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GENERIC UNIQUENESS AND SIZE ESTIMATES IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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1. The inverse conductivity problem.

Inverse boundary value problems are those in which one wants to determine physical parameters associated to the interior of a certain body, or region, from measurements taken from the exterior. We shall deal with the so called problem of *electrical impedance tomography* or *inverse conductivity problem*. Suppose Ω is an electrical conducting isotropic body, and let $\sigma = \sigma(x) > 0$ be its (scalar) conductivity. The associated direct boundary value problem is:

Given σ and suitable boundary data, for instance the current density η, find the electrical potential u inside Ω.

This corresponds to the solution of the Neumann problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in } \Omega, \\ \nabla u \cdot \nu = \eta & \text{on } \partial\Omega. \end{cases} \tag{1}$$

To each current profile $\eta\left(\int_{\partial\Omega}\eta = 0\right)$, one can associate the boundary values of the potential $u|_{\partial\Omega}$ (the Dirichlet data). That is, given σ, the linear map

$$N_\sigma : \eta = \sigma\nabla u \cdot \nu \to u|_{\partial\Omega}$$

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called the Neumann-to-Dirichlet map, is known. The inverse boundary value problem is then:

> *Given N_σ (or equivalently, the Dirichlet-to-Neumann map $\Lambda_\sigma = N_\sigma^{-1}$) find σ.*

Essentially under this setting, this problem was first raised by A. P. Calderón [13] in 1980. The *uniqueness issue*, that is, *whether N_σ uniquely determines σ*, was first attacked by Kohn and Vogelius [23], [24] and solved, in the case when the space dimension n is greater than or equal 3, by Sylvester and Uhlmann [29].
The uniqueness for the case $n = 2$ was finally resolved by Nachman [26] in '95.

Allied to uniqueness, stands the *stability issue*, which is especially important for the practical purpose of reconstruction. Namely:

> *If N_σ is incompletely known and if its measurement is affected by errors, what kind of information about σ can be extracted from such incomplete and noisy data?*

There are indeed several pieces of evidence that the dependence of σ on N_σ is not stable. One argument suggesting instability goes as follow.

Let σ_1, σ_2 be two conductivity coefficient and let N_1, N_2 be the corresponding Neumann-to-Dirichlet maps. We have the following identity

$$\int_\Omega (\sigma_1 - \sigma_2)\nabla u_1 \cdot \nabla u_2 = -\int_{\partial\Omega} \eta_1 (N_1 - N_2)\eta_2 \tag{2}$$

here u_i, $i = 1, 2$ is the solution to (1),when $\sigma = \sigma_i$ and $\eta = \eta_i$, and η_1, η_2 are any two current profiles $\left(\int_{\partial\Omega} \eta_i = 0\right)$.

From the weak formulation of (1), it can be seen that the above identity (2) carries all the information about the connection between σ and N_σ.

By formally differentiating the mapping $\sigma \to N_\sigma$, let us linearize the identity (2) around the conductivity $\sigma_1 \equiv 1$. Namely set

$$\sigma_1 \equiv 1 , \quad \sigma_2 = 1 + t\delta\sigma$$

and

$$\delta N = \frac{d}{dt} N_2|_{t=0}.$$

We obtain

$$\int_\Omega \delta\sigma \nabla h_1 \cdot \nabla h_2 = -\int_{\partial\Omega} \eta_1 \delta N \eta_2 \tag{3}$$

where h_i, $i = 1, 2$, is the harmonic function in Ω having Neumann data η_i on $\partial\Omega$.

Let us specialize the geometry to the case of the unit disk $\Omega = B_1 \subset \mathbb{R}^2$. Since the class of harmonic functions in the disk is spanned by the polynomials $r^{|n|}e^{\imath n\theta}$, $u = 0, \pm 1, \pm 2, \ldots$, it suffices to consider in (3)

$$h_1 = r^{|n|}e^{\imath n\theta}, h_2 = r^{|m|}e^{-\imath m\theta}, \qquad n, m = 0, \pm 1, \pm 2, \ldots$$

