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A REMARK ON THE PLANAR NON LINEAR ELLIPTIC OBLIQUE DERIVATIVE PROBLEM

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We prove that, if l is an unit vector field tangential to the boundary of Ω, $\partial\Omega$, at a finite number of points, the planar non linear elliptic derivative problem

$$\begin{cases} A(x, H(u)) - \lambda u = f & \text{a.e. in } \Omega \subset \mathbb{R}^2 \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \dots, n, \end{cases}$$

admits a unique solution in the Sobolev space $W^{2,2}(\Omega)$.

1. Introduction.

In this paper we are concerned with the “strong” solvability, namely in the Sobolev space $W^{2,2}(\Omega)$, of the oblique derivative problem

$$\text{(1.1)} \qquad \begin{cases} \mathcal{A}(x, H(u)) - \lambda u = f(x) & \text{a.e. in } \Omega \subset R^2 \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \dots, n, \end{cases}$$

where λ is a number greater than zero; $\mathcal{A}$ is a mapping satisfying Carathèodory’s condition and Campanato’s (A)-condition (see[1]); l is an unit vector field tangential to the boundary of Ω, $\partial\Omega$, at a finite number of points.

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For $\lambda = 0$ the linear planar problem for operator

$$\mathcal{L}u = \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

has been well studied by G. Talenti [16], who established a $W^{2,2}(\Omega)$ solvability, assuming a_{ij} to be measurable functions and $\frac{d\theta}{d\varphi} - \chi > 0$. Here $\emptyset$ denotes the angle between the unit vector l and the normal n, φ the curvilinear parameter relative to $\partial\Omega$ and χ the mean curvature of $\partial\Omega$.
In the multidimensional case the strong solvability of linear oblique regular derivative problem for complete operators

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

has been studied by C. Miranda [12], M. Chicco [5] and G. Viola [17] if $a_{ij} \in W^{1,n}(\Omega)$ and by M. Chicco [4] and F. Nicolosi [14] if a_{ij} are measurable functions satisfying the Cordes's condition and other additional assumptions.
G. Di Fazio-D.K. Palagachev [6] and A. Maugeri-D.K. Palagachev [11] have studied the linear regular oblique derivative problem, assuming the coefficients of the principal part of the operator $\mathcal{L}$ to belong to the space VMO of functions with vanishing mean oscillations. They generalize so the previous results, because if $a_{ij} \in C^0(\overline{\Omega})$, or $a_{ij} \in W^{1,n}(\Omega)$, then $a_{ij} \in VMO$. The previous results have been successfully applied to the study of oblique derivative problems for quasi-linear elliptic operators with VMO principal coefficients (cfr. G. Di Fazio-D.K. Palagachev [7]).
Recently the regular planar oblique derivative problem has been studied for discontinuous nonlinear operators by S. Giuffrè [9], who has studied also the tangential oblique derivative problem for nonlinear discontinuous operators in the plane (cfr. S. Giuffrè [10]).
In the present paper we shall prove, in Theorem 2.1, the solvability of problem (1.1) in the Sobolev Space $W^{2,2}(\Omega)$; also we shall prove, in Theorem 2.2, that there exists a number $q_0 > 2$ such that for each $q \in [2, q_0)$, and for each $f \in L^q(\Omega)$, problem (1.1) admits a unique solution $u \in W^{2,q}(\Omega)$. At last, in Theorem 2.3, we give a result on the eigen-values of the operator $\mathcal{A}(x, H(u))$.

