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ON BERNOULLI BOUNDARY VALUE PROBLEM

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We consider the boundary value problem:

$$\begin{cases} x^{(m)}(t) = f(t,\bar{x}(t)), & a \le t \le b, \quad m > 1 \\ x(a) = \beta_0 & \\ \Delta x^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = \beta_{k+1}, & k = 0,...,m-2 \end{cases}$$

where $\bar{x}(t) = (x(t), x^{'}(t), ..., x^{(m-1)}(t))$, $\beta_i \in \mathbf{R}$, $i = 0,...,m-1$, and f is continuous at least in the interior of the domain of interest.
We give a constructive proof of the existence and uniqueness of the solution, under certain conditions, by Picard's iteration. Moreover Newton's iteration method is considered for the numerical computation of the solution.

1. Introduction

In this paper we consider the following boundary problem:

$$\begin{cases} (1a) \quad x^{(m)}(t) = f\left(t, \bar{x}(t)\right), & a \le t \le b, \quad m > 1 \\ (1b) \quad x(a) = \beta_0, \quad \Delta x_a^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = \beta_{k+1} & k = 0,...,m-2 \end{cases} \tag{1}$$

where $\bar{x}(t) = (x(t), x^{'}(t), ..., x^{(m-1)}(t))$, f is defined and continuous at least in the domain of interest included in $[a,b] \times \mathbf{R}^m$; $[a,b] \subset \mathbf{R}$, and $\beta_i \in \mathbf{R}$, $i =$

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$0,...,m-1$.
This problem is called the *Bernoulli* boundary value ([4],[6]) problem.
The boundary conditions in $(1a)-(1b)$ are new and it is easy to give them physical and engineering interpretations [1]; this is the motivation of our investigation.
In [6] the authors give a non constructive proof of the existence and uniqueness of the solution of $(1a)-(1b)$, while in this work they prove the convergence of Picard's iteration under certain conditions and, therefore, supply a constructive proof.
The outline of the paper is the following: in section 2 we give the preliminaries, in section 3 we investigate the existence and uniqueness of the solution by Picard's iteration; finally, in section 4 we consider the Newton's iterations method for the numerical calculation of the solution.

2. Definitions and preliminaries

If $B_n(x)$ is the Bernoulli polynomial of degree n defined by [7]

$$\begin{cases} B_0(x)=1 \\ B_n^{'}(x)=nB_{n-1}(x) & n\geq 1 \\ \int_0^1 B_n(x)dx=0 & n\geq 1 \end{cases} \tag{2}$$

in a recent paper Costabile [5] proved the following theorems.
Theorem 1. Let $f\in \mathbf{C}^{(\nu)}[a,b]$ we have

$$f(x)=f(a)+\sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\frac{h^{(k-1)}}{k!}\Delta f_a^{(k-1)}-R_\nu[f](x) \tag{3}$$

where
$h=b-a, \quad S_k(t)=1B_k(t)-B_k(0)\ , \quad f_a=f(a), \quad \Delta f_a^{(k)}=f^{(k)}(b)-f^{(k)}(a)$

$$R_\nu[f](x)=\frac{h^{(\nu-1)}}{\nu!}\cdot\int_a^b\left(f^{(\nu)}(t)\left(B_\nu^*\left(\frac{x-t}{h}\right)+(-1)^{\nu+1}B_\nu\left(\frac{t-a}{h}\right)\right)\right)dt$$

and

$$B_m^*(t)=B_m(t)\quad 0\leq t\leq 1, \qquad B_m^*(t+1)=B_m^*(t)$$

Theorem 2. Putting

$$P_\nu[f](x)=f_a+\sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\frac{h^{(k-1)}}{k!}\Delta f_a^{(k-1)} \tag{4}$$

the following equalities are true

$$\begin{cases} P_\nu[f](a) = f(a) \equiv f_a \\ P_\nu[f](b) = f(b) \equiv f_b \\ \Delta P_\nu^{(k)} \equiv P_\nu^{(k)}(b) - P_\nu^{(k)}(a) = f^{(k)}(b) - f^{(k)}(a) \equiv \Delta f_a^{(k)}, \quad k = 1, ..., \nu - 1 \end{cases} \tag{5}$$

The conditions (5) in the previous equalities are called *Bernoulli interpolatory conditions* analogously to Lidstone interpolatory conditions [3],[4].

