{k\in N_0} \{\lambda_k(\rho)\}$, we have the thesis. □

5. Proof of Theorem I.

Let $g \in L^2(R_z^+)$ be, put

$$u(\tau, z) = T_{\tau,\rho} g(z);$$

by results of previous section we deduce that

$$u(\tau, z) \in C^0(\overline{R}_\tau^+, L^2(R_z^+)) \cap C^\infty(R_\tau^+ \times \overline{R}_z^+)$$

and $u(\tau, z)$ is solution of problem (I). If now $\Phi(\tau, z, z';\rho)$ is the distribution defined in (II) and (III), by Theorem 4.5 we obtain

$$u(\tau, z) = \int_0^{+\infty} \Phi(\tau, z, z';\rho) g(z')\,dz' \quad \tau > 0.$$

Now we are able to prove Theorem I.

Since $\lambda_k(\rho) = -(2\nu_k(\rho)+1) \to -\infty$, $\forall \rho \in R$, from Proposition 2.1 follows that series in (III) converges in $C^\infty(R^+_\tau \times \overline{R}^+_z \times \overline{R}^+_{z'})$. Moreover, denoting by $<,>$ the duality pairing between $C_0^\infty(R_z^+)$ and $\mathcal{D}'(R_z^+)$, if $g \in C_0^\infty(R_z^+)$ we have

$$< \Phi(\tau, z, z'; \rho), g(z') >= \sum_{k=0}^{+\infty} e^{\lambda_k(\rho)\tau} \varphi_k(z;\rho) g_k \qquad \tau \geq 0,\ z \geq 0,$$

where

$$g_k = \int_0^{+\infty} g(z)\varphi_k(z;\rho)\, dz.$$

Being $g \in C_0^\infty(R_z^+)$,

$$g_k = \int_0^{+\infty} g(z) \frac{A^m \varphi_k(z;\rho)}{\lambda_k^m(\rho)}\, dz = \int_0^{+\infty} A^m g(z) \frac{\varphi_k(z;\rho)}{\lambda_k^m(\rho)}\, dz \quad \forall m \in N_0 \ \forall k \in N,$$

follows $\forall\ k \in N$

$$|g_k| \leq\ c \left\| A^m g \right\| k^{-m} \quad \forall m \in N_0,$$

so

$$\left| < \Phi(\tau, z, z'; \rho), g(z') > \right| \leq\ c \left\| A^m g \right\| \quad \forall \tau \geq 0$$

and

$$< \Phi(\tau, z, z'; \rho), g(z') > \ \in C^0(\overline{R}^+_\tau \times \overline{R}^+_z) \qquad \forall g \in C_0^\infty(R_z^+).$$

In this way we have proved that $\Phi(\tau, z, z'; \rho) \in C^0(\overline{R}^+_\tau \times \overline{R}^+_z, \mathcal{D}'(R^+_{z'}))$.
Straight through one proves $(IV)_1$ and $(IV)_2$; moreover being

$$< \Phi(\tau, z, z'; \rho), g(z') >= T_{\tau,\rho} g(z) \qquad (\tau, z) \in R^+_\tau \times \overline{R}^+_z \quad \forall g \in C_0^\infty(R^+_{z'})$$

we have

$$\lim_{\tau \to 0} < \langle\Phi(\tau, z, z'; \rho), g(z') >= T_{0,\rho} g = g(z),$$

and so $(IV)_3$. This concludes the proof of Theorem. $\square$

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