A REMARK ON THE PLANAR NON LINEAR
ELLiptIC OBLIQUE DERIVATIVE PROBLEM

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We prove that, if \( l \) is an unit vector field tangential to the boundary of \( \Omega, \partial \Omega \), at a finite number of points, the planar non linear elliptic derivative problem

\[
\begin{aligned}
A(x, H(u)) - \lambda u &= f \quad \text{a.e. in } \Omega \subset \mathbb{R}^2 \\
n\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n,
\end{aligned}
\]

admits a unique solution in the Sobolev space \( W^{2,2}(\Omega) \).

1. Introduction.

In this paper we are concerned with the “strong” solvability, namely in the Sobolev space \( W^{2,2}(\Omega) \), of the oblique derivative problem

\[
\begin{aligned}
A(x, H(u)) - \lambda u &= f(x) \quad \text{a.e. in } \Omega \subset \mathbb{R}^2 \\
n\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n,
\end{aligned}
\]

where \( \lambda \) is a number greater than zero; \( A \) is a mapping satisfying Carathéodory’s condition and Campanato’s (A)-condition (see[1]); \( l \) is an unit vector field tangential to the boundary of \( \Omega, \partial \Omega \), at a finite number of points.

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For \( \lambda = 0 \) the linear planar problem for operator

\[
\mathcal{L}u = \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}
\]

has been well studied by G. Talenti [16], who established a \( W^{2,2}(\Omega) \) solvability, assuming \( a_{ij} \) to be measurable functions and \( \frac{\partial u}{\partial n} - \chi > 0 \). Here \( \emptyset \) denotes the angle between the unit vector \( l \) and the normal \( n, \varphi \) the curvilinear parameter relative to \( \partial \Omega \) and \( \chi \) the mean curvature of \( \partial \Omega \).

In the multidimensional case the strong solvability of linear oblique regular derivative problem for complete operators

\[
\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u
\]

has been studied by C. Miranda [12], M. Chicco [5] and G. Viola [17] if \( a_{ij} \in W^{1,\alpha}(\Omega) \) and by M. Chicco [4] and F. Nicolosi [14] if \( a_{ij} \) are measurable functions satisfying the Cordes’s condition and other additional assumptions.

G. Di Fazio-D.K. Palagachev [6] and A. Mauger-D.K. Palagachev [11] have studied the linear regular oblique derivative problem, assuming the coefficients of the principal part of the operator \( \mathcal{L} \) to belong to the space \( VMO \) of functions with vanishing mean oscillations. They generalize so the previous results, because if \( a_{ij} \in C^1(\Omega) \), or \( a_{ij} \in W^{1,\alpha}(\Omega) \), then \( a_{ij} \in VMO \). The previous results have been successfully applied to the study of oblique derivative problems for quasi-linear elliptic operators with \( VMO \) principal coefficients (cfr. G. Di Fazio-D.K. Palagachev [7]).

Recently the regular planar oblique derivative problem has been studied for discontinuous nonlinear operators by S. Giuffrè [9], who has studied also the tangential oblique derivative problem for nonlinear discontinuous operators in the plane (cfr. S. Giuffrè [10]).

In the present paper we shall prove, in Theorem 2.1, the solvability of problem (1.1) in the Sobolev Space \( W^{2,2}(\Omega) \); also we shall prove, in Theorem 2.2, that there exists a number \( q_0 > 2 \) such that for each \( q \in (2, q_0) \), and for each \( f \in L^q(\Omega) \), problem (1.1) admits a unique solution \( u \in W^{2,q}(\Omega) \). At last, in Theorem 2.3, we give a result on the eigen-values of the operator \( A(x, H(u)) \).
2. Notations and Hypothesis.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set with boundary $\partial \Omega$ of class $C^2$; let us assume that $\partial \Omega$ be a closed curve and let $x_1 = x_1(\varphi), x_2 = x_2(\varphi) \in [0, L]$ be parametric equations of $\partial \Omega$, with $\varphi$ the curvilinear parameter. Let $l = (Y_1(\varphi), Y_2(\varphi))$ be a unit vector field such that $Y_i(0) = Y_i(L), i = 1, 2$. Setting $n = (X_1, X_2)$ for the unit outward normal to $\partial \Omega$, and denoting by $\theta$ the angle between the unit vector $l$ and the normal $n$, we assume that