2. Notations and Hypothesis.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set with boundary $\partial\Omega$ of class C^2; let us assume that $\partial\Omega$ be a closed curve and let $x_1 = x_1(\varphi)$, $x_2 = x_2(\varphi) \in [0, L]$ be parametric equations of $\partial\Omega$, with φ the curvilinear parameter. Let $l = (Y_1(\varphi), Y_2(\varphi))$ be a unit vector field such that $Y_i(0) = Y_i(L)$, $i = 1, 2$. Setting $n = (X_1, X_2)$ for the unit outward normal to $\partial\Omega$, and denoting by θ the angle between the unit vector l and the normal n, we assume that

$$\cos\theta = \sum_{i=1}^{2} X_i(\varphi) Y_i(\varphi) \geq 0 \quad \forall\varphi \in [0, L] \tag{2.1}$$

with $\cos\theta = 0$ at a finite number of points $\varphi_j \in]0, L[$, $j = 1, \dots, n$, with $\varphi_1 < \varphi_2 < \dots < \varphi_n$.
Assuming $\varphi_0 = 0$, $\varphi_{n+1} = L$, we suppose

$$\begin{cases} \theta \in C^1([\varphi_0, \varphi_1)),\ \theta \in C^1((\varphi_j, \varphi_{j+1})), \\ \qquad\qquad j = 1, \dots, n-1,\ \theta \in C^1((\varphi_n, L]); \\ \lim\limits_{\varphi\to\varphi_j^-} \theta(\varphi) = -\dfrac{\pi}{2} - 2(j-1)\pi; \\ \lim\limits_{\varphi\to\varphi_j^+} \theta(\varphi) = -\dfrac{3}{2}\pi - 2(j-1)\pi;\ j = 1, \dots, n \\ Y_i \in C^1([\varphi_0, \varphi_1)),\ Y_i \in C^1((\varphi_j, \varphi_{j+1})), \\ \qquad\qquad j = 1, \dots, n-1,\ Y_i \in C^1((\varphi_n, L]), \\ \lim\limits_{\varphi\to\varphi_j^-} Y_i(\varphi) = -\lim\limits_{\varphi\to\varphi_j^+} Y_i(\varphi),\ i = 1, 2;\ j = 1, \dots, n. \end{cases} \tag{2.2}$$

Denoting by χ the mean curvature of $\partial\Omega$, we also suppose

$$\begin{cases} \chi \leq \dfrac{d\theta}{d\varphi} \leq 0; \quad \forall\varphi \in [\varphi_{j-1}, \varphi_j],\ j = 1, \dots, n \\ \dfrac{d\theta(\varphi_j^-)}{d\varphi} < 0; \quad \dfrac{d\theta(\varphi_j^+)}{d\varphi} < 0, \quad j = 1, \dots, n. \end{cases} \tag{2.3}$$

Futhermore we impose the following requirements:
i_1) let $\mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^{2\times 2}$ be a mapping measurable in $x \in \Omega$ for each $\xi \in \mathbb{R}^{2\times 2}$, continuous in ξ for almost all $x \in \Omega$, such that $\mathcal{A}(x, 0) = 0$ and verifying the next condition (A) introduced by S. Campanato [1]:

(A) there exist three positive constants α, γ, δ, with $\gamma + \delta < 1$, such that, $\forall \xi,\ \tau \in \mathbb{R}^{2\times 2}$ and for almost all $x \in \Omega$, it results:

$$\left|\sum_{i=1}^{2} \xi_{ii} - \alpha[\mathcal{A}(x, \xi, +\tau) - \mathcal{A}(x, \tau)]\right| \leq \gamma \|\xi\|_{2\times 2} + \delta \left|\sum_{i=1}^{2} \xi_{ii}\right|$$

where by $\|\cdot\|_{2\times 2}$ we denote the usual euclidean norm in $\mathbb{R}^{2\times 2}$.
Let us note that Campanato's (A)-condition is equivalent to a condition of pseudomonotonicity and ensures that the derivatives $\dfrac{\partial \mathcal{A}}{\partial \xi_{ij}}(x, \xi)$ exist almost everywhere. Moreover they are essentially bounded.
The following theorems hold:

