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TRIANGULAR PROJECTIVE PLANES OF ORDER q AND $(q + 1)$-ARCS

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We suitably define the triangular projective planes of order q and connect them with the $(q + 1)$-arcs. In particular, a finite projective plane is either triangular, or contains a lot of $(q + 1)$-arcs.

1. Introduction.

We define 4-*triangle* of an affine plane α_q the set $T = \{V, B_1, B_2, B_3, \}$, where B_1, B_2, and B_3 are three distinct points lying on a line b and V is a point outside b. Let d be a direction of α_q. A 4-*triangular d-family* of α_q is a family $\mathcal{T}$ of 4- triangles satysfying three suitable conditions involving the direction d of α_q which we call *triangular direction*. The plane α_q is *triangular* if any direction is triangular. A projective plane π_q is *triangular* if every affine plane obtained by deleting a line of π_q is triangular. The reason of defining the triangular planes is that either π_q is triangular, or it contains a point through which the number of $(q+1)$-arcs is at least $(q-1)!$. In desarguesian planes the triangularity is satisfied if q is odd and $q \geq 9$.

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2. Finite Triangular Planes.

Let α_q be a finite affine plane of order $q \geq 3$. Let b be a line of α_q and B_1, B_2, B_3 three distinct points of b. Let V be a point of $\alpha_q - b$. We call *4-triangle* of α_q the set $T = \{V, B_1, B_2, B_3\}$. The set $\mathcal{B} = \{B_1, B_2, B_3\}$ is called *base* of T and the line b is called *base-line* of T. The point V is called *vertex* of T. The line $l_j = VB_j$, $j = 1, 2, 3$, is called *edge* of T and the point B_j is called *base-point* of the edge l_j in T, $j = 1, 2, 3$. Obviously the notion of 4-triangle is invariant under the affinities of α_q. Let d be a direction of α_q. We call 4-*triangular d-family* of α_q a family $\mathcal{T}$ of 4-triangles such that the following conditions hold:

(1) Every point of α_q is the vertex of a unique element of $\mathcal{T}$ and therefore $\mathcal{T}$ is a covering of α_q. Two distinct elements of $\mathcal{T}$ meet in at most one point. The edges and the base-lines of any $T \in \mathcal{T}$ have directions distinct from d.

(2) Let V be a base-point of $T' \in \mathcal{T}$. If V' is the vertex of T' and $\mathcal{B}$ is the base of the element of $\mathcal{T}$ whose vertex is V, then $\mathcal{B} \cap VV' = \emptyset$.

(3) Let l be an edge of $T \in \mathcal{T}$ and let l' be an edge of $T' \in \mathcal{T}$, $T \neq T'$. Let B, B' be the base-points of l and l' in T and T' respectively. Then $B = B'$ if and only if $l = l'$. If $B \neq B'$ (and then $l \neq l'$), the edges l and l' are parallel, if and only if the direction of the line BB' is d. If $B = B'$ (and then $l = l'$), let V'' and V''' be two distinct points of l. Let T'' and T''' be the elements of $\mathcal{T}$ whose vertices are V'' and V'''. Then either $T'' \cap T''' = \emptyset$ or $T'' \cap T''' = \{B\}$.

The notion of 4-triangular d-family is invariant under the affinities of α_q. From (1), (2), (3) the following properties of the family $\mathcal{T}$ hold.

Theorem 1. *Let s be a line of α_q with direction d and let V' and V'' be two distinct points of s. Let T' and T'' be the 4-triangles of vertices V' and V''. Then $T' \cap T'' = \emptyset$.*

A direction d of α_q is called *triangular* if in α_q a 4-triangular d-family $\mathcal{T}$ exists. We say that α_q is *triangular* if any direction of α_q is triangular. A projective plane π_q is called *triangular* if any affine plane α_q embedded in π_q is triangular. It is easy to check that

Theorem 2. *The affine plane AG(2, q) is triangular if and only if there is a triangular direction in AG(2, q).*

From Theorem 2 it follows that the notion of triangular affine plane is significant if the plane is non-desarguesian. Obviously we get

Theorem 3. *The plane PG(2, q) is triangular if and only if AG(2, q) is triangular.*

From Theorem 3 it follows that the notion of triangular projective plane is significant if the plane is non-desarguesian.