\begin{equation}
\cos \theta = \sum_{i=1}^{2} X_i(\varphi)Y_i(\varphi) \geq 0 \quad \forall \varphi \in [0, L]
\end{equation}

with $\cos \theta = 0$ at a finite number of points $\varphi_j \in ]0, L[, j = 1, \ldots, n$, with $\varphi_1 < \varphi_2 < \ldots < \varphi_n$.

Assuming $\varphi_0 = 0, \varphi_{n+1} = L$, we suppose

\begin{align}
\begin{cases}
\theta \in C^1([\varphi_0, \varphi_1)), & \theta \in C^1((\varphi_j, \varphi_{j+1})), \\
& j = 1, \ldots, n-1, \quad \theta \in C^1((\varphi_n, L]); \\
\lim_{\varphi \to \varphi_j^+} \theta(\varphi) &= -\frac{\pi}{2} - 2(j-1)\pi; \\
\lim_{\varphi \to \varphi_j^-} \theta(\varphi) &= -\frac{3}{2} \pi - 2(j-1)\pi; \\
Y_i \in C^1([\varphi_0, \varphi_1]), & Y_i \in C^1((\varphi_j, \varphi_{j+1})), \\
& j = 1, \ldots, n-1, \quad Y_i \in C^1((\varphi_n, L]).
\end{cases}
\end{align}

Denoting by $\chi$ the mean curvature of $\partial \Omega$, we also suppose

\begin{align}
\begin{cases}
\chi \leq \frac{d\theta}{d\varphi} \leq 0; & \forall \varphi \in [\varphi_{j-1}, \varphi_j], \; j = 1, \ldots, n \\
d\frac{\theta(\varphi_j^-)}{d\varphi} < 0; & \frac{\theta(\varphi_j^+)}{d\varphi} < 0, \quad j = 1, \ldots, n.
\end{cases}
\end{align}

Furthermore we impose the following requirements:

$i_i)$ let $\mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^{2\times 2}$ be a mapping measurable in $x \in \Omega$ for each $\xi \in \mathbb{R}^{2\times 2}$, continuous in $\xi$ for almost all $x \in \Omega$, such that $\mathcal{A}(x, 0) = 0$ and verifying the next condition (A) introduced by S. Campanato [1]:


(A) there exist three positive constants $\alpha, \gamma, \delta$, with $\gamma + \delta < 1$, such that, 
\[ \forall \xi, \tau \in \mathbb{R}^{2 \times 2} \text{ and for almost all } x \in \Omega, \text{ it results:} \]
\[ \left| \sum_{i=1}^{2} \xi_{ii} - \alpha[A(x, \xi, +\tau) - A(x, \tau)] \right| \leq \gamma \| \xi \|_{2 \times 2} + \delta \left| \sum_{i=1}^{2} \xi_{ii} \right| \]
where by $\| \cdot \|_{2 \times 2}$ we denote the usual euclidean norm in $\mathbb{R}^{2 \times 2}$. Let us note that Campanato’s (A)-condition is equivalent to a condition of pseudomonotonicity and ensures that the derivatives $\frac{\partial A}{\partial \xi_{ij}}(x, \xi)$ exist almost everywhere. Moreover they are essentially bounded.