Theorem 2.1. *Under assumptions* (2.1), (2.2), (2.3), i_1), $\forall f \in L^2(\Omega)$, $\forall \lambda > 0$, *the problem*

$$\text{(2.4)} \qquad \begin{cases} \mathcal{A}(x, H(u)) - \lambda u = f(x) & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \ldots, n \end{cases}$$

is uniquely solvable in $W^{2,2}(\Omega)$ *and it results*

$$\text{(2.5)} \qquad \|H(u)\|_{L^2(\Omega)} \leq \frac{\alpha}{1-k} \|f\|_{L^2(\Omega)}, \quad 0 < k < 1.$$

Theorem 2.2. *Under assumptions* (2.1), (2.2), (2.3), i_1) *there exists a number* $q_0 > 2$, *such that* $\forall q \in [2, q_0)$, $\forall f \in L^q(\Omega)$, $\forall \lambda > 0$, *the problem*

$$\text{(2.6)} \qquad \begin{cases} \mathcal{A}(x, H(u)) - \lambda u = f(x) & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \ldots, n \end{cases}$$

admits a unique solution $u \in W^{2,q}(\Omega)$.

Theorem 2.3. *Under assumptions* (2.1), (2.2), (2.3), i_1) *for each* $\lambda > 0$, *the problems*

$$\begin{cases} \mathcal{A}(x, H(u)) = \lambda u & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \ldots, n \end{cases} \quad \text{and} \quad \begin{cases} \Delta u = \lambda u & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \ldots, n \end{cases}$$

admit the same solutions and the possible eigen-values of the first problem are all numbers less or equal to zero.

Remark 2.1. Let us note that from the identities

$$\frac{\partial u}{\partial l}(\varphi_j) = \sum_{i=1}^{2} u_{x_i}(\varphi_j) Y_i(\varphi_j) = 0 \qquad j = 1, \dots, n$$

$$\cos\theta(\varphi_j) = \sum_{i=1}^{2} X_i(\varphi_j) Y_i(\varphi_j) = 0 \qquad j = 1, \dots, n,$$

it follows

$$\frac{\partial u}{\partial \varphi}(\varphi_j) = 0 \qquad j = 1, \dots, n. \tag{2.7}$$

In fact, it must be

$$\begin{vmatrix} u_{x_1}(\varphi_j) & u_{x_2}(\varphi_j) \\ X_1(\varphi_j) & X_2(\varphi_j) \end{vmatrix} = 0 \qquad j = 1, \dots, n$$

and then

$$u_{x_1}(\varphi_j) X_2(\varphi_j) - u_{x_2}(\varphi_j) X_1(\varphi_j) = -\frac{\partial u}{\partial \varphi}(\varphi_j) = 0 \qquad j = 1, \dots, n.$$

3. Preliminary results.

We recall some auxiliary results. Let us start with the following estimate due to G. Talenti [16].

Lemma 3.1. *Under assumptions* (2.2), (2.3)*, for every function* $u \in C^2(\overline{\Omega}) \cap$ $\cap C^3(\Omega)$ *such that* $\frac{\partial u}{\partial l} = 0$ *on* $\partial\Omega$*, it results*

$$\|H(u)\|_{L^2(\Omega)} \le \|\Delta u\|_{L^2(\Omega)}. \tag{3.1}$$

For the reader's convenience, we give the proof of Lemma 3.1, previously proved by G. Talenti, under the more general assumptions $Y_i \in C^1([0, L])$, $i = 1, 2$ and $\frac{d\theta}{d\varphi} - \chi \ge 0$, $\forall \varphi \in [0, L]$. We recall that in [16] are given some examples from which the necessity of the condition $\frac{d\theta}{d\varphi} - \chi \ge 0$ can be derived.