Theorem 4. *In* $AG(2, 3)$ *triangular directions do not exist. Therefore* $AG(2, 3)$ *is not triangular.*

Proof. Assume that d is a triangular direction in $AG(2, 3)$ and let $\mathcal{T}$ be a 4-triangular d-family in $AG(2, 3)$. From (1) the directions of the edges and of the base-line of $T \in \mathcal{T}$ are distinct and different from d. Then there are five distinct directions in $AG(2, 3)$. A contradiction, since in $AG(2, 3)$ there are exactly four directions. So the theorem is proved.

From theorem 3 and Theorem 4 it follows

Theorem 5. *The plane* $PG(2, 3)$ *is not triangular.*

Theorem 6. *The plane* $AG(2, 4)$ *is triangular.*

Proof. The points and the lines of $AG(2, 4)$ are the following.

Points of $AG(2, 4)$:

$$\{V, V', V'', V''', A, A', A'', A''', B, B', B'', B''', C, C', C'', C'''\}.$$

Lines of $AG(2, 4)$:

$$\{V, V', V'', V'''\}, \{A, A', A'', A'''\}, \{B, B', B'', B'''\}, \{C, C', C'', C'''\}, \{V, A', B', C'\},$$

$$\{V', A, B, C\}, \{V'', A''', B''', C'''\}, \{V''', A'', B'', C''\}, \{V, A, B''', C''\}, \{V''', A''', B, C'\},$$

$$\{V, B, C''', A''\}, \{V, C, B'', A'''\}, \{A', V', C''', B''\}, \{A', C, B''', V'''\}, \{A', B, V'', C''\},$$

$$\{B'C, A'', V''\}, \{A, B', C''', V'''\}, \{B', V', A''', C''\}, \{C', V', A'', B'''\}, \{C', B'', V'', A\},$$

Let d be the direction of the line $\{V, V', V'', V'''\}$ and let $\mathcal{T}$ be the following family of 4-triangles whose vertices are the first ones of every following quadruple of points:

$$\{V, A, B, C\}, \{V', A', B', C'\}, \{V'', A'', B'', C''\}, \{V''', A''', B''', C'''\},$$

$$\{A', V, B''', C''\}, \{B', V, A'', C'''\}, \{C', V, B'', A'''\}, \{A, V', B'', C'''\},$$

$$\{B, V', A''', C''\}, \{C, V', A'', B'''\}\{A'', V''', B, C'\}, \{B'', V''', A', C\},$$

$$\{C'', V''', B', A\}, \{A''', V'', B', C\}, \{B''', V'', A, C'\}, \{C''', V'', A', B\}.$$

It is easy to check that $\mathcal{T}$ is a 4-triangular d-family of $AG(2, 4)$. Since in $AG(2, 4)$ the direction d is triangular and from Theorem 2 the proof follows.

From Theorem 3 and Theorem 6 it follows that

Theorem 7. *The plane $PG(2, 4)$ is triangular.*

3. Triangular Planes and $(q + 1)$-arcs.

A k-arc of α_q is a set of k points three by three non-collinear. In α_q a line l is called *tangent* to a set S, if $|l \cap S| = 1$. Let d be a direction of α_q. We say that a q-arc $\mathcal{C}$ is *d-tangent* if every line with direction d is tangent to $\mathcal{C}$.

The following main Theorem holds.