The following theorems hold:

**Theorem 2.1.** Under assumptions (2.1), (2.2), (2.3), $i_1$, $\forall f \in L^2(\Omega), \forall \lambda > 0$, the problem
\[
\begin{align*}
A(x, H(u)) - \lambda u &= f(x) \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]
is uniquely solvable in $W^{2,2}(\Omega)$ and it results
\[
\| H(u) \|_{L^2(\Omega)} \leq \frac{\alpha}{1 - k} \| f \|_{L^2(\Omega)}, \quad 0 < k < 1.
\]

**Theorem 2.2.** Under assumptions (2.1), (2.2), (2.3), $i_1$ there exists a number $q_0 > 2$, such that $\forall q \in [2, q_0), \forall f \in L^q(\Omega), \forall \lambda > 0$, the problem
\[
\begin{align*}
A(x, H(u)) - \lambda u &= f(x) \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]
admits a unique solution $u \in W^{2,q}(\Omega)$.

**Theorem 2.3.** Under assumptions (2.1), (2.2), (2.3), $i_1$ for each $\lambda > 0$, the problems
\[
\begin{align*}
A(x, H(u)) &= \lambda u \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega
\end{align*}
\quad \text{and} \quad
\begin{align*}
\Delta u &= \lambda u \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]
admit the same solutions and the possible eigen-values of the first problem are all numbers less or equal to zero.
Remark 2.1. Let us note that from the identities
\[
\frac{\partial u}{\partial l}(\varphi_j) = \sum_{i=1}^{2} u_{x_i}(\varphi_j)Y_i(\varphi_j) = 0 \quad j = 1, \ldots, n
\]
\[
\cos \theta(\varphi_j) = \sum_{i=1}^{2} X_i(\varphi_j)Y_i(\varphi_j) = 0 \quad j = 1, \ldots, n,
\]
it follows
\[
(2.7) \quad \frac{\partial u}{\partial \varphi}(\varphi_j) = 0 \quad j = 1, \ldots, n.
\]
In fact, it must be
\[
\begin{vmatrix}
 u_{x_1}(\varphi_j) & u_{x_2}(\varphi_j) \\
 X_1(\varphi_j) & X_2(\varphi_j)
\end{vmatrix} = 0 \quad j = 1, \ldots, n
\]
and then
\[
u_{x_1}(\varphi_j)X_2(\varphi_j) - u_{x_2}(\varphi_j)X_1(\varphi_j) = -\frac{\partial u}{\partial \varphi}(\varphi_j) = 0 \quad j = 1, \ldots, n.
\]

3. Preliminary results.

We recall some auxiliary results. Let us start with the following estimate due to G. Talenti [16].

Lemma 3.1. Under assumptions (2.2), (2.3), for every function \( u \in C^{2}(\Omega) \cap C^{3}(\Omega) \) such that \( \frac{\partial u}{\partial l} = 0 \) on \( \partial \Omega \), it results
\[
(3.1) \quad \|H(u)\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)}.
\]