Proof. We will suppose $\cos\theta = 0$ in the unique point $\varphi = \varphi_1 \in]0, L[$. Let us set $p_i = \dfrac{\partial u}{\partial x_i}$, $i = 1, 2$, $p_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, 2$.
In order to obtain estimate (3.1), taking into account the identity

$$\sum_{i,k=1}^{2} p_{ik}^2 + \sum_{i,k=1}^{2} (p_{ii}p_{kk} - p_{ik}^2) = (\Delta u)^2,$$

it is enough to prove that

$$\int_\Omega (p_{11}p_{22} - p_{12}^2)\, dx \geq 0.$$

From the identity

$$p_{11}p_{22} - p_{12}^2 = \frac{1}{2}\frac{\partial}{\partial x_1}(p_1 p_{22} - p_2 p_{12}) - \frac{1}{2}\frac{\partial}{\partial x_2}(p_1 p_{21} - p_2 p_{11}),$$

by the Gauss-Green Theorem, we obtain

$$\int_\Omega (p_{11}p_{22} - p_{12}^2)\, dx = \frac{1}{2}\int_0^L [(p_1 p_{22} - p_2 p_{12})X_1 - (p_1 p_{21} - p_2 p_{11})X_2]\, d\varphi.$$

We consider the system

$$\text{(3.2)} \qquad \begin{cases} p_1 Y_1 + p_2 Y_2 = 0 \\ -p_1 Y_2 + p_2 Y_1 = c(\varphi), \end{cases}$$

it results

$$\text{(3.3)} \qquad \begin{cases} p_1 = -c(\varphi)Y_2(\varphi) \\ p_2 = c(\varphi)Y_1(\varphi) \end{cases} \qquad \forall \varphi \in [0, L]$$

and also $c^2(\varphi) = p_1^2 + p_2^2 \quad \forall \varphi \in [0, L]$.
Deriving the first equation of system (3.2) in $[0, \varphi_1]$ and in $[\varphi_1, L]$, we obtain

$$p_{11}X_2Y_1 - p_{12}X_1Y_1 + p_{21}X_2Y_2 - p_{22}X_1Y_2 = p_1Y_1' + p_2Y_2'.$$

Taking into account (3.3) and the last identity, we obtain in $[0, \varphi_1]$ and in $[\varphi_1, L]$

$$\int_\Omega (p_{11}p_{22} - p_{12}^2)\, dx = \\ = \frac{1}{2}\int_0^{\varphi_1} c(\varphi)[p_1Y_1' + p_2Y_2']\, d\varphi + \frac{1}{2}\int_{\varphi_1}^{L} c(\varphi)[p_1Y_1' + p_2Y_2']\, d\varphi.$$

From (3.3) and bearing in mind that $Y_2'Y_1 - Y_1'Y_2 = \frac{d\theta}{d\varphi} - \chi$, $\forall \varphi \in [0, \varphi_1]$ and $\forall \varphi \in [\varphi_1, L]$, we have

$$\int_\Omega (p_{11}p_{22} - p_{12}^2)\, dx =$$
$$= \frac{1}{2}\int_0^{\varphi_1} (p_1^2 + p_2^2)\left(\frac{d\theta}{d\varphi} - \chi\right) d\varphi + \frac{1}{2}\int_{\varphi_1}^{L} (p_1^2 + p_2^2)\left(\frac{d\theta}{d\varphi} - \chi\right) d\varphi.$$

Then, by condition (2.3) we obtain estimate (3.1). □

We observe that estimate (3.1) holds true also for a more extended class of functions W_l, where W_l is the closure in $W^{2,2}(\Omega)$ of the set of functions u belonging to $C^2(\overline{\Omega}) \cap C^3(\Omega)$ such that $\frac{\partial u}{\partial l} = 0$ on $\partial\Omega$.

In the general case $q \in [2, +\infty)$ the following result holds (cfr. M. Chicco [4]):

Lemma 3.2. *For each function $u \in W^{2,q}(\Omega)$ such that $\frac{\partial u}{\partial l} = 0$ on $\partial\Omega$, we have*

$$\|H(u)\|_{L^q(\Omega)} \leq c(q)\|\Delta u\|_{L^q(\Omega)}, \tag{3.4}$$

where $c(q) : [2, +\infty) \to [1, +\infty)$ is a continuous function at $q = 2$ and $c(2) = 1$.