Let us represent $\delta\sigma$ by the Fourier series

$$\delta\sigma = \sum_{n=0}^{\infty} \left(c_n(r)e^{\imath n\theta} + \overline{c_n(r)}e^{-\imath n\theta}\right)$$

and let us pose:

$$N_{n,m} = \int_0^{2\pi} e^{\imath n\theta}\delta N\left[e^{-\imath m\theta}\right]d\theta$$

in such a way that $\{N_{n,m}\}$ is the infinite matrix representing the operator δN with respect to the Fourier basis $\left\{e^{\imath n\theta}\right\}$. Taking now $n \geq m \geq 0$ and $k = n - m$, we obtain from (3)

$$2\int_0^1 r^{k+2m-1}c_k(r)dr = -N_{k+m,m} \quad \text{for every } k, m \geq 0. \tag{4}$$

That is, for each Fourier coefficient c_k of $\delta\sigma$, an infinite sequence of its Hausdorff moments is given. It is well-known that the determination of a function from its moments is a severely ill-posed problem, see for instance Talenti [30].

The evident limitation of the above argument is that it shows the instability of a linearization of the inverse conductivity problem and not of the original nonlinear problem, which, in principle, could be better conditioned than any of its linearizations. At the same time, this argument poses a warning to an easy use of standard inversion procedures, which typically involve, in one way or another, some form of linearization.

On the positive side, let us mention that stability results of conditional type (that is, under prior assumptions on the regularity of the unknown conductivity σ) have been obtained, [1], [2], [3]. The essence of these results is as follows:

suppose we a priori know $||\sigma||_{C^2} \leq E$ for some $E > 0$, then the mapping

$$N_\sigma \to \sigma \tag{5}$$

is continuous (in the natural topologies), with an estimated modulus of continuity of logarithmic type.

An interesting, still unanswered, question is whether such logarithmic rate is the best possible.

2. The inverse conductivity problem with one measurement.

In the experimental practice only an incomplete knowledge of N_σ will be available, since it will be sampled on finitely many profiles of the current density η.Therefore it is reasonable to incorporate among the data additional prior information on the unknown conductivity σ which concerns its structure, rather than its smoothness, so to reduce the dimension of the undetermined parameter. One basic case of this sort is the so called *transmission problem*.

Assume that σ has the following structure

$$\sigma = 1 + (k-1)X_D \tag{6}$$

where D is a set compactly contained in Ω and $k > 0, k \neq 1$ is a given constant.

The *inverse transmission problem* (also known as the *inverse conductivity problem with one measurement*) consists in finding D given one (or finitely many) pair of nontrivial boundary measurements

$$\left\{ u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega} \right\}$$

where u is a solution of the equation in (1).

There have been numerous attempts in proving uniqueness, but still it remains an open problem. Partial results are of the following kinds.

(i) (Local uniqueness) *Assume D, D' are sufficiently smooth ($C^{1,\alpha}$) simply connected domains, which give rise to the same boundary measurement $\left\{u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}|_{\partial\Omega}\right\}$, if D, D' are sufficiently close, then they coincide.*

Results of this type have been obtained when the space dimension $n = 2$. Cherednichenko [14], Bellout, Friedman and Isakov [11] proved such kind of result under analyticity assumptions on $\partial D, \partial D'$. The regularity assumption was reduced to $C^{1,\alpha}$ by Alessandrini, Isakov and Powell [6].

A typical assumption in the above papers is that the prescribed boundary data on u (for instance the Neumann data η in (1)) must be such that, in a generalized sense, ∇u never vanishes in the interior. A constructive procedure, which exhibits such boundary data, was presented in [7], and consequently applied in [6].

(ii) (Uniqueness for special geometries). *If D is a priori known to be a convex polygon or polyhedron then it is uniquely determined by one pair* $\left\{u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega}\right\}$. Results of this kind have been obtained by Friedman and Isakov [17], Barcelò, Fabes and Seo [10] and by Alessandrini and Isakov [5].
A remarkable result is also due to Seo which says that if D is a priori known to be a polygon, then it is uniquely determined by *two* suitable pairs of boundary measurements [28]. Also other special geometries (disks, balls cylinders) have been investigated, see for instance [20], [15].

