LE MATEMATICHE
Vol. LIV (1999) – Fasc. II, pp. 353–360

HOLOMORPHIC SELF-MAPS OF SINGULAR PROJECTIVE CURVES

EDOARDO BALLICO

Here we classify all complex singular irreducibile projective curves. X, such that there exists a holomorphic map $f : X \to X$ with f locally biholomorphic at each point of X_{reg} and with $\deg(f) \geq 2$: X is rational and either it has a unique singular point with two branches or it has exactly two singular points, both unibranch.

Let $f : X \to X$ be a "locally invertible" moephism in a category. Must f be invertible? The corresponding problem was studies in [1] for the categories of compact differentiable manifolds and (but only if $\dim(X) = 2$) of compact complex manifolds. The problem arised from [2] in wich it was studied the corresponding problem for maps $h : A \to B$ in wich A and B may be different differentiable manifolds. The motivation behind [3] was explained at the end of the introduction of [3] and in [3], sections 4.2, 4.3 and 4.4; key words: Market Equilibrium, Limited Arbitrage and Uniqueness with Short Sales. Here we will never met such words. Both from the classification point of view and for the applications it seems important to assume X compact. However, if X is compact and smooth the existence of such f with $\deg(f) \geq 2$ is very restrictive. If X is a singular compact complex space, we cannot hope to have such non-trivial pairs (X, f) with f locally biholomorphic at each point of X (see Remark 1.2

Entrato in Redazione il 13 febbraio 2000.
The author was partially supported by MURST (Italy).

and 1.3). The best we can hope is that f is locally invertible at each smooth point of X. Our main result is the following theorem.

Theorem 0.1. *Let X be an integral projective curve with* $\mathrm{Sing}(X) \neq \phi$ *and such there exists a finite morphism $f : X \to X$ of degree $d \geq 2$ which is locally biholomorphic at each point of R_{reg}. Let $\pi : Y \to X$ be the normalization. We have $Y \cong \mathbf{P}^1$ and* $\mathrm{card}(\pi^1(\mathrm{Sing}(X)) = 2$. *There is a morphism $f' : \mathbf{P}^1 \to \mathbf{P}^1$ with $f' \circ \pi = \pi \circ f$ and in particular* $\deg(f') = d$. *The map f' is locally biholomorphic at each point of $\pi^1(X_{\mathrm{reg}})$ and for every $Q \in \pi^1(\mathrm{Sing}(X))$ we have* $\mathrm{card}(f^{-1}(Q)) = 1$, *i.e. f is totally ramified at Q. Two cases may occur:*

Case A) $\mathrm{card}(\mathrm{Sing}(X)) = 2$ *and each singular point of X is unibranch.*

Case B) $\mathrm{card}(\mathrm{Sing}(X)) = 1$ *and the singular point of X has exactly two branches.*

Viceversa, given any such curve X there is an integer $d \geq 2$ and a degree d holomorphic map $f : X \to X$ such that X is locally biholomorphic at each point of X_{reg}.

We are even able to classify all such map f or, equivalently, all such maps f'.

Remark 0.2. Up to an element of $\mathrm{Aut}(\mathbf{P}^1)$ and a non-zero multiplicative constant the map f' in the statement of 0.1 is uniquely determined by the integer d and, possibly,the interchange of the two points of $\pi^{-1}(\mathrm{Sing}(X))$. Taking $\pi^{-1}(\mathrm{Sing}(X)) = \{0, \infty\}$, we will see in 1.5 that $f'(z) = cz^d$ (resp. $f'(z) = cz^{-d}$) for some $c \in \mathbf{C}\backslash\{0\}$ if $f'(0) = 0$ and $f'(\infty) = \infty$ (resp if $f'(0) = \infty$ and $f'(\infty) = 0$).
For more details on the solutions in Cases A) and B) and in particular on the possible integers $d := \deg(f)$, see Propositions 1.8, 1.9 and 1.10.

1.– Here we will prove Theorem 0.1 and Remark 0.2. Then we will show how to solve two related problems on self-maps of pointed projective curves (see 1.11 and 1.12).