We will need also the following result:

Lemma 3.3. *For each function $f \in L^q(\Omega)$, $q > 1$, and for each $\alpha > 0$, the oblique derivative problem*

$$\begin{cases} \Delta u - \alpha\lambda u = f(x) & \text{a.e. in } \Omega \\ \frac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \ldots, n \end{cases} \tag{3.5}$$

is uniquely solvable in the Sobolev space $W^{2,q}(\Omega)$.

Proof. We consider the case of a unique discontinuity point φ_1 such that

$$\lim_{\varphi \to \varphi_1^-} \theta(\varphi) = -\frac{\pi}{2}, \quad \lim_{\varphi \to \varphi_1^+} \theta(\varphi) = -\frac{3}{2}\pi.$$

Let us consider the vector field $l^* = (Y_1^*, Y_2^*)$ such that

$$Y_i^*(\varphi) = \begin{cases} Y_i(\varphi) & \text{for } \varphi \in [0, \varphi_1], \\ -Y_i(\varphi) & \text{for } \varphi \in [\varphi_1, L] \end{cases} \qquad i = 1, 2.$$

By denoting $\theta^*(\varphi)$ the angle between l^* and n, it results

$$\cos\theta^*(\varphi) = \begin{cases} \cos\theta(\varphi) & \text{for } \varphi \in [0, \varphi_1], \\ -\cos\theta(\varphi) = \cos(\theta(\varphi) + \pi) & \text{for } \varphi \in [\varphi_1, L]. \end{cases}$$

$l^*(\varphi)$ turns to be a continuous field (recall also condition (2.7) $\dfrac{\partial u}{\partial l^*}(\varphi_1) = \dfrac{\partial u}{\partial l}(\varphi_1) = \dfrac{\partial u}{\partial \varphi}(\varphi_1) = 0$).

The problem

$$\text{(3.6)} \qquad \begin{cases} \Delta u - \alpha\lambda u = f(x) & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l^*} = 0 & \text{on } \partial\Omega \\ u(\varphi_1) = 0 \end{cases}$$

admits a unique solution in $W^{2,q}(\Omega)$. In fact the problem

$$\text{(3.6)}' \qquad \begin{cases} \Delta u - \alpha\lambda u = f(x) & \text{a.e. in } \Omega \\ \dfrac{\partial u}{\partial l^*} = 0 & \text{on } \partial\Omega \end{cases}$$

is always a non-degenerate one. Then the problem has a finite index, i.e. finite dimensional kernel and cokernel. The kernel is not trivial, because the field l^* makes a turn around the normal n. Prescribing, then, the value of the solution u at the point, $u(\varphi_1) = 0$, we obtain that the operator

$$T : u \in W_l^q = \overline{\left\{u \in C^2(\overline{\Omega}) \cap C^3(\Omega) : \frac{\partial u}{\partial l^*} = 0 \text{ on } \partial\Omega, u(\varphi_1) = 0\right\}}^{W^{2,q}(\Omega)}$$
$$\to Tu = \Delta u - \alpha\lambda u \in L^q(\Omega)$$

has a trivial kernel and cokernel, and closed range, namely $L^q(\Omega)$. Then (see [13], Teorema 82. I), problem (3.6)$'$ is uniquely solvable in $W^{2,q}(\Omega)$. Therefore the assertion follows by observing that the solution u to the problem (3.6) is also solution to the problem (3.5). □

We now recall some definitions and results, which will be useful for the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3.

Definition 3.1. *Let $\mathcal{B}$ be a set and $\mathcal{B}_1$ a real Banach space. We consider two mappings A and B defined on $\mathcal{B}$ with values in $\mathcal{B}_1$. The mapping A is said to be near B if there exist two positive constants α and k, $0 < k < 1$, such that for all $u, v \in \mathcal{B}$ we have*

$$\left\| B(u) - B(v) - \alpha[A(u) - A(v)] \right\|_{\mathcal{B}_1} \le k \left\| B(u) - B(v) \right\|_{\mathcal{B}_1}.$$

The next Theorem is proved by S. Campanato [3].