For the reader’s convenience, we give the proof of Lemma 3.1, previously proved by G. Talenti, under the more general assumptions \( Y_i \in C^{1}([0, L]), \)
\( i = 1, 2 \) and \( \frac{\partial v}{\partial \varphi} - \chi \geq 0, \forall \varphi \in [0, L] \). We recall that in [16] are given some examples from which the necessity of the condition \( \frac{\partial v}{\partial \varphi} - \chi \geq 0 \) can be derived.
Proof. We will suppose $\cos \theta = 0$ in the unique point $\varphi = \varphi_1 \in [0, L]$. Let us set
\[ p_i = \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \quad p_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i, j = 1, 2. \]
In order to obtain estimate (3.1), taking into account the identity
\[ \sum_{i,k=1}^{2} p^2_{ik} + \sum_{i,k=1}^{2} (p_{ii} p_{kk} - p^2_{ik}) = (\Delta u)^2, \]
it is enough to prove that
\[ \int_{\Omega} (p_{11} p_{22} - p^2_{12}) \, dx \geq 0. \]
From the identity
\[ p_{11} p_{22} - p^2_{12} = \frac{1}{2} \frac{\partial}{\partial x_1} (p_1 p_{22} - p_2 p_{12}) \quad \text{and} \quad \frac{1}{2} \frac{\partial}{\partial x_2} (p_1 p_{22} - p_2 p_{12}), \]
by the Gauss-Green Theorem, we obtain
\[ \int_{\Omega} (p_{11} p_{22} - p^2_{12}) \, dx = \frac{1}{2} \int_{0}^{L} [(p_1 p_{22} - p_2 p_{12}) X_1 - (p_1 p_{21} - p_2 p_{11}) X_2] \, d\varphi. \]
We consider the system
\[ \begin{aligned}
    & p_1 Y_1 + p_2 Y_2 = 0 \\
    & -p_1 Y_2 + p_2 Y_1 = c(\varphi),
\end{aligned} \tag{3.2} \]
it results
\[ \begin{aligned}
    & p_1 = -c(\varphi) Y_2(\varphi) \\
    & p_2 = c(\varphi) Y_1(\varphi) \quad \forall \varphi \in [0, L]
\end{aligned} \tag{3.3} \]
and also $c^2(\varphi) = p^2_1 + p^2_2 \quad \forall \varphi \in [0, L]$.
Deriving the first equation of system (3.2) in $[0, \varphi_1]$ and in $[\varphi_1, L]$, we obtain
\[ p_{11} X_2 Y_1 - p_{12} X_1 Y_1 + p_{21} X_2 Y_2 - p_{22} X_1 Y_2 = p_1 Y_1' + p_2 Y_2'. \]
Taking into account (3.3) and the last identity, we obtain in $[0, \varphi_1]$ and in $[\varphi_1, L]$
\[ \int_{\Omega} (p_{11} p_{22} - p^2_{12}) \, dx = \]
\[= \frac{1}{2} \int_{0}^{\varphi_1} c(\varphi) [p_1 Y_1' + p_2 Y_2'] \, d\varphi + \frac{1}{2} \int_{\varphi_1}^{L} c(\varphi) [p_1 Y_1' + p_2 Y_2'] \, d\varphi. \]
From (3.3) and bearing in mind that $Y_2^BY_1^I - Y_1^BY_2^I = \frac{d\theta}{d\varphi} - \chi$, $\forall \varphi \in [0, \varphi_1]$ and $\forall \varphi \in [\varphi_1, L]$, we have

$$\int_\Omega (p_{11}p_{22} - p_{12}^2) \, dx =$$

$$= \frac{1}{2} \int_0^{\varphi_1} (p_{12}^2 + p_{22}^2) \left( \frac{d\theta}{d\varphi} - \chi \right) \, d\varphi + \frac{1}{2} \int_{\varphi_1}^L (p_{12}^2 + p_{22}^2) \left( \frac{d\theta}{d\varphi} - \chi \right) \, d\varphi.$$ 

Then, by condition (2.3) we obtain estimate (3.1).

We observe that estimate (3.1) holds true also for a more extended class of functions $W_i$, where $W_i$ is the closure in $W^{2,2}(\Omega)$ of the set of functions $u$ belonging to $C^2(\overline{\Omega}) \cap C^3(\Omega)$ such that $\partial_u = 0$ on $\partial \Omega$.

In the general case $q \in [2, +\infty)$ the following result holds (cfr. M. Chicco [4]):

**Lemma 3.2.** For each function $u \in W^{2,q}(\Omega)$ such that $\frac{\partial u}{\partial \varphi} = 0$ on $\partial \Omega$, we have

$$\|H(u)\|_{L^1(\Omega)} \leq c(q) \|\Delta u\|_{L^1(\Omega)},$$

where $c(q) : [2, +\infty) \rightarrow [1, +\infty)$ is a continuous function at $q = 2$ and $c(2) = 1$.

