HOLOMORPHIC SELF-MAPS OF SINGULAR
PROJECTIVE CURVES

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Here we classify all complex singular irreducible projective curves.
X, such that there exists a holomorphic map \( f : X \to X \) with \( f \) locally
biholomorphic at each point of \( X_{\text{reg}} \) and with \( \deg(f) \geq 2 \): \( X \) is rational and
either it has a unique singular point with two branches or it has exactly two
singular points, both unibranched.

Let \( f : X \to X \) be a “locally invertible” morphism in a category. Must \( f \)
be invertible? The corresponding problem was studies in [1] for the categories
of compact differentiable manifolds and (but only if \( \dim(X) = 2 \)) of compact
complex manifolds. The problem arised from [2] in wich it was studied the
Corresponding problem for maps \( h : A \to B \) in wich \( A \) and \( B \) may be different
differentiable manifolds. The motivation behind [3] was explained at the end of
the introduction of [3] and in [3], sections 4.2, 4.3 and 4.4; key words: Market
Equilibrium, Limited Arbitrage and Uniqueness with Short Sales. Here we will
never met such words. Both from the classification point of view and for the
applications it seems important to assume \( X \) compact. However, if \( X \) is compact
and smooth the existence of such \( f \) with \( \deg(f) \geq 2 \) is very restrictive. If \( X \)
is a singular compact complex space, we cannot hope to have such non-trivial
pairs \((X, f)\) with \( f \) locally biholomorphic at each point of \( X \) (see Remark 1.2

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and 1.3). The best we can hope is that $f$ is locally invertible at each smooth point of $X$. Our main result is the following theorem.

**Theorem 0.1.** Let $X$ be an integral projective curve with $\text{Sing}(X) \neq \emptyset$ and such there exists a finite morphism $f : X \to X$ of degree $d \geq 2$ which is locally biholomorphic at each point of $R_{\text{reg}}$. Let $\pi : Y \to X$ be the normalization. We have $Y \cong \mathbb{P}^1$ and $\text{card}(\pi^{-1}(\text{Sing}(X))) = 2$. There is a morphism $f' : \mathbb{P}^1 \to \mathbb{P}^1$ with $f' \circ \pi = \pi \circ f$ and in particular $\text{deg}(f') = d$. The map $f'$ is locally biholomorphic at each point of $\pi^{-1}(X_{\text{reg}})$ and for every $Q \in \pi^{-1}(\text{Sing}(X))$ we have $\text{card}(f'^{-1}(Q)) = 1$, i.e. $f$ is totally ramified at $Q$. Two cases may occur:

Case A) $\text{card}(\text{Sing}(X)) = 2$ and each singular point of $X$ is unibranch.

Case B) $\text{card}(\text{Sing}(X)) = 1$ and the singular point of $X$ has exactly two branches.

Viceversa, given any such curve $X$ there is an integer $d \geq 2$ and a degree $d$ holomorphic map $f : X \to X$ such that $X$ is locally biholomorphic at each point of $X_{\text{reg}}$.

We are even able to classify all such map $f$ or, equivalently, all such maps $f'$.

**Remark 0.2.** Up to an element of Aut($\mathbb{P}^1$) and a non-zero multiplicative constant the map $f'$ in the statement of 0.1 is uniquely determined by the integer $d$ and, possibly, the interchange of the two points of $\pi^{-1}(\text{Sing}(X))$. Taking $\pi^{-1}(\text{Sing}(X)) = \{0, \infty\}$, we will see in 1.5 that $f'(z) = cz^d$ (resp. $f'(z) = cz^{-d}$) for some $c \in \mathbb{C}\setminus\{0\}$ if $f'(0) = 0$ and $f'(\infty) = \infty$ (resp if $f'(0) = \infty$ and $f'(\infty) = 0$).

For more details on the solutions in Cases A) and B) and in particular on the possible integers $d := \text{deg}(f)$, see Propositions 1.8, 1.9 and 1.10.

1.– Here we will prove Theorem 0.1 and Remark 0.2. Then we will show how to solve two related problems on self-maps of pointed projective curves (see 1.11 and 1.12).