Theorem 3.1. *The mapping $A : \mathcal{B} \to \mathcal{B}_1$ is injective or surjective or bijective if and only if it is near to a mapping $B : \mathcal{B} \to \mathcal{B}_1$ which is injective or surjective or bijective respectively. Moreover, there is the estimate:*

$$\left\| B(u) - B(0) \right\|_{\mathcal{B}_1} \leq \frac{\alpha}{1-k} \left\| A(u) - A(0) \right\|_{\mathcal{B}_1}.$$

4. Proof of the theorems.

Let us recall that

$$W_l = \overline{\{u \in C^2(\overline{\Omega}) \cap C^3(\Omega) : \frac{\partial u}{\partial l} = 0 \text{ on } \partial\Omega, u(\varphi_j) = 0, \ j = 1, \dots, n\}}^{W^{2,2}(\Omega)}$$

and let us prove the following:

Lemma 4.1. *Under assumptions* (2.1), (2.2), (2.3), *for every $u \in W_l$ it results*

$$(\Delta u/u)_{L^2(\Omega)} \leq 0. \tag{4.1}$$

Proof. We shall prove Lemma 4.1 supposing $\cos\theta = 0$ in the unique point $\varphi = \varphi_1 \in]0, L[$.
The inequality (4.1) is equivalent to

$$\int_\Omega u \Delta u \, dx \leq 0 \qquad \forall u \in W_l.$$

Taking into account the identity

$$u\Delta u = \left[\frac{\partial}{\partial x_1}(up_1) + \frac{\partial}{\partial x_2}(up_2) \right] - (p_1^2 + p_2^2)$$

and the Gauss-Green formula, we get

$$\int_\Omega u \Delta u \, dx = \int_0^L u[p_1 X_1 + p_2 X_2] \, d\varphi - \int_\Omega (p_1^2 + p_2^2) \, dx.$$

We observe that

$$\int_0^L u[p_1X_1 + p_2X_2]\,d\varphi =$$
$$= \lim_{\varepsilon\to 0^+}\left(\int_0^{\varphi_1-\varepsilon} u[p_1X_1 + p_2X_2]\,d\varphi + \int_{\varphi_1+\varepsilon}^{L} u[p_1X_1 + p_2X_2]\,d\varphi\right).$$

In $[0, \varphi_1 - \varepsilon]$ and $[\varphi_1 + \varepsilon, L]$ it results $\cos\theta > 0$, then from the system

$$\begin{cases} p_1Y_1 + p_2Y_2 = 0 \\ p_1x_1' + p_2x_2' = \dfrac{du}{d\varphi} \end{cases}$$

we get

$$p_1 = -\frac{Y_2}{\cos\theta}\frac{du}{d\varphi}\,, \quad p_2 = \frac{Y_1}{\cos\theta}\frac{du}{d\varphi}\,.$$

Then

$$\int_0^L u[p_1X_1 + p_2X_2]\,d\varphi =$$
$$= -\lim_{\varepsilon\to 0^+}\left[\int_0^{\varphi_1-\varepsilon} u\frac{du}{d\varphi}\tan\theta\,d\varphi + \int_{\varphi_1+\varepsilon}^{L} u\frac{du}{d\varphi}\tan\theta\,d\varphi\right].$$

In virtue of (2.7), we obtain for all $\varphi \in [0, \varphi_1]$ and for a suitable point $\overline{\varphi} \in]\varphi, \varphi_1[$:

$$\frac{du(\varphi)}{d\varphi} = \frac{du(\varphi)}{d\varphi} - \frac{du(\varphi_1)}{d\varphi} = (\varphi - \varphi_1)\frac{d^2u(\overline{\varphi})}{d\varphi^2}\,.$$