Theorem 8. *Let d be a direction of α_q. If in α_q d-tangent q-arcs do not exist, then the direction d is triangular. It follows that, if d is not triangular, there is a d-tangent q-arc in α_q*

Proof. Assume that d-tangent q-arcs do not exist in α_q. Let $s_1, s_2, \dots, s_q$ be the lines of α_q whose common direction is d and let $d_1, d_2, \dots, d_q$ be the directions of α_q different from d. Let $\mathcal{S} = \{s_1, s_2, \dots, s_q\}$ and $\Delta = \{d_1, d_2, \dots, d_q\}$. Consider the following bijection

$$\varphi : s_j \in \mathcal{S} \to \; d_j \in \Delta.$$

Let P be a point of α_q. Then there is a unique index j, $1 \le j \le q$, such that $P \in s_j$. Let $r(P)$ be the line of α_q through P whose direction is $d_j = \varphi(s_j)$. The direction of $r(P)$ is different from d. Let $\mathcal{R}$ be the set of lines of α_q. Consider the following mapping:

$$r : P \in \alpha_q \to r(P) \in \mathcal{R} - S.$$

It is easy to prove that r is a bijection. We call the line $r(P)$ the *pseudopolar* of P and P the *pseudopole* of $r(P)$. Let V be a point of α_q and s_j the line of $\mathcal{S}$ through V. Let $l_1, l_2, \ldots, l_q$ be the lines of α_q through V different from s_j and let L_i be the pseudopole of l_i, $i = 1, 2, \ldots, q$. Let $L = \{L_1, L_2, \ldots, L_q\}$. Since the lines $l_1, l_2, \ldots, l_q$ are distinct and r is a bijection, the points $L_1, L_2, \ldots, L_q$ are distinct. It follows that $|L| = q$. We remark that every line of $\mathcal{S}$ contains a unique point of L. Moreover, every point $L_i \in L$, $L_i \neq V$, is the pseudopole of the line $L_i V$. The set L cannot be an arc, otherwise L is a d-tangent q-arc of α_q, a contradiction. Therefore there are distinct points $L_{i_1}, L_{i_2}, L_{i_3}$ of L belonging to a line b whose direction is obviously distinct from d. Let us prove that $V \notin b$. Assume $V \in b$. Then at least two points X, Y of the set $\{L_{i_1}, L_{i_2}, L_{i_3}\}$ are distinct from V. The points X and Y are two distinct pseudopoles of b: a contradiction, since r is a bijection. This proves that $V \notin b$. It follows that the set $\{V, L_{i_1}, L_{i_2}, L_{i_3}\}$ is a 4-triangle with vertex V and base $\{L_{i_1}, L_{i_2}, L_{i_3}\}$. The point L_{i_s} is the pseudopole of the edge $L_{i_s} V$, $s = 1, 2, 3$, and the base-line has not the direction d. For any $V \in \alpha_q$ we construct a 4-triangle as above. In such a way we obtain a family $\mathcal{T}$ of 4-triangles.

Let us prove that $\mathcal{T}$ is a 4-triangular d-family of α_q. Every point of α_q is the vertex of a unique element of $\mathcal{T}$ by construction. We remark that the pseudopolar of every point of $T \in \mathcal{T}$ contains the vertex of T. Now let $T, T' \in \mathcal{T}$, $T \neq T'$. Assume $|T \cap T'| \geq 2$ and let X, Y be two distinct points of $T \cap T'$. The line XY does not belong to $\mathcal{S}$, otherwise X and Y are two distinct points of T belonging to a line of $\mathcal{S}$, but the edges and the base of T have not the direction d. It follows that the lines $s_X, s_Y \in \mathcal{S}$ through X, Y respectively are distinct. Since $s_X \neq s_Y$, it follows that $\varphi(s_X) \neq \varphi(s_Y)$ and the lines $r(X), r(Y)$ are not parallel. Let $Z = r(X) \cap r(Y)$. Since $X \in T$, $Y \in T$, it follows that Z is the vertex of T, because we remarked that in any $T \in \mathcal{T}$ the pseudopolars of the points of T contain the vertex of T. Similarly, from $X \in T'$, $Y \in T'$, it follows that Z is the vertex of T'. Since $T \neq T'$ and their vertices are distinct, we have a contradiction which proves that $|T \cap T'| \geq 2$ is impossible. So $|T \cap T'| \leq 1$. The directions of the edges and of the base-line of any $T \in \mathcal{T}$ are distinct from d. So (1) is proved.