**Lemma 1.1.** Let $X$ be a complex compact irreducible variety with $\text{Sing}(X)$ finite and $f : X \to X$ a finite morphism of degree $d > 1$ such that $f|X_{\text{reg}}$ is locally biholomorphic. Let $\pi : Y \to X$ be the normalization map. Then there exists a finite morphism of degree $d > 1$ $f' : Y \to Y$ such that $f' \circ \pi = \pi \circ f'$ and $f'|\pi^{-1}(X_{\text{reg}})$ is locally biholomorphic. If $\dim(X) \geq 2$, $f'$ is locally biholomorphic at each point of $Y_{\text{reg}}$. 

Proof. The holomorphic map $f \circ \pi : Y \to X$ is finite and surjective. By the universal property of the normalization there is a unique holomorphic map $f' : Y \to Y$ such that $f \circ \pi = \pi \circ f'$. The holomorphic map $f'$ is locally biholomorphic at each point of $Y \setminus \pi^{-1}(\text{Sing}(X))$. Assume $\dim(X) \geq 2$, i.e. $\dim(Y) \geq 2$. Since the discriminant locus of a finite holomorphic map between complex manifolds of the same dimension is either empty or a pure one-codimensional hypersurface and $\dim(\pi^{-1}(\text{Sing}(X))) = 0 \leq \dim(Y) - 2$, $f'$ is locally biholomorphic at each point of $Y_{\text{reg}}$.

Remark 1.2. Let $X$ be a complex compact irreducible variety with $\text{Sing}(X)$ finite and $f : X \to X$ a finite morphism of degree $d > 1$ such that $f|_{X_{\text{reg}}}$ is locally biholomorphic. Hence for every $P \in X_{\text{reg}}$, $f(P) \notin \text{Sing}(X)$. Since $X$ is compact, $f$ is surjective. Since $\text{Sing}(X)$ is finite, $\text{Sing}(X) = f^{-1}(\text{Sing}(X))$ and $f$ induces a permutation of the finite set $\text{Sing}(X)$. Hence there is an integer $k \geq 1$ such that the iteration $f^k := f \circ \cdots \circ f (k \text{ times})$ of $f$ fixes every point of $\text{Sing}(X)$.

Remark 1.3. Remark 1.2 explains why we do not assume that $f$ is locally biholomorphic for every $P \in X$: since $\text{Sing}(X)$ is finite and $\deg(f) > 1$, this would force $\text{Sing}(X) = \emptyset$. The case $\text{Sing}(X) = \emptyset$ is obvious by the next well-known remark.

Remark 1.4. Let $C$ be a smooth projective curve of genus $q$ and $u : C \to C$ a finite morphism of degree $d > 1$. By the Riemann-Hurwitz formula ([4], IV. 2.4) we have $2q - 2 = d(2q - 2) + b$, with $b = 0$ if $u$ is locally biholomorphic and $b > 0$, otherwise. Hence if $u$ is locally biholomorphic, then $q = 1$, i.e. $C$ is an elliptic curve, while if $u$ is not locally biholomorphic, then $q = 0$, i.e. $C \cong \mathbb{P}^1$. Vice versa, for every elliptic curve $C$ and all integers $t, n$ with $t \geq 2$ and $n \geq 2$ there are holomorphic maps $u : C \to C$ and $v : \mathbb{P}^1 \to \mathbb{P}^1$ with $\deg(u) = t^2$ and $\deg(v) = n$. By Riemann-Hurwitz formula every such $u$ is locally biholomorphic and no such $v$ is locally biholomorphic.