Bearing in mind condition (2.3)

$$\lim_{\varphi\to\varphi_1^-}\left|u(\varphi)\frac{du}{d\varphi}\tan\theta\right| = \lim_{\varphi\to\varphi_1^-}\left|u(\varphi)(\varphi - \varphi_1)\frac{d^2u(\overline{\varphi})}{d\varphi^2}\tan\theta\right| \leq$$

$$\leq M\lim_{\varphi\to\varphi_1^-}\left|u(\varphi)\frac{\varphi - \varphi_1}{\cos\theta}\right| = M\left|\frac{\frac{1}{-d\theta(\varphi_1^-)}}{d\varphi}\right|\lim_{\varphi\to\varphi_1^-}|u(\varphi)| = 0$$

and then

$$\lim_{\varepsilon\to 0^+}\int_0^{\varphi_1-\varepsilon} u\frac{du}{d\varphi}\tan\theta\,d\varphi = \int_0^{\varphi_1} u\frac{du}{d\varphi}\tan\theta\,d\varphi.$$

Similarly, we get

$$\lim_{\varepsilon\to 0^+}\int_{\varphi_1+\varepsilon}^{L} u\frac{du}{d\varphi}\tan\theta\, d\varphi = \int_{\varphi_1}^{L} u\frac{du}{d\varphi}\tan\theta\, d\varphi.$$

On the other hand

$$\int_0^{\varphi_1-\varepsilon} u\frac{du}{d\varphi}\tan\theta\, d\varphi = \frac{1}{2}[u^2\tan\theta]_0^{\varphi_1-\varepsilon} - \frac{1}{2}\int_0^{\varphi_1-\varepsilon}\frac{u^2(\varphi)}{\cos^2\theta}\frac{d\theta}{d\varphi}\,d\varphi =$$

$$= \frac{1}{2}\left\{u^2(\varphi_1-\varepsilon)\tan\theta(\varphi_1-\varepsilon) - u^2(0)\tan\theta(0) - \int_0^{\varphi_1-\varepsilon}\frac{u^2(\varphi)}{\cos^2\theta}\frac{d\theta}{d\varphi}\,d\varphi\right\}.$$

As above, we have for all $\varphi \in [0, \varphi_1]$ and for a suitable $\tilde{\varphi} \in]\varphi, \varphi_1[$:

$$u(\varphi) = u(\varphi) - u(\varphi_1) = (\varphi - \varphi)\frac{du(\tilde{\varphi})}{d\varphi};$$

so we obtain

$$\lim_{\varepsilon\to 0^+}\left|u^2(\varphi_1-\varepsilon)\tan\theta(\varphi_1-\varepsilon)\right| \le K\lim_{\varepsilon\to 0^+}\left|\frac{\varepsilon^2}{\cos\theta(\varphi_1-\varepsilon)}\right| = 0;$$

and similarly

$$\lim_{\varepsilon\to 0^+}\left|u^2(\varphi_1+\varepsilon)\tan\theta(\varphi_1+\varepsilon)\right| = 0.$$

In an analogous way we get that $\lim_{\varphi\to\varphi_1^-}\dfrac{u^2(\varphi)}{\cos^2\theta}$ is finite.

Taking into account that $Y_i(0) = Y_i(L), X_i(0) = X_i(L)$, $i = 1, 2$ implies $\tan\theta(0) = \tan\theta(L)$, we get

$$\int_\Omega u\Delta u\, dx = \frac{1}{2}\left\{\int_0^{\varphi_1}\frac{u^2}{\cos^2\theta}\frac{d\theta}{d\varphi}\,d\varphi + \int_{\varphi_1}^{L}\frac{u^2}{\cos^2\theta}\frac{d\theta}{d\varphi}\,d\varphi\right\} - \int_\Omega \left(p_1^2+p_2^2\right)dx$$

and by condition (2.3), we obtain estimate (4.1). □

Taking into account Lemma 4.1, we get (cfr. S. Campanato [2])

Lemma 4.2. *Under the same assumptions of Lemma 4.1, if the mapping $\mathcal{A}(x, H(u))$ is near to Δu with constants α and k, then for all $\lambda \ge 0$ the mapping $[\mathcal{A}(x, H(u)) - \lambda u]$ is near to the mapping $[\Delta u - \alpha\lambda u]$ both considered as mappings from $W_l \to L^2(\Omega)$.*

Now we are in position to prove Theorem 2.1.