Let us prove the condition (2). Assume $\mathcal{B} \cap VV' \neq \emptyset$. Then $\mathcal{B}$ and VV' meet in a unique point X. Obviously $X \neq V$. The points X and V are two distinct points having the same pseudopolar VV', since V is the pseudopole of VV' and X is the pseudopole of $XV = VV'$: a contradiction, because r is a bijection. So (2) is proved.

Now let us prove (3). The first statement of (3) follows easily since two points of α_q coincide, if and only if they have the same pseudopolar. The second statement follows since two distinct lines of $\mathcal{R} - S$ are parallel, if and only if their pseudopoles belong to the same line of $\mathcal{S}$. In order to prove the third statement, assume $T'' \cap T''' \neq \emptyset$. Then, either $T'' \cap T''' \subset \{V'', V'''\}$, or $T'' \cap T''' \not\subset \{V'', V'''\}$. If $T'' \cap T''' = \{V''\}$, the point V'' belongs to the base of T''' and then V'' is the pseudopole of l. Therefore $\{B\} = \{V''\} = T'' \cap T'''$ (the point B is the pseudopole of l). Similary, if $T'' \cap T''' = \{V'''\}$, we get $\{B\} = \{V'''\} = T'' \cap T'''$. If $T'' \cap T''' \not\subset \{V'', V'''\}$, the point $P = T'' \cap T'''$ belongs to the bases of the above triangles. Then, from the first statement, $P \in V''V''' = l$. Moreover $P = B$, since B is the pseudopole of $l = l'$. It follows that $T'' \cap T''' = \{B\}$. So (3) is proved.

From Theorem 8 it follows

Theorem 9. *If in α_q q-arcs do not exist, then every direction of α_q is triangular and therefore α_q is triangular. It follows that, if α_q is not triangular, then q-arcs in α_q do exist.*

For projective planes the following result holds.

Theorem 10. *If in π_q there are not $(q+1)$-arcs, then π_q is triangular. It follows that, if π_q is not triangular, then π_q contains $(q+1)$-arcs.*

Proof. Assume that π_q does not contain $(q+1)$-arcs. Let $\bar{r}$ be a line of π_q and $\alpha_q = \pi_q - \bar{r}$. Let d be a direction of α_q. The plane α_q does not contain any d-tangent q-arc $\mathcal{C}$, otherwise the set $\mathcal{C} \cup \{P\}$, where P is the direction d, is a $(q+1)$-arc of π_q and this contradicts the hypothesis. From Theorem 8 it follows that the direction d is triangular and therefore α_q is triangular and also π_q is triangular. So the theorem is proved.

From Theorem 10 it follows

Theorem 11. *Let π_q be a finite projective plane of order q. Then, either π_q is triangular, or π_q contains $(q+1)$-arcs.*

4. Triangular planes and their automorphisms.

We recall that a *semilinear space* is a pair $(\mathcal{S}, L)$, where $\mathcal{S}$ is a non-empty set whose elements are called points and L is a family of parts of $\mathcal{S}$ whose elements are called lines, such that

$$L \text{ is a covering of } \mathcal{S},$$

$$|l| \geq 2, \qquad \forall\, l \in L,$$

there is at most one line through two distinct points.

Two points x, y are *joinable* (and we write $x \sim y$), if the line through them exists, otherwise they are *unjoinable* (and we write $x \not\sim y$). If $\forall x, y \in \mathcal{S}, x \sim y$, the space $(\mathcal{S}, L)$ is called *linear space*, otherwise it is called *proper semilinear space*.