We will need also the following result:

**Lemma 3.3.** For each function $f \in L^q(\Omega), q > 1$, and for each $\alpha > 0$, the oblique derivative problem

$$\begin{cases}
\Delta u - \alpha \lambda u = f(x) & \text{a.e. in } \Omega \\
\frac{\partial u}{\partial \varphi} = 0 & \text{on } \partial \Omega \\
u(\varphi_j) = 0 & j = 1, \ldots, n
\end{cases}$$

is uniquely solvable in the Sobolev space $W^{2,q}(\Omega)$.

**Proof.** We consider the case of a unique discontinuity point $\varphi_1$ such that

$$\lim_{\varphi \to \varphi_1^-} \theta(\varphi) = -\frac{\pi}{2}, \quad \lim_{\varphi \to \varphi_1^+} \theta(\varphi) = -\frac{3\pi}{2}.$$ 

Let us consider the vector field $l^* = (Y_1^*, Y_2^*)$ such that

$$Y_i^*(\varphi) = \begin{cases}
Y_i(\varphi) & \text{for } \varphi \in [0, \varphi_1], \\
-Y_i(\varphi) & \text{for } \varphi \in [\varphi_1, L]
\end{cases} \quad i = 1, 2.$$
By denoting \( \theta^*(\varphi) \) the angle between \( l^* \) and \( n \), it results
\[
\cos \theta^*(\varphi) = \begin{cases} 
\cos \theta(\varphi) & \text{for } \varphi \in [0, \varphi_1], \\
- \cos \theta(\varphi) = \cos(\theta(\varphi) + \pi) & \text{for } \varphi \in [\varphi_1, L].
\end{cases}
\]

\( l^*(\varphi) \) turns to be a continuous field (recall also condition (2.7) \( \frac{\partial u}{\partial l^*}(\varphi_1) = \frac{\partial u}{\partial \varphi}(\varphi_1) = 0 \)).

The problem
\[
(3.6)
\begin{align*}
\Delta u - \alpha \lambda u &= f(x) \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l^*} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_1) &= 0
\end{align*}
\]

admits a unique solution in \( W^{2,q}(\Omega) \). In fact the problem
\[
(3.6)' \begin{align*}
\Delta u - \alpha \lambda u &= f(x) \quad \text{a.e. in } \Omega \\
\frac{\partial u}{\partial l^*} &= 0 \quad \text{on } \partial \Omega 
\end{align*}
\]
is always a non-degenerate one. Then the problem has a finite index, i.e. finite dimensional kernel and cokernel. The kernel is not trivial, because the field \( l^* \) makes a turn around the normal \( n \). Prescribing, then, the value of the solution \( u \) at the point, \( u(\varphi_1) = 0 \), we obtain that the operator
\[
T : u \in W^q_1 \rightarrow W^{2,q}(\Omega)
\]
has a trivial kernel and cokernel, and closed range, namely \( L^q(\Omega) \). Then (see [13], Teorema 82.1), problem (3.6)' is uniquely solvable in \( W^{2,q}(\Omega) \). Therefore the assertion follows by observing that the solution \( u \) to the problem (3.6) is also solution to the problem (3.5).

We now recall some definitions and results, which will be useful for the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3.

**Definition 3.1.** Let \( \mathcal{B} \) be a set and \( \mathcal{B}_1 \) a real Banach space. We consider two mappings \( A \) and \( B \) defined on \( \mathcal{B} \) with values in \( \mathcal{B}_1 \). The mapping \( A \) is said to be near \( B \) if there exist two positive constants \( \alpha \) and \( k \), \( 0 < k < 1 \), such that for all \( u, v \in \mathcal{B} \) we have
\[
\| B(u) - B(v) - \alpha [A(u) - A(v)] \|_{\mathcal{B}_1} \leq k \| B(u) - B(v) \|_{\mathcal{B}_1}.
\]
The next Theorem is proved by S. Campanato [3].