3. Uniqueness results of generic type.

In [5] another approach to the inverse conductivity problem with one measurement was taken, aimed at showing that, should non uniqueness ever occur, it would be an exceptional event. The result in this directions is as follows. We suppose $n \geq 2$, and ∂D sufficiently smooth ($C^{1,\alpha}$).

> *If D is not uniquely determined by one boundary measurement $\left\{u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega}\right\}$ then there exists some portion of its boundary which is made of an $(n-2)$-parameter family of analytic curves.*

As noted above, this result can be viewed in a broad sense, as a result of generic uniqueness, but also it shows that there exists indeed domains $D \subset\subset \Omega$ which are uniquely determined by one boundary measurements.

Let us sketch the main ideas of the proof.
If we set $u^e = u|_{\Omega\setminus\overline{D}}$, $u^i = u|_D$, then it is well known that u^e, u^i are separately harmonic and satisfy the transmission conditions:

$$
\begin{cases} u^i = u^e & \text{on } \partial D \\ k\dfrac{\partial u^i}{\partial \nu} = \dfrac{\partial u^e}{\partial \nu} & \text{on } \partial D \end{cases} \tag{7}
$$

Suppose $D' \neq D$ is another subdomain of Ω that gives rise to the same boundary measurement. It is known that we must have $D\setminus\overline{D'} \neq \emptyset$, $D'\setminus\overline{D} \neq \emptyset$, see [2].
Let us pick $x^o \in \partial D\setminus\overline{D'}$ such that $\nabla u^e(x^o) \neq 0$ (this is possible, because otherwise $u^e \equiv$ const, that is the boundary data should be trivial). Let B be a ball centered at x^o, small enough so that $B \cap D' = \emptyset$. Let u' be the potential corresponding to D' such that $u' = u$, $\frac{\partial u'}{\partial \nu} = \frac{\partial u}{\partial \nu}$ on $\partial\Omega$.

By the uniqueness for the Cauchy problem, we have $u^e = u'$ in $B\setminus\overline{D}$. That is, u^e harmonically continues to u' throughout B. Set $v = u^i - u'$ in $D \cap B$, we have

$$
\begin{cases} \Delta v = 0 & \text{in } D \cap B, \\ v = 0 & \text{on } \partial D \cap B, \\ k\dfrac{\partial v}{\partial \nu} = (1-k)\dfrac{\partial u'}{\partial \nu} & \text{on } \partial D \cap B. \end{cases} \tag{8}
$$

This is an overdetermined problem for which the existence of a solution v poses constraints on the boundary $\partial D \cap B$. In other words it is a free boundary problem. In particular we have that, if at some point $y \in \partial D \cap B$, $\frac{\partial u'}{\partial \nu}(y) \neq 0$, by a slight adaptation of a result by Kinderlehrer and Nirenberg [?], then ∂D is analytic near y.

Viceversa, if $\frac{\partial u'}{\partial \nu}(y) = 0$ for every $y \in \partial D \cap B$, then $\partial D \cap B$ is made of stream lines of u'. That is, locally, $\partial D \cap B$ is an $(n-2)$-parameter family of analytic curves.

Let us also notice that when $n = 2$, the a-priori assumption $\partial D \in C^{1,\alpha}$ can be relaxed by merely assuming that ∂D is a Jordan arc. An interesting open problem is the study of the regularity of the free boundary in (8) under reduced a priori assumption. (After this lecture was presented, we have learned that Athanasopulos, Caffarelli and Salsa [27] have proven regularity under the a priori assumption that ∂D is Lipschitz.)

4. Size estimates.

Finally let us discuss another direction of research: instead of determining the exact shape and location of D, can we at least evaluate its size in terms of the data?

Attempts of this sort are due to Friedman [16], Bryan [12], Lusin [25], Alessandrini, Rosset [8], Kang, Sheen and Seo [21] and Ikehata [19]. Let us outline the most recent results by Alessandrini, Rosset and Seo [9]. For simplicity, let us assume $k = 2$, but let us stress that the method in [28] enables to treat also anisotropic equations of the form

$$
\operatorname{div}\Big(\big(A\chi_{\Omega \setminus D} + B\chi_D\big)\nabla u\Big) = 0
$$

with A, B uniformly elliptic matrices, with A Lipschitz continuous and either $A - B > 0$ or $B - A < 0$.