A subset $\mathcal{S}' \subset \mathcal{S}$ is a *subspace* of $(\mathcal{S}, L)$, if and only if for any $x, y \in \mathcal{S}'$ we have $x \sim y$, $xy \subset \mathcal{S}'$. A subspace is *maximal*, if it is not properly contained in a subspace. A *clique* is a set of $\mathcal{S}$ consisting of two by two joinable points. An *anticlique* is a set consisting of two by two unjoinable points. An *ovoid* is a subset of $\mathcal{S}$ meeting any maximal subspace in a unique point. If $\mathcal{S}$ is finite and the lines have the same size, $(\mathcal{S}, L)$ is a *partial Steiner system.* A partial Steiner system is *homogeneous*, if the number of lines through every point is the same. In [3] M. Scafati and G. Tallini proved that, if $(\mathcal{S}, L)$ is a homogeneous partial Steiner system, with $|\mathcal{S}| = v$ and $k = |l|$, $\forall l \in L$, then for every anticlique A the following holds:

$$|A| \leq v/k, \tag{4}$$
$$|A| = v/k \iff A \text{ is an ovoid and the maximal subspaces are the lines.}$$

Let π_q be a finite projective plane of order q. A line t of π_q is *triangular*, if the affine plane $\alpha_q = \pi_q - t$ is triangular. Obviously every automorphism of π_q preserves the set of triangular lines, which we denote by $\mathcal{R}_t$. We prove the following theorem:

Theorem 12. *Let $\mathcal{R}$ be the set of the lines of π_q. If $\mathcal{R}_t \neq \emptyset$, $\mathcal{R}_t \neq \mathcal{R}$, then the automorphism group $\mathcal{G}$ of π_q is not transitive on the points. It follows that, if $\mathcal{G}$ is transitive on the points, then either $\mathcal{R}_t = \emptyset$, or $\mathcal{R}_t = \mathcal{R}$.*

Proof. Assume $\mathcal{R}_t \neq \emptyset$, $\mathcal{R}_t \neq \mathcal{R}$ and $\mathcal{G}$ transitive on the points. By the assumption, it follows that in π_q there are a triangular line r and a non-triangular line s, $r \neq s$. Let $\{P\} = r \cap s$. Since $\mathcal{G}$ is transitive on the points, it follows that the number n, $1 \leq n < q+1$, of triangular lines through every point of π_q is the same. So the pair $(\mathcal{S}, R_t)$ is a homogeneous partial Steiner system. Moreover, in $(\mathcal{S}, R_t)$ the maximal spaces are the lines. Obviously the line s is an anticlique and also an ovoid of $(\mathcal{S}, R_t)$, since every line of $\mathcal{R}_t$ is tangent to s. From (4), it follows

$$|s| = (q^2 + q + 1)/(q + 1) = 1/(q + 1) + q,$$

a contradiction, since the right hand side of the above equation is not an integer. The contradiction proves that $\mathcal{G}$ is not transitive.

We say that π_q is *totally non-triangular*, if π_q does not contain triangular lines. From Theorem 12 it follows:

Theorem 13. *Let π_q be non-triangular. If the automorphism group is transitive on the points, then π_q is totally non-triangular.*

Proof. From the hypothesis, $\mathcal{R}_t \neq \mathcal{R}$ and $\mathcal{G}$ is transitive on the points. Then, from Theorem 12, it follows $\mathcal{R}_t = \emptyset$.

For instance, in $PG(2,3)$, which is not triangular (see Theorem 5), the group $\mathcal{G}$ is point transitive and therefore $PG(2,3)$ is totally non-triangular. Moreover, $PG(2,4)$ is triangular (see Theorem 7) and $\mathcal{G}$ is point transitive.