**Theorem 3.1.** The mapping \( A : \mathcal{B} \to \mathcal{B}_1 \) is injective or surjective or bijective if and only if it is near to a mapping \( B : \mathcal{B} \to \mathcal{B}_1 \) which is injective or surjective or bijective respectively. Moreover, there is the estimate:

\[
\| B(u) - B(0) \|_{\mathcal{B}_1} \leq \frac{\alpha}{1 - k} \| A(u) - A(0) \|_{\mathcal{B}_1},
\]

4. **Proof of the theorems.**

Let us recall that

\[
W_i = \{ u \in C^2(\overline{\Omega}) \cap C^3(\Omega) : \frac{\partial u}{\partial l} = 0 \text{ on } \partial \Omega, u(\varphi_j) = 0, j = 1, \ldots, n \}
\]

and let us prove the following:

**Lemma 4.1.** Under assumptions (2.1), (2.2), (2.3), for every \( u \in W_i \) it results

\[
(\Delta u/u)_{W^2,2(\Omega)} \leq 0.
\]

**Proof.** We shall prove Lemma 4.1 supposing \( \cos \theta = 0 \) in the unique point \( \varphi = \varphi_1 \in \]0, L[. The inequality (4.1) is equivalent to

\[
\int_{\Omega} u \Delta u \, dx \leq 0 \quad \forall u \in W_i.
\]

Taking into account the identity

\[
u \Delta u = \left[ \frac{\partial}{\partial x_1} (up_1) + \frac{\partial}{\partial x_2} (up_2) \right] - (p_1^2 + p_2^2)
\]

and the Gauss-Green formula, we get

\[
\int_{\Omega} u \Delta u \, dx = \int_0^L u[p_1 X_1 + p_2 X_2] \, d\varphi - \int_{\Omega} (p_1^2 + p_2^2) \, dx.
\]
We observe that
\[
\int_0^L u[p_1 X_1 + p_2 X_2] \, d\varphi = \\
= \lim_{\varepsilon \to 0^+} \left( \int_0^{\varphi_1 - \varepsilon} u[p_1 X_1 + p_2 X_2] \, d\varphi + \int_{\varphi_1 + \varepsilon}^{L} u[p_1 X_1 + p_2 X_2] \, d\varphi \right).
\]
In \([0, \varphi_1 - \varepsilon]\) and \([\varphi_1 + \varepsilon, L]\) it results \(\cos \theta > 0\), then from the system
\[
\begin{aligned}
p_1 Y_1 + p_2 Y_2 &= 0 \\
p_1 x_1' + p_2 x_2' &= \frac{du}{d\varphi}
\end{aligned}
\]
we get
\[
p_1 = -\frac{Y_2}{\cos \theta} \frac{du}{d\varphi}, \quad p_2 = \frac{Y_1}{\cos \theta} \frac{du}{d\varphi}.
\]
Then
\[
\int_0^L u[p_1 X_1 + p_2 X_2] \, d\varphi = \\
= -\lim_{\varepsilon \to 0^+} \left[ \int_0^{\varphi_1 - \varepsilon} u \frac{du}{d\varphi} \tan \theta \, d\varphi + \int_{\varphi_1 + \varepsilon}^{L} u \frac{du}{d\varphi} \tan \theta \, d\varphi \right].
\]
In virtue of (2.7), we obtain for all \(\varphi \in [0, \varphi_1]\) and for a suitable point \(\overline{\varphi} \in ]\varphi, \varphi_1[:\)
\[
\frac{du(\varphi)}{d\varphi} = \frac{du(\varphi_1)}{d\varphi} - \frac{du'(\varphi_1)}{d\varphi} = (\varphi - \varphi_1) \frac{d^2 u(\overline{\varphi})}{d\varphi^2}.
\]
Bearing in mind condition (2.3)
\[
\lim_{\varphi \to \varphi_1^+} \frac{u(\varphi) \frac{du}{d\varphi} \tan \theta}{\varphi - \varphi_1} = \lim_{\varphi \to \varphi_1^+} \frac{u(\varphi)(\varphi - \varphi_1) \frac{d^2 u(\overline{\varphi})}{d\varphi^2} \tan \theta}{\varphi - \varphi_1} \leq \\
\leq M \lim_{\varphi \to \varphi_1^+} \frac{|u(\varphi)(\varphi - \varphi_1)|}{\cos \theta} = M \left| \frac{\frac{1}{d\theta(\varphi_1)}}{\cos \theta} \right| \lim_{\varphi \to \varphi_1^+} |u(\varphi)| = 0
\]
and then
\[
\lim_{\varepsilon \to 0^+} \int_0^{\varphi_1 - \varepsilon} u \frac{du}{d\varphi} \tan \theta \, d\varphi = \int_0^{\varphi_1} u \frac{du}{d\varphi} \tan \theta \, d\varphi.
\]
Similarly, we get
\[
\lim_{\epsilon \to 0^+} \int_{\varphi_1}^{L} u \frac{d \theta}{d \varphi} \tan \theta \, d \varphi = \int_{\varphi_1}^{L} u \frac{d \theta}{d \varphi} \tan \theta \, d \varphi.
\]