Lemma 1.5. Let $X$ be an integral projective curve with $\text{Sing}(X) \neq \emptyset$ and such that there exists a finite morphism $u : X \to X$ of degree $d > 1$ which is locally biholomorphic at each point of $X_{\text{reg}}$. We have $f^{-1}(\text{Sing}(X)) = \text{Sing}(X)$ and $f$ induces a bijection of $\text{Sing}(X)$ onto itself. Let $\pi : Y \to X$ be the normalization. We have $Y \cong \mathbb{P}^1$. There is a morphism $f' : \mathbb{P}^1 \to \mathbb{P}^1$ with $f' \circ \pi = \pi \circ f$ and in particular $\deg(f) = d > 1$. The map $f'$ is locally biholomorphic at each point of $\pi^{-1}(X_{\text{reg}})$ and for every $Q \in \pi^{-1}(\text{Sing}(X))$ we have $\operatorname{card}(f^{-1}(Q)) = 1$, i.e. $f'$ is totally ramified at $Q$. We have $\operatorname{card}(\pi^{-1}(\text{Sing}(X))) = 2$. Up to an element of $\operatorname{Aut}(\mathbb{P}^1)$ the map $f'$ is uniquely determined, up to a non-zero constant,
by the integer $d$ and $f'|\pi^1(\text{Sing}(X))$: taking $\pi^1(\text{Sing}(X)) = \{0, \infty\}$, we have $f'(z) = cz^d$ (resp. $f'(z) = cz^n$) with $c \in \mathbb{C}\setminus\{0\}$ if $f'(0) = 0$ and $f'(') = \infty$ (resp. if $f'(0) = \infty$ and $f'(') = 0$).

**Proof.** Since $Y$ is the normalization of $X$, the degree $d$ morphism $\pi \circ f : Y \to X$ factors through the normalization map, $\pi$, of the target, proving the existence of $f'$. By Remark 1.4 either $Y \cong \mathbb{P}^1$ or $Y$ is an elliptic curve. Since $f$ is locally biholomorphic at each point of $X_{\text{reg}}$, $f(X_{\text{reg}}) \subseteq X_{\text{reg}}$. Since $X$ is irreducible and compact, $u$ is surjective. Hence $\text{Sing}(X) \subseteq f(\text{Sing}(X))$. Since $\text{Sing}(X)$ is finite, we have $f^{-1}(\text{Sing}(X)) = \text{Sing}(X)$ and $u$ induces a bijection of $\text{Sing}(X)$ onto itself ( Remark 1.2). From the finiteness of $\pi^{-1}(\text{Sing}(X))$ and the relation $\pi \circ f' = f \circ \pi$, we obtain $f'^{-1}(\pi^{-1}(\text{Sing}(X))) = \pi^{-1}(\text{Sing}(X))$ and $f'$ induces a bijection of $\pi^{-1}(\text{Sing}(X))$ onto itself. Hence $f'$ is totally ramified at every point of $\pi^{-1}(\text{Sing}(X))$, i.e. the ramification order of $f$ at $X$ is at least $d - 1$. Thus the degree, $z$, of the ramification divisor of is at least $(d - 1) \cdot \text{card}(\pi^{-1}(\text{Sing}(X)))$. Since $Y \cong \mathbb{P}^1$, the Riemann-Hurwitz formula gives the relation $-2 = -2d + z$. Hence $\text{card}(\pi^{-1}(\text{Sing}(X))) \leq 2$. Since for every $P \in \mathbb{P}^1$, $\mathbb{P}^1\setminus\{P\} \cong \mathbb{C}$ is simply connected, we have $\text{card}(\pi^{-1}(\text{Sing}(X))) = 2$ and hence $1 \leq \text{card}(\text{Sing}(X)) \leq 2$. Now the last assertion is elementary. We assume that $Y$ is an elliptic curve. By the Riemann-Hurwitz formula (see Remark 1.3) every non-constant holomorphic map between two elliptic curves is locally biholomorphic. As in the previous case $f$ induces a degree $d$ morphism $f' : Y \to Y$ which induces a permutation of $\pi^{-1}(\text{Sing}(X))$. Since $\text{Sing}(X) \neq \emptyset$ and $\text{deg}(f') = d > 1$, $f'$ cannot be locally biholomorphic, contradiction.