5. The number of $(q+1)$-arcs in a non-triangular projective plane.

Assume π_q is not triangular. Then there is a non-triangular affine plane $\alpha_q \subset \pi_q$ and therefore in α_q there is a non-triangular direction d. If $d_1, d_2, \dots, d_q$ are the directions of α_q different from d and $s_1, s_2, \dots, s_q$ are the lines of α_q whose common direction is d, we choose an arbitrary bijection

$$\phi : \{s_1, \dots, s_q\} \to \{d_1, \dots, d_q\}.$$

Since d is not triangular, in α_q no 4-triangular d-families exist. So at least one of the sets L (see Theorem 8) is a d-tangent q-arc. To show this, assume that L is not a q-arc. Then L contains three collinear points L_1, L_2, L_3 on a line not through V (since $L_j V$, $j = 1, 2, 3$ is the pseudopolar of L_j and the pseudopolarity is a bijection). So L contains a 4-triangle T whose vertex is V. If all the sets L (depending on V) are not d-tangent q-arcs, the 4-triangles T are a 4-triangular d-family. A contradiction, since d is not triangular. In conclusion, every bijection ϕ gives rise to at least one d- tangent q-arc. For every ϕ we choose one of such d-tangent q-arcs. The number of the bijections is $q!$, so we get $q!$ d-tangent q-arcs (not necessarily distinct).

Denote by $\mathcal{L} = \{L_j\}_{j=1,\dots,M}$ the family of the distinct d-tangent q-arcs we have chosen ($M \leq q!$). We remark that if we choose the same L_j for m_j bijections ϕ, then $m_j \leq q$.

To show this, let $L_j = \{A_1, \dots, A_q\}$ be an element of $\mathcal{L}$, where $A_i \in s_i$, $i = 1, \dots, q$. Consider a point A_h, $h = 1, \dots, q$ in L_j. Associate with each line s_i, $i \neq h$, the direction of the line $A_i A_h$ and with the line s_h the direction of the tangent line of L_j at the point A_h, different from s_h. In such a way we construct a bijection ϕ. If we repeat the previous construction for the point $A_k \in L_j - A_h$, we get a bijection $\phi' \neq \phi$ ($\phi' \neq \phi$, since ϕ and ϕ' associate with the line s_k different directions). In such a way we obtain q different bijections $\phi_1, \dots, \phi_q$ which are all the bijections giving rise to the same L_j, according to

Theorem 8. So $m_j \le q$. Since $\sum_{j=1}^{M} m_j = q!$ and $m_j \le q$, $j = 1, \ldots, M$, it follows $q! = \sum_{j=1}^{M} m_j \le \sum_{j=1}^{M} q = qM$, hence $M \ge (q-1)!$.

In π_q the union of a q-arc L_j and the direction $d = \{P\}$ is a $(q+1)$-arc, so in π_q the number of $(q+1)$-arcs through $\{P\}$ is at least $(q-1)!$. So we get

Theorem 14. *If π_q is not triangular, then there is a point $P \in \pi_q$ such that the number N of $(q+1)$-arcs through it is such that $N \ge (q-1)!$*

In $PG(2, q)$ we easy compute that the number b of irreducible conics is

$$b = q^2(q-1)(q^2+q+1).$$

Let q be odd. Then b is also the number of $(q+1)$-arcs, since each $(q+1)$-arc is an irreducible conic and conversely. Denote by $\mathcal{S}$ the point set of $PG(2, q)$ and by $\mathcal{C}$ the family of the $(q+1)$-arcs of $PG(2, q)$. The pair $(\mathcal{S}, C)$ is a 2 - $(q^2+q+1, q+1, \lambda_2)$ design (see [1]), where λ_2 is the number of $(q+1)$-arcs through two distinct points. Denoting by λ_1 the number of $(q+1)$-arcs through a point, we get

$$\lambda_1 = q^2(q^2-1).$$

Since $q \ge 9$ implies $q^2(q^2-1) < (q-1)!$, from Theorem 14 we get

Theorem 15. *The plane $PG(2, q)$, q odd and $q \ge 9$, is triangular.*

From Theorem 15 we obtain:

Theorem 16. *If π_q, q odd and $q \ge 9$, is not triangular, then π_q is non-desarguesian and, if $q \ge 11$, it contains a number of $(q+1)$-arcs which is greater than b.*

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