On the other hand
\[
\int_0^{\varphi_1 - \epsilon} u \frac{d \theta}{d \varphi} \tan \theta \, d \varphi = \frac{1}{2} \left[ u^2 \tan \theta \right]_{0}^{\varphi_1 - \epsilon} - \frac{1}{2} \int_0^{\varphi_1 - \epsilon} u^2(\varphi) \frac{d \theta}{\cos^2 \theta} \, d \varphi =
\]
\[
= \frac{1}{2} \left\{ u^2(\varphi_1 - \epsilon) \tan \theta(\varphi_1 - \epsilon) - u^2(0) \tan \theta(0) - \int_0^{\varphi_1 - \epsilon} u^2(\varphi) \frac{d \theta}{\cos^2 \theta} \, d \varphi \right\}.
\]
As above, we have for all \( \varphi \in [0, \varphi_1] \) and for a suitable \( \tilde{\varphi} \in ]\varphi, \varphi_1] \):
\[
u(\varphi) = u(\varphi) - u(\varphi_1) = (\varphi - \varphi) \frac{d u(\tilde{\varphi})}{d \varphi};
\]
so we obtain
\[
\lim_{\epsilon \to 0^+} \left| u^2(\varphi_1 - \epsilon) \tan \theta(\varphi_1 - \epsilon) \right| \leq K \lim_{\epsilon \to 0^+} \left| \frac{\epsilon^2}{\cos \theta(\varphi_1 - \epsilon)} \right| = 0;
\]
and similarly
\[
\lim_{\epsilon \to 0^+} \left| u^2(\varphi_1 + \epsilon) \tan \theta(\varphi_1 + \epsilon) \right| = 0.
\]

In an analogous way we get that \( \lim_{\varphi \to \varphi_1^+} \frac{u^2(\varphi)}{\cos^2 \theta} \) is finite.

Taking into account that \( Y_i(0) = Y_i(L), X_i(0) = X_i(L), i = 1, 2 \) implies \( \tan \theta(0) = \tan \theta(L) \), we get
\[
\int_\Omega u \Delta u \, dx = \frac{1}{2} \left\{ \int_0^{\varphi_1} u^2 \frac{d \theta}{\cos^2 \theta} \, d \varphi + \int_{\varphi_1}^L u^2 \frac{d \theta}{\cos^2 \theta} \, d \varphi \right\} - \int_\Omega (p_1^2 + p_2^2) \, dx
\]
and by condition (2.3), we obtain estimate (4.1). \( \square \)

Taking into account Lemma 4.1, we get (cfr. S. Campanato [2])