Lemma 1.1. *Let X be a complex compact irreducibile variety with* $\mathrm{Sing}(X)$ *finite and $f : X \to X$ a finite morphism of degree $d > 1$ such that $f|X_{\mathrm{reg}}$ is locally biholomorphic. Let $\pi : Y \to X$ be the normalization map. Then there exists a finite morphism of degree $d > 1$ $f' : Y \to Y$ such that $f \circ \pi = \pi \circ f'$ and $f'|\pi^1(X_{\mathrm{reg}})$ is locally biholomorphic. If* $\dim(X) \geq 2$, *f' is locally biholomorphic at each point of Y_{reg}.*

Proof. The holomorphic map $f \circ \pi : Y \to X$ is finite and surjective . By the universal property of the normalization there is a unique holomorphic map $f' : Y \to Y$ such that $f \circ \pi = \pi \circ f'$. The holomorphic map f' is locally biholomorphic at each point of $Y \backslash \pi^{-1}(\mathrm{Sing}(X)$. Assume $\dim(X) \geq 2$, i.e. $\dim(Y) \geq 2$. Since the discriminant locus of a finite holomorphic map between complex manifolds of the same dimension is either empty or a pure one-codimensional hypersurface and $\dim(\pi^{-1}(\mathrm{Sing}(X)) = 0 \leq \dim(Y) - 2$, f' is locally biholomorphic at each point of Y_{reg}.

Remark 1.2. Let X be a complex compact irreducible variety with $\mathrm{Sing}(X)$ finite and $f : X \to X$ a finite morphism of degree $d > 1$ such that $f|X_{\mathrm{reg}}$ is locally biholomorphic. Hence for every $P \in X_{\mathrm{reg}}$, $f(P) \notin \mathrm{Sing}(X)$. since X is compact, f is surjective. Since $\mathrm{Sing}(X)$ is finite, $\mathrm{Sing}(X) = f^{-1}(\mathrm{Sing}(X))$ and f induces a permutation of the finite set $\mathrm{Sing}(X)$. Hence there is an integer $k \geq 1$ such that the iteration $f^k := f \circ \cdots \circ f$ (k times) of f fixes every point of $\mathrm{Sing}(X)$.

Remark 1.3. Remark 1.2 explains why we do not assume that f is locally biholomorphic for every $P \in X$: since $\mathrm{Sing}(X)$ is finite and $\deg(f) > 1$, this would force $\mathrm{Sing}(X) = \phi$. The case $\mathrm{Sing}(X) = \phi$ is obvious by the next well-known remark.

Remark 1.4. Let C be a smooth projective curve of genus q and $u : C \to C$ a finite morphism of degree $d > 1$. By the Riemann-Hurwitz formula ([4], IV. 2.4) we have $2q - 2 = d(2q - 2) + b$, with $b = 0$ if u is locally biholomorphic and $b > 0$, otherwise. Hence if u is locally biholomorphic, then $q = 1$, i.e. C is an elliptic curve, while if u is not locally biholomorphic, then $q = 0$, i.e. $C \cong \mathbf{P}^1$. Viceversa, for every elliptic curve C and all integers t, n with $t \geq 2$ and $n \geq 2$ there are holomorphic maps $u : C \to C$ and $\nu : \mathbf{P}^1 \to \mathbf{P}^1$ with $\deg(u) = t^2$ and $\deg(\nu) = n$. By Riemann-Hurwitz formula every such u is locally biholomorphic and no such ν is locally biholomorphic.