**Definition 1.6.** Fix an integer $d \geq 2$. Let $(X, P)$ the germ of a unibranch singularity of curves and $R$ the completion of the associated local ring. Since $(X, P)$ is unibranch, $R$ may be embedded as a unitary $\mathbb{C}$-local ring in the power series ring $\mathbb{C}[[t]]$ in one variable. We will say that $(X, P)$ has Property $(\$; \ ; \ )$ if there is an embedding $j : R \to \mathbb{C}[[t]]$ such that $j(R)$ has generators $1, p_1(t), \ldots, p_e(t)$ ($e := \dim(T_p X)$) and $j(R)$ contains $p_i(t^d)$ for every $i$ with $1 \leq i \leq e$. Property $(\$; \ ; \ )$ depends only on the one-dimensional domain $R$ and hence we are allowed to say that $R$ has Property $(\$; \ ; \ )$ or not.

**Remark 1.7.** See $\mathbb{C}[[t]]$ as the completion of the local ring of the affine line $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$ at 0 and call $u : \mathbb{A}^1 \to \mathbb{A}^1$ the morphism with $u(z) := z^d$. Let $R$ be the local ring of a unibranch curve singularity and fix an embedding $j$ of $R$ in $\mathbb{C}[[t]]$. This embedding may be used to prove that $R$ has Property $(\$; \ ; \ )$ if and only if $u^e(j(R)) \subseteq R$. Iterating the morphism $u$ we see that if $R$ has Property $(\$; \ ; \ )$, then it has Property $(\$; \ ; \ )$. 

Proposition 1.8. Fix an integer $d \geq 2$. Let $X$ be an integral projective curve with $\text{card}(\text{Sing}(X)) \geq 2$ such that there exists a degree $d$ holomorphic map $f : X \to X$ which is locally biholomorphic at each point of $X_{\text{reg}}$. Then $\text{card}(\text{Sing}(X)) = 2$. The normalization of $X$ is $\mathbb{P}^1$, each singular point of $X$ is unibranch and $f$ induces a bijection of $\text{Sing}(X)$ onto itself. Each singular point of $X$ has Property $(\$; d)$. If this bijection is not the identity, then $f^2 : X \to X$ has degree $d^2$, it is locally biholomorphic at each point of $X_{\text{reg}}$ and $f^2(P) = P$ for every $P \in \text{Sing}(X)$. Viceversa, for every integral curve $X$ with $\text{card}(\text{Sing}(X)) = 2$, only unibranch singularities, with $\mathbb{P}^1$ as normalization and such that every singular point of $X$ has Property $(\$; d)$ there is a holomorphic map $f : X \to X$ with $\text{deg}(f) = d$, locally biholomorphic at each point of $X_{\text{reg}}$ and such that $f|\text{Sing}(X)$ is the identity; such a map is unique up to $\text{a}$ non-zero constant. Furthermore, there is a unique, up to a non-zero constant, holomorphic map $f^\prime : X \to X$ with $\text{deg}(f^\prime) = d$, $f^\prime$ locally biholomorphic at each point of $X_{\text{reg}}$ and such that $f^\prime$ interchange the two points of $\text{Sing}(X)$.

Proof. The proof of Lemma 1.5 gives the first part except Property $(\$; d)$. The second part, i.e. the viceversa part, is very easy using Remark 1.7; for the last assertion, just use any morphism $\mathbb{P}^1 \to \mathbb{P}^1$ which interchanges $0$ and $\infty$. Notice that, for any fixed $X$ and $u : \mathbb{P}^1 \to \mathbb{P}^1$ with $\text{deg}(u) = d$, there is at most one morphism $f : X \to X$ inducing $u$. Hence from the viceversa part and the explicit description of $u$ and Remark 1.7 we obtain that all singular points of $X$ have Property $(\$; d)$ or Property $(\$, d^2)$. 

Proposition 1.9. Let $R$ be the analytic or formal local ring of a unibranch curve singularity. Let $k$ be its multiplicity. Then $R$ has Property $(\$; k)$.