**Lemma 4.2.** Under the same assumptions of Lemma 4.1, if the mapping \( \mathcal{A}(x, H(u)) \) is near to \( \Delta u \) with constants \( \alpha \) and \( k \), then for all \( \lambda \geq 0 \) the mapping \( [\mathcal{A}(x, H(u)) - \lambda u] \) is near to the mapping \( [\Delta u - \alpha \lambda u] \) both considered as mappings from \( W_1 \to L^2(\Omega) \).
Now we are in position to prove Theorem 2.1. The problem

\[
\begin{align*}
\Delta u - \alpha \lambda u &= f(x) \in L^2(\Omega) \quad \text{a. e. in } \Omega \\
\frac{\partial u}{\partial t} &= 0 \quad \text{on } \partial \Omega \\
u(\phi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]

admits only one solution \( u \in W^{2,2}(\Omega) \) (cfr. Lemma 3.3), then the mapping \( \Delta u - \alpha \lambda u \) is bijective from \( W_1 \) in \( L^2(\Omega) \). The Campanato’s (A)-condition and Lemma 3.1 ensures us that the mapping \( \mathcal{A}(x, H(u)) \) is near to the Laplacian \( \Delta u \), both considered as mappings from \( W_1 \) in \( L^2(\Omega) \) (cfr. [3]). Then, from Lemma 4.1 and Lemma 4.2, we get that the mapping \( \mathcal{A}(x, H(u)) - \lambda u \) is near to mapping \( \Delta u - \alpha \lambda u \). By Theorem 3.2 the mapping \( \mathcal{A}(x, H(u)) - \lambda u \) is also bijective from \( W_1 \) in \( L^2(\Omega) \), so we derive the existence and the uniqueness of the solution \( u \in W^{2,2}(\Omega) \) of problem (2.4).

To prove estimate (2.5) let \( u \in W^{2,2}(\Omega) \) be the solution of problem (2.4); taking into account Lemma 3.1, estimate (4.1) and the nearness of the mappings \( \Delta u - \alpha \lambda u \) and \( \mathcal{A}(x, H(u)) - \lambda u \), with constants \( \alpha \) and \( k, 0 < k < 1 \), we get

\[
\|H(u)\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)} \leq \|\Delta u - \alpha \lambda u\|_{L^2(\Omega)} \leq \|H(u)\|_{L^2(\Omega)} \leq \|\Delta u - \alpha \lambda u\|_{L^2(\Omega)} \leq k\|\Delta u - \alpha \lambda u\|_{L^2(\Omega)} + \alpha\|f\|_{L^2(\Omega)}.
\]

From this, we obtain estimate (2.5). □

**Proof of Theorem 2.2.** The proof of Theorem 2.2 is analogous to the proof of Theorem 2.2 by S. Giuffrè [9]. □

**Proof of Theorem 2.3.** The mappings \([\mathcal{A}(x, H(u)) - \lambda u]\) and \([\Delta u - \alpha \lambda u]\) from \( W_1 \) in \( L^2(\Omega) \) are near. Then there exists \( k \in (0, 1) \) such that

\[
\|\Delta u - \alpha \lambda u - \alpha[\mathcal{A}(x, H(u)) - \lambda u]\|_{L^2(\Omega)} \leq k\|\Delta u - \alpha \lambda u\|_{L^2(\Omega)}.
\]

Hence, for all \( \lambda > 0 \), the function \( u \in W^{2,2}(\Omega) \) is solution of the problem

\[
\begin{align*}
\mathcal{A}(x, H(u)) &= \lambda u \quad \text{a. e. in } \Omega \\
\frac{\partial u}{\partial t} &= 0 \quad \text{on } \partial \Omega \\
u(\phi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]
if and only if \( u \) is solution of the problem

\[
\begin{align*}
\Delta u &= \alpha \lambda u \quad \text{a. e. in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \\
u(\varphi_j) &= 0 \quad j = 1, \ldots, n
\end{align*}
\]

that is \( u \equiv 0 \).

Then the mapping

\[\mathcal{A}(x, H(\cdot)) : W_1 \rightarrow L^2(\Omega)\]

have all possible eigenvalues less or equal to zero. \(\square\)

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