Lemma 1.5. *Let X be an integral projective curve with* $\mathrm{Sing}(X) \neq \phi$ *and such that there exists a finite morphism $u : X \to X$ of degree $d > 1$ which is locally biholomorphic at each point of X_{reg}. We have f^{-1}*$(\mathrm{Sing}(X)) = \mathrm{Sing}(X)$ *and f induces a bijection of* $\mathrm{Sing}(X)$ *onto itself. Let $\pi : Y \to X$ be the normalization. We have $Y \cong \mathbf{P}^1$. There is a morphism $f' : \mathbf{P}^1 \to \mathbf{P}^1$ with $f' \circ \pi = \pi \circ f$ and in particular* $\deg(f) = d > 1$. *The map f' is locally biholomorphic at each point of $\pi^1(X_{\mathrm{reg}})$ and for every $Q \in \pi^{-1}$*$(\mathrm{Sing}(X))$ *we have card*$(f'^{-1}(Q)) = 1$, *i.e. f' is totally ramified at Q. We have* $\mathrm{card}(\pi^{-1}(\mathrm{Sing}(X)) = 2$. *Up to an element of* $\mathrm{Aut}(\mathbf{P}^1)$ *the map f' is uniquel determined, up to a non-zero constant,* *by the integer d and* $f'|\pi^1(\mathrm{Sing}(X))$*: taking* $\pi^1(\mathrm{Sing}(X)) = \{0, \infty\}$*, we have* $f'(z) = cz^d$ *(resp.* $f'(z) = cz^d$*) with* $c \in \mathbf{C}\backslash\{0\}$ *if* $f'(0) = 0$ *and* $f'(\infty) = \infty$ *(resp. if* $f'(0) = \infty$ *and* $f'(\infty) = 0$*).*

Proof. Since Y is the normalization of X, the degree d morphism $\pi \circ f : Y \to X$ factors through the normalization map, π, of the target, proving the existence of f'. By Remark 1.4 either $Y \cong \mathbf{P}^1$ or Y is an elliptic curve. Since f is locally biholomorphic at each point of X_{reg}, $f(X_{\mathrm{reg}}) \subseteq X_{\mathrm{reg}}$. Since X is irreducibile and compact, u is surjective. Hence $\mathrm{Sing}(X) \subseteq f(\mathrm{Sing}(X))$. Since $\mathrm{Sing}(X)$ is finite, we have $f^{-1}(\mathrm{Sing}(X)) = \mathrm{Sing}(X)$ and u induces a bijection of $\mathrm{Sing}(X)$ onto itself (Remark 1.2). From the finiteness of $\pi^{-1}(\mathrm{Sing}(X))$ and the relation $\pi \circ f' = f \circ \pi$, we obtain $f'^{-1}(\pi^{-1}(\mathrm{Sing}(X)) = \pi^{-1}(\mathrm{Sing}(X))$ and f' induces a bijection of $\pi^{-1}(\mathrm{Sing}(X))$ onto itself. Hence f' is totally ramified at every point of $\pi^{-1}(\mathrm{Sing}(X))$, i.e. the ramification order of f at X is at least $d - 1$. Thus the degree, z, of the ramification divisor of is at least $(d-1)\cdot \mathrm{card}(\pi^{-1}(\mathrm{Sing}(X))$. Since $Y \cong \mathbf{P}^1$, the Rieman-Hurwitz formula gives the relation $-2 = -2d + z$. Hence $\mathrm{card}(\pi^{-1}(\mathrm{Sing}(X)) \leq 2$. Since for every $P \in \mathbf{P}^1$, $\mathbf{P}^1\backslash\{P\} \cong \mathbf{C}$ is simply connected, we have $\mathrm{card}(\pi^{-1}(\mathrm{Sing}(X)) = 2$ and hence $1 \leq \mathrm{card}(\mathrm{Sing}(X)) \leq 2$. Now the last assertion is elementary. We assume that Y is an elliptic curve. By the Riemann-Hurwitz formula (see Remark 1.3) every non-constant holomorphic map between two elliptic curves is locally biholomorphic. As in the previous case f induces a degree d morphism $f' : Y \to Y$ wich induces a permutation of $\pi^{-1}(\mathrm{Sing}(X))$. Since $\mathrm{Sing}(X) \neq \phi$ and $\deg(f') = d > 1$, f' cannot be locally biholomorphic, contradiction.

Definition 1.6. Fix an integer $d \geq 2$. Let (X, P) the germ of a unibranch singularity of curves and R the completion of the associated local ring. Since (X, P) is unibranch, R may embedded as a unitary $\mathbf{C}$-local ring in the power series ring $\mathbf{C}[[t]]$ in one variable. We will say that (X, P) has Property $(\$\,; d)$ if there is an embedding $j : R \to \mathbf{C}[[t]]$ such that $j(R)$ has generators $1, p_1(t), \ldots, p_e(t)$ ($e := \dim(T_pX)$) and $j(R)$ contains $p_i(t^d)$ for every i with $1 \leq i \leq e$. Property $(\$\,; d)$ depends only on the one-dimensional domain R and hence we are allowed to say that R has Property $(\$\,; d)$ or not.