Proof. Let $m$ be the maximal ideal of $R$ and $e := \text{dim}_{\mathbb{C}}(m/m^2)$ the embedding dimension of $R$. The normalization of $R$ is $\mathbb{C}[[t]]$ (or take convergent power series in the analytic case) and, up to a change of coordinates, there are $e$ power series $f_1(t), \ldots, f_e(t)$ such that $R$ is the completion of the $\mathbb{C}$-subalgebra of $\mathbb{C}[[t]]$ generated by $1, f_1(t), \ldots, f_e(t)$ and such that $f_1(t) = t^k$ and all other powers of $t$ appearing with non-zero coefficient in some $f_i(t)$, $i \geq 2$, have order at least $k + 1$ ([3], Remark 2.1.1). Every element of $\mathbb{C}[[t^k]]$ is of the form $g(f_1(t))$ with $g \in \mathbb{C}[[t]]$.

Proposition 1.10. Fix an integer $d \geq 2$. Let $X$ be an integral projective curve with $\text{card}(\text{Sing}(X)) = 1$, say $\text{Sing}(X) = \{P\}$, such that there exists a degree $d$ holomorphic map $f : X \to X$ which is locally biholomorphic at each point of $X_{\text{reg}}$. Then the normalization of $X$ is $\mathbb{P}^1$. $X$ has two branches, say $B'$ and $B''$, and $f$ induces a permutation of these two branches. Then each branch of $X$ has Property $(\$; d)$. If this permutation is not the identity, then
Proof. By Lemma 1.5 to prove the first part it is sufficient to show that the two branches of $X$ at $P$ have Property ($\mathcal{S}$ : $d$). Let $R$ be formal (or analytic) local ring of the germ $(X, P)$. Since $X$ has two branches and it is reduced, $R$ has two minimal prime ideals $p_1$ and $p_2$ with $p_1 \cap p_2 = \{0\}$. Set $R_i := R/p_i$. Let $X'$ be the partial normalization of $X$ in which we have just separated the two branches, i.e. $\text{card(Sing}(X')) = 2$, say $\text{Sing}(X') = \{P_1, P_2\}$, and $X'$ has formal local ring $R_i$ at $P_i$. Since $f(P) = P$, the map $f$ induces an injective $K$-homorphism $\Psi(p_i)$, $i = 1, 2$, is contained in a minimal prime of $R$, the map $f$ induces a degree $d$ map $f'' : X' \to X'$ which coincides with $f$ on $X'_{\text{reg}} \cong X_{\text{reg}}$. Hence we may apply Proposition 1.8 to $X'$ and conclude the proof of the first part. Take $(X, P)$ as in the second part. Let $R$ be formal (or analytic) local ring of the germ $(X, P)$. Let $X'$ be the partial normalization of $X$ in which we have just separated the two branches, i.e. $\text{card(Sing}(X')) = 2$, say $\text{Sing}(X') = \{P_1, P_2\}$, and $X'$ has formal local ring $R_i$ at $P_i$. There is a degree $d$ morphism $f_1 : X' \to X'$ which is locally biholomorphic over $X'_{\text{reg}} \cong X_{\text{reg}} = X \setminus \{P\}$; furthermore there is such morphism, $f_2$, which fixes the two points of $\text{Sing}(X')$ and another one, $f_3$, which interchanges the two points; up to a non-zero multiplicative constant $f_2$ and $f_3$ are uniquely determined. It is sufficient to prove that every such $f_1$ induces a morphism $f : X \to X$ with $f|X_{\text{reg}} = f_1|X'_{\text{reg}}$. This is obvious set-theoretically and even topologically, but we need to check that the set-theoretic map is holomorphic at $P$. We have an inclusion $j : R \to R_1 \oplus R_2 \subset \mathbb{C}[[t_1]] \oplus \mathbb{C}[[t_2]]$, where the latter ring is the semilocal ring of the normalization of $R$. The conductor, $J$, of $R$ in $\mathbb{C}[[t_1]] \oplus \mathbb{C}[[t_2]]$ is of the form $((t_1^{a_1}), (t_2^{a_2}))$ for some integer $a_1 \geq 0$ and $a_2 \geq 0$. The homomorphism $f^{*e}$ sends $J$ into $((t_1^a), (t_2^{da}))$. Since $da_1 \geq a_1$ and $da_2 \geq a_2$, $((t_1^{da}), (t_2^{da}))$ is contained in the conductor $J$. Hence $f^{*e}$ descends to a homomorphism $f^* : R \to R$, showing that $R$ is holomorphic at $P$. The structure of the set of all possible maps $f$ follows from the structure of all possible maps $f' : X' \to X'$ considered in 1.8.
Motivated by the theories of orbifolds, of algebraic pairs \((Y, S)\) and of algebraic stacks, we show why our work gives a solution of the following two related problems.