Suppose that for a given $F > 0$ the Neumann data η in (1) satisfies

$$
\frac{||\eta||_{L^2(\partial\Omega)}}{||\eta||_{H^{-\frac{1}{2}}(\partial\Omega)}} \leq F
$$

and let u_0 be the solution of the problem

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu} = \eta & \text{on } \partial\Omega \end{cases} \tag{9}$$

that is u_0 is the electrostatic potential in case the inclusion D is absent.

Let us set

$$W = \int_{\partial\Omega} \eta u\,, \quad W_0 = \int_{\partial\Omega} \eta u_0$$

These numbers correspond to the electrical power needed to mantain the current η when the inclusion D is present or absent, respectively. Such powers can be easily evaluated (either measured or computed) in terms of the boundary measurement.

The first result is as follows.

There exist constants $C_1, C_2 > 0$ and an exponent $p > 1$ such that

$$C_1 \frac{W_0 - W_1}{W_0} \le |D| \le C_2 \left(\frac{W_0 - W_1}{W_0} \right)^{\frac{1}{p}} \tag{10}$$

Here $|D|$ denotes the measure of D. These estimates apply whenever D is a measurable (possibly disconnected) subset of Ω. Such estimates can be improved if we are willing to assume that D is open and satisfies the following *fatness* condition (11).
Namely, given $h_1 > 0$, and denoting $D_h = \{x \in D | \mathrm{dist}(x, \partial D) > h\}$,

If we assume

$$|D_{h_1}| \ge \frac{1}{2}|D| \tag{11}$$

then we have

$$C_1 \frac{W_0 - W_1}{W_0} \le |D| \le C_2 \frac{W_0 - W_1}{W_0}. \tag{12}$$

Let us illustrate the main tools which are required in order to obtain the above estimates.

(I) From the weak formulations of the boundary value problems (9) and (1) under (6) it is possible to obtain

$$\frac{1}{2} \int_D |\nabla u_0|^2 \le W_0 - W \le \int_D |\nabla u_0|^2 \tag{13}$$

and thus we are reduced to estimate $\int_D |\nabla u_0|^2$ in terms of $|D|$ from above and below. While the upper estimate follows easily from a standard interior gradient estimate, the lower estimate requires a deeper analysis since, in general, ∇u_0 may vanish at interior points.

(II) If we fix $\rho > 0$, we can find a constant $C > 0$ depending on ρ and on F, such that, for any $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) > 4\rho$ we have

$$\int_{B_\rho(x)} |\nabla u_0|^2 \geq C \int_\Omega |\nabla u_0|^2. \tag{14}$$

This is, in a disguised form, a stability estimate for for the harmonic continuation from $B_\rho(x)$ in Ω. Its proof comes from the iterated use of three spheres inequalities and regularity estimates near the boundary (see [9], Lemma 2.2).

From such an estimate, inequalities (12), those for "fat domains", easily follow by covering D_{h_1} by nonoverlapping cubes of side $\varepsilon = O(h_1)$.

(III) The case of a general measurable set D requires a further deeper argument. Garofalo and Lin [18] in their proof of the unique continuation property for elliptic operators, obtained that solutions satisfy very powerful local properties of homogeneity in the average. Namely we have $|\nabla u_0|^2 \in A_p$ for some p, where A_p is the class of Muckenhoupt weights. More precisely, by combining the results in [18] and the arguments leading to (11) we obtain the following.

Given $\rho > 0$, there exist constants $C > 0$, $p > 1$ depending on ρ and F, such that, for any x satisfying dist$(x, \partial\Omega) > 4\rho$, and for any $r < \rho$

$$\left(\frac{1}{|B_r|}\int_{B_r(x)} |\nabla u_0|^2\right)\left(\frac{1}{|B_r|}\int_{B_r(x)} |\nabla u_0|^{-\frac{2}{p-1}}\right)^{p-1} \leq C.$$

Thus $|\nabla u_0|^{-\frac{2}{p-1}}$ is locally integrable, hence we readily obtain

$$|D| \leq C W_0^{-\frac{1}{p}} \Big(\int_D |\nabla u_0|^2\Big)^{\frac{1}{p}}$$

and (10) follows.

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