Remark 1.7. See $\mathbf{C}[[t]]$ as the completion of the local ring of the affine line $A^1 = \mathrm{Spec}(\mathbf{C}[t])$ at 0 and call $u : \mathbf{A}^1 \to \mathbf{A}^1$ the morphism with $u(z) := z^d$. Let R be the local ring of a unibranch curve singularity and fix an embedding j of R in $\mathbf{C}[[t]]$. This embedding may be used to prove that R has Property $(\$\,; d)$ if and only if $u^*(j(R)) \subseteq R$. Iterating the morphism u we see that if R has Property $(\$\,; d)$, then it has Property $(\$\,; d^k)$.

Proposition 1.8. *Fix an integer $d \geq 2$. Let X be an integral projective curve with* card(Sing(X)) ≥ 2 *such that there exists a degree d holomorphic map $f : X \to X$ wich is locally biholomorphic at each point of X_{reg}. Then* card(Sing(X)) $= 2$. *The normalization of X is $\mathbf{P}^1$, each singular point of X is unibranch and f induces a bijection of* Sing(X) *onto itself. Each singular point of X has Property* ($\$; d$). *If this bijection is not the identity, then $f^2 : X \to X$ has degree d^2, it is locally biholomorphic at each point of X_{reg} and $f^2(P) = P$ for every $P \in$* Sing(X). *Viceversa, for every integral curve X with* card(Sing(X)) $= 2$, *only unibranch singularities, with $\mathbf{P}^1$ as normalization and such that every singular point of X has Property* ($\$; d$) *there is a holomorphic map $f : X \to X$ with* $\deg(f) = d$, *f locally biholomorphic at each point of X_{reg} and such that $f|$Sing(X) is the identity; such map is unique up to a non-zero constant. Furthermore, there is a unique, up to a non-zero constant, holomorphic map $f' : X \to X$ with* $\deg(f) = d$, *f' locally biholomorphic at each point of X_{reg} and such that f' interchange the two points of* Sing(X)*;*

Proof. The proof of Lemma 1.5 gives the first part except Property ($\$; d$). The second part, i.e. the viceversa part, is very easy using Remark 1.7; for the last assertion, just use any morphism $\mathbf{P}^1 \to \mathbf{P}^1$ which interchanges 0 and ∞. Notice that, for any fixed X and $u : \mathbf{P}^1 \to \mathbf{P}^1$ with $\deg(u) = d$, there is at most one morphism $f : X \to X$ inducing u. Hence from the viceversa part and the explicit description of u and Remark 1.7 we obtain that all singular points of X have Property ($\$; d$) or Property ($\$, d^2$).

Proposition 1.9. *Let R be the analytic or formal local ring of a unibranch curve singularity. Let k be its multiplicity. Then R has Property* ($\$; k$).

Proof. Let $\mathbf{m}$ be the maximal ideal of R and $e := \dim_{\mathbf{C}}(\mathbf{m}/\mathbf{m}^2)$ the embedding dimension of R. The normalization of R is $\mathbf{C}[[t]]$ (or take convergent power series in the analytic case) and, up to a change of coordinates, there are e power series $f_1(t), \ldots, f_e(t)$ such that R is the completion of the $\mathbf{C}$-subalgebra of $\mathbf{C}[[4]]$ generated by $1, f_1(t), \ldots, f_e(t)$ and such that $f_1(t) = t^k$ and all other powers of t appearing with non-zero coefficient in some $f_i(t)$, $i \geq 2$, have order at leask $k+1$ ([3], Remark 2.1.1). Every element of $\mathbf{C}[[t^k]]$ is of the form $g(f_1(t))$ with $g \in \mathbf{C}[[t]]$.