The problem

$$\begin{cases} \Delta u - \alpha\lambda u = f(x) \in L^2(\Omega) & \text{a. e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \dots, n \end{cases}$$

admits only one solution $u \in W^{2,2}(\Omega)$ (cfr. Lemma 3.3), then the mapping $\Delta u - \alpha\lambda u$ is bijective from W_l in $L^2(\Omega)$. The Campanato's (A)-condition and Lemma 3.1 ensures us that the mapping $\mathcal{A}(x, H(u))$ is near to the Laplacian Δu, both considered as mappings from W_l in $L^2(\Omega)$ (cfr. [3]). Then, from Lemma 4.1 and Lemma 4.2, we get that the mapping $\mathcal{A}(x, H(u)) - \lambda u$ is near to mapping $\Delta u - \alpha\lambda u$. By Theorem 3.2 the mapping $\mathcal{A}(x, H(u)) - \lambda u$ is also bijective from W_l in $L^2(\Omega)$, so we derive the existence and the uniqueness of the solution $u \in W^{2,2}(\Omega)$ of problem (2.4).

To prove estimate (2.5) let $u \in W^{2,2}(\Omega)$ be the solution of problem (2.4); taking into account Lemma 3.1, estimate (4.1) and the nearness of the mappings $\Delta u - \alpha\lambda u$ and $\mathcal{A}(x, H(u)) - \lambda u$, with constants α and k, $0 < k < 1$, we get

$$\|H(u)\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)} \leq \|\Delta u - \alpha\lambda u\|_{L^2(\Omega)} \leq$$

$$\leq \|\Delta u - \alpha\lambda u - \alpha[\mathcal{A}(x, H(u)) - \lambda u]\|_{L^2(\Omega)} + \alpha\|f\|_{L^2(\Omega)} \leq$$

$$\leq k\|\Delta u - \alpha\lambda u\|_{L^2(\Omega)} + \alpha\|f\|_{L^2(\Omega)}.$$

From this, we obtain estimate (2.5). □

Proof of Theorem 2.2. The proof of Theorem 2.2 is analogous to the proof of Theorem 2.2 by S. Giuffrè [9]. □

Proof of Theorem 2.3. The mappings $[\mathcal{A}(x, H(u)) - \lambda u]$ and $[\Delta u - \alpha\lambda u]$ from W_l in $L^2(\Omega)$ are near. Then there exists $k \in (0, 1)$ such that

$$\|\Delta u - \alpha\lambda u - \alpha[\mathcal{A}(x, H(u)) - \lambda u]\|_{L^2(\Omega)} \leq k\|\Delta u - \alpha\lambda u\|_{L^2(\Omega)}.$$

Hence, for all $\lambda > 0$, the function $u \in W^{2,2}(\Omega)$ is solution of the problem

$$\begin{cases} \mathcal{A}(x, H(u)) = \lambda u & \text{a. e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \dots, n \end{cases}$$

if and only if u is solution of the problem

$$\begin{cases} \Delta u = \alpha\lambda u & \text{a. e. in } \Omega \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial\Omega \\ u(\varphi_j) = 0 & j = 1, \dots, n \end{cases}$$

that is $u \equiv 0$.
Then the mapping

$$\mathcal{A}(x, H(\cdot)) : W_l \to L^2(\Omega)$$

have all possible eigenvalues less or equal to zero. □

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