**Proposition 1.11.** Let \(Y\) be a smooth projective curve over \(\mathbb{C}\) and \(S \subset Y\) with \(S\) finite and \(S \neq \emptyset\). Assume the existence of a holomorphic map \(f : Y \to Y\) with \(\deg(f) = d \geq 2\) such that \(S = f^{-1}(f(S))\) and that \(f\) is locally biholomorphic at each point of \(Y \setminus S\). Then \(Y \cong \mathbb{P}^1\), \(\text{card}(S) = 2\). Up to an element of \(\text{Aut}(\mathbb{P}^1)\) and a non-zero multiplicative constant the map \(f\) is uniquely determined by the integer \(d\) and by the condition "\(f\) exchanges the two points of \(S\) or not": taking \(S = \{0, \infty\}\), we have \(f(z) = cz^d\) with \(c \in \mathbb{C} \setminus \{0\}\) for every \(z \in \mathbb{C}\) if \(f(0) = 0\) and \(f(\infty) = \infty\) and \(f(z) = cz^d\) with \(c \in \mathbb{C} \setminus \{0\}\) if \(f(0) = \infty\) and \(f(\infty) = 0\).

**Proof.** By Remark 1.4 we have \(Y \cong \mathbb{P}^1\). The second part follows from Riemann-Hurwitz formula exactly as in the proof of Proposition 1.8.

**Proposition 1.12.** Let \(X\) be an integral projective curve over \(\mathbb{C}\) with \(\text{Sing}(X) \neq \emptyset\) and \(S \subset X_{\text{reg}}\) with \(S\) finite and \(S \neq \emptyset\). Assume the existence of a holomorphic map \(f : X \to X\) with \(\deg(f) = d \geq 2\) such that \(S = f^{-1}(f(S))\) and that \(f\) is locally biholomorphic at each point of \(X \setminus (S \cup \text{Sing}(X))\); Let \(\pi : Y \to X\) be the normalization. Then \(Y \cong \mathbb{P}^1\), \(\text{card}(S) = \text{card}(\text{Sing}(X)) = 1\) and \(X\) is unibranch at unique point, \(P\), of \(\text{Sing}(X)\). Furthermore, \(X\) satisfies condition \((S; d)\) at \(P\). Vice versa, given any such pair \((X, S)\), up to a normalization by \(\text{Aut}(\mathbb{P}^1)\), i.e. taking \(S = \{0\}\) and \(\text{Sing}(X) = \{\infty\}\), there exists exactly one such holomorphic map \(f\), up to a non-zero multiplicative constant.

**Proof.** By Remark 1.4 we have \(Y \cong \mathbb{P}^1\). By Remark 1.2 we have \(\text{Sing}(X) = f^{-1}(f(\text{Sing}(X)))\). The morphism \(f\) induces a degree \(d\) morphism \(f' : Y \to Y\) with \(f'\) locally biholomorphic at each point of \(\pi^{-1}(S \cup \text{Sing}(X))\). Since \(S \subset Y_{\text{reg}}, S \neq \emptyset\) and \(\text{Sing}(X) \neq \emptyset\), we obtain \(\text{card}(S) = \text{card}(\text{Sing}(X)) = \text{card}(\pi^{-1}(\text{Sing}(X))) = 1\). Up to a normalization we assume \(S = \{0\}\) and \(\text{Sing}(X) = \{\infty\}\). We have \(f'(z) = cz^d\) with \(c \in \mathbb{C} \setminus \{0\}\) for every \(z \in \mathbb{C}\). Now everything follows from the discussion of Property \((S; d)\) made in the proof of Proposition 1.8.
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