Proposition 1.10. *Fix an integer $d \geq 2$. Let X be an integral projective curve with* card(Sing(X)) $= 1$, *say* Sing(X) $= \{P\}$, *such that there exists a degree d holomorphic map $f : X \to X$ which is locally biholomorphic at each point of X_{reg}. Then the normalization of X is $\mathbf{P}^1$, X has two branches, say B' and B'', and f induces a permutations of these two branches. Then each branch of X has Property* ($\$; d$). *If this permutation is not the identity, then*

$f^2 : X \to X$ has degree d^2, it is locally biholomorphic at each point of $X_{\rm reg}$ and f^2 fixes the two branches B' and B''. Viceversa, for every integral curve X with card(Sing(X)) $= 1$, *say* Sing(X) $= \{P\}$, *with exactly two branches at P, with $\mathbf{P}^1$ as normalization and such that every branch of X has Property* ($\$; d$) *there is a holomorphic map $f : X \to X$ with* $\deg(f) = d$, *f locally biholomorphic at each point of $X_{\rm reg}$ and such that f fixes the branches; such map is unique up to a non-zero constant. Furthermore, there is a unique, up to a non-zero constant, holomorphic map $f : X \to X$ with* $\deg(f') = d$, *f' locally biholomorphic at each point of $X_{\rm reg}$ and such that f' interchanges the two branches of X at P.*

Proof. By Lemma 1.5 to prove the first part it is sufficient to show that the two branches of X at P have Property ($\$; d$). Let R be formal (or analytic) local ring of the germ (X, P). Since X has two branches and it is reduced, R has two minimal prime ideals p_1 and p_2 with $p_1 \cap p_2 = \{0\}$. Set $R_i := R/p_i$. Let X' be the partial normalization of X in which we have just separated the two branches, i.e. card(Sing(X')) $= 2$, say Sing(X') $= \{P_1, P_2\}$, and X' has formal local ring R_i at P_i. since $f(P) = P$, the map f induces an injective $\mathbf{K}$-homorphism $\Psi(p_i)$, $i = 1, 2$, is contained in a minimal prime of R, the map f induces a degree d map $f'' : X' \to X'$ which coincides with f on $X'_{\rm reg} \cong X_{\rm reg}$. Hence we may apply Proposition 1.8 to X' and conclude the proof of the first part. Take (X, P) as in the second part. Let R be formal (or analytic) local ring of the germ (X, P). Let X' be the partial normalization of X in which we have just separated the two branches, i.e. card(Sing(X')) $= 2$, say Sing(X') $= \{P_1, P_2\}$, and X' has formal local ring R_i at R_i. There is a degree d morphism $f_1 : X' \to X'$ which is locally biholomorphic over $X'_{\rm reg} \cong X_{\rm reg} = X\backslash\{P\}$; furthermore there is such morphism, f_2, which fixes the two points of Sing(X') and another one, f_3, which interchanges the two points; up to a non-zero multiplicative constant f_2 and f_3 are uniquely determined. It is sufficient to prove that every such f_1 induces a morphim $f : X \to X$ with $f|X_{\rm reg} = f_1|X'_{\rm reg}$. This is obvious set-theoretically and even topologically, but we need to check that the set-theoretic map is holomorphic at P. We have an inclusion $j : R \to R_1 \oplus R_2 \subset \mathbf{C}[[t_1]] \oplus \mathbf{C}[[t_2]]$, where the latter ring is the semilocal ring of the normalization of R. The conductor, J, of R in $\mathbf{C}[[t_1]] \oplus \mathbf{C}[[t_2]]$ is of the form $((t_1^{a_1}), (t_2^{a_2}))$ for somr integer $a_1 \geq 0$ and $a_2 \geq 0$. The homomorphism f'^* sends J into $((t_1^{da_1}, (t_2^{da_2}))$. Since $da_1 \geq a_1$ and $da_2 \geq a_2$, $((t_1^{da_1}), (t_2^{da_2}))$ is contained in the conductor J. Hence f'^* descends to a homomorphism $f^* : R \to R$, showing that R is holomorphic at P. The structure of the set of all possible maps f follows from the structure of all possible maps $f' : X' \to X'$ considered in 1.8.

Motivated by the theories of orbifolds, of algebraic pairs (Y, S) and of algebraic stacks, we show why our work gives a solution of the following two related problems.

Proposition 1.11. *Let Y be a smooth projective curve over $\mathbf{C}$ and $S \subset Y$ with S finite and $S \neq \phi$. Assume the existence of a holomorphic map $f : Y \to Y$ with $\deg(f) = d \geq 2$ such that $S = f^{-1}(f(S))$ and that f is locally biholomorphic at each point of $Y \backslash S$. Then $Y \cong \mathbf{P}^1$, $\operatorname{card}(S) = 2$. Up to an element of $\operatorname{Aut}(\mathbf{P}^1)$ and a non-zero multiplicative constant the map f is uniquely determined by the integer d and by the condition "f exchanges the two points of S or not": taking $S = \{0, \infty\}$, we have $f(z) = cz^d$ with $c \in \mathbf{C}\backslash\{0\}$ for every $z \in \mathbf{C}$ if $f(0) = 0$ and $f(\infty) = \infty$ and $f(z) = cz^d$ with $c \in \mathbf{C}\backslash\{0\}$ if $f(0) = \infty$ and $f(\infty) = 0$.*

Proof. By Remark 1.4 we have $Y \cong \mathbf{P}^1$. The second part follows from Riemann-Hurwitz formula exactly as in the proof of Proposition 1.8.

Proposition 1.12. *Let X be an integral projective curve over $\mathbf{C}$ with $\operatorname{Sing}(X) \neq \phi$ and $S \subset X_{\text{reg}}$ with S finite and $S \neq \phi$. Assume the existence of a holomorphic map $f : X \to X$ with $\deg(f) = d \geq 2$ such that $S = f^{-1}(f(S))$ and that f is locally biholomorphic at each point of $X \backslash (S \cup \operatorname{Sing}(X))$; Let $\pi : Y \to X$ be the normalization. Then $Y \cong \mathbf{P}^1$, $\operatorname{card}(S) = \operatorname{card}(\operatorname{Sing}(X)) = 1$ and X is unibranch at unique point, P, of $\operatorname{Sing}(X)$. Furthermore, X satisfies condition $(\$; d)$ at P. Viceversa, given any such pair (X, S), up to a normalization by $\operatorname{Aut}(\mathbf{P}^1)$, i.e. taking $S = \{0\}$ and $\operatorname{Sing}(X) = \{\infty\}$, there exists exactly one such holomorphic map f, up to a non-zero multiplicative constant.*

Proof. By Remark 1.4 we have $Y \cong \mathbf{P}^1$. By Remark 1.2 we have $\operatorname{Sing}(X) = f^{-1}(f(\operatorname{Sing}(X))$. The morphism f induces a degree d morphism $f' : Y \to Y$ with f' locally biholomorphic at each point of $\pi^{-1}(S \cup \operatorname{Sing}(X))$. Since $S \subset Y_{\text{reg}}$, $S \neq \phi$ and $\operatorname{Sing}(X) \neq \phi$, we obtain $\operatorname{card}(S) = \operatorname{card}(\operatorname{Sing}(X)) = \operatorname{card}(\pi^{-1}(\operatorname{Sing}(X))) = 1$. Up to a normalization we assume $S = \{0\}$ and $\operatorname{Sing}(X) = \{\infty\}$. We have $f'(z) = cz^d$ with $c \in \mathbf{C}\backslash\{0\}$ for every $z \in \mathbf{C}$. Now everything follows from the discussion of Property $(\$; d)$ made in the proof of Proposition 1.8.

REFERENCES

[1] E. Ballico, *Globally invertible differentiable or holomorphic maps,* Rend. Sem. Mat. Univ. Padova (to appear).

[2] A. Campillo, *Algebroid Curves in Positive Characteristic,* Lect. Notes in Math. 813, Springer-Verlag, Berlin, 1980.

[3] G. Chichilnisky, *Topology and invertible maps,* Advance Appl. Math. 21, (1998) pp. 113–123.

[4] R. Hartshorne, *Algebraic Geometry,* Springer-Verlag, 1977.

Dipartimento di Matematica,
Università di Trento,
38050 Povo Trento, (Italy)
e-mail: ballico@science.unitn.it