Theorem 3. If $f \in C^{(\nu+1)}[a,b]$ we have

$$R_\nu[f](x) = \int_a^b G(x,t) f^{(\nu+1)}(t) dt$$

where

$$G(x,t) = \frac{1}{\nu!}\left[(x-t)_+^\nu - \sum_{k=1}^{\nu} S_k\left(\frac{x-a}{h}\right)\frac{h^{(k-1)}}{k!}\binom{\nu}{k-1}(b-t)^{\nu-k+1}\right] \tag{6}$$

with

$$(x)_+^k = \begin{cases} x^k & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases}$$

Theorem 4. For $f \in C^{(\nu)}[a,b]$ we have

$$|R_\nu[f](x)| \leq \frac{h^{\nu-1}}{6(2\pi)^{\nu-2}} \int_a^b \left| f^{(\nu)}(t) \right| dt \tag{7}$$

For the following, we need

Lemma 1.[6] If $f \in C^{(\nu)}[a,b]$ and satisfies the homogeneous Bernoulli interpolatory conditions i.e:

$$\begin{cases} f(a) = 0 \\ f^{(k)}(b) - f^{(k)}(a) = 0 \qquad k = 0, ..., \nu - 2 \end{cases} \tag{8}$$

putting

$$M_\nu = \max_{a \leq t \leq b} \left| f^{(\nu)}(t) \right|$$

the following inequalities hold

$$\left| f^{(k)}(t) \right| \leq C_{\nu,k} \cdot M_\nu \cdot (b-a)^{\nu-k} \qquad 0 \leq k \leq \nu - 1 \tag{9}$$

where

$$\begin{cases} C_{\nu,0} = \dfrac{1}{3(2\pi)^{\nu-2}} \\ C_{\nu,k} = \dfrac{1}{6(2\pi)^{\nu-k-2}} \qquad k = 1, 2, .., \nu - 1 \end{cases}$$

Proof >From (8) the expansion (3) becomes

$$f(t) = \frac{h^{\nu-1}}{\nu!}\left[B_\nu\left(\frac{t-a}{h}\right) - B_\nu\right]\Delta f_a^{(\nu-1)} - R_\nu[f](t) \tag{10}$$

We also have

$$f^{(\nu-1)}(t) = f^{(\nu-1)}(a) + \int_a^t f^{(\nu)}(s)ds$$

from which

$$\left|\Delta f_a^{(\nu-1)}\right| \equiv \left|f^{(\nu-1)}(b) - f^{(\nu-1)}(a)\right| \leq M_\nu(b-a) \tag{11}$$

Using the known inequalities in [7]

$$|B_l(x)| \leq \frac{l!}{12(2\pi)^{l-2}} \qquad l \in N, \quad l \geq 0, \quad 0 \leq x \leq 1$$

and (7),(11) we have from (10)

$$|f(t)| \leq \frac{h^\nu \cdot M_\nu}{3(2\pi)^{\nu-2}}$$

that is (9) for $k = 0$.
With a successive derivation of (10) and by applying (8) we have

$$\begin{aligned} f^{(k)}(t) = {} & \frac{h^{\nu-(k+1)}}{(\nu-k)!}\Delta f_a^{(\nu-1)} B_{\nu-k}\left(\frac{t-a}{h}\right) + \\ & -\frac{h^{\nu-(k+1)}}{(\nu-k)!}\int_a^b f^{(\nu)}(t) B^*_{\nu-k}\left(\frac{t-s}{h}\right) ds \quad k = 1,2,...,\nu-1 \end{aligned} \tag{12}$$

and applying the previous inequalities we get

$$\left|f^{(k)}(t)\right| \leq \frac{h^{\nu-k} \cdot M_\nu}{6(2\pi)^{\nu-k-2}} \qquad k = 1,2,...,\nu-1$$

that is (9) for $k = 1,2,...,\nu-1$.

3. Existence and uniqueness

To the boundary value problem $(1a)-(1b)$ we associate the homogeneous boundary value problem

$$\begin{cases} x^{(m)}(t) = f(t,\bar{x}(t)), & a \leq t \leq b, \quad m > 1 \\ x(a) = x(b) = 0 & \\ \Delta x^{(k)} \equiv x^{(k)}(b) - x^{(k)}(a) = 0 & k = 1,...,m-2 \end{cases} \tag{13}$$

From Theorem 3, the solution of the boundary value problem (13) is

$$x(t) = \int_a^b G(t,s)\, f(s,\overline{x}(s))\, ds \tag{14}$$

where $G(t,s)$ is the *Green* function [8] defined by (6), with $\nu = m-1$. The polynomial $P_{m-1}[x](t)$ defined by (4) with $x(a)=\beta_0,\ x^{(k)}(b)-x^{(k)}(a)=\beta_{k+1},\quad k=0,...,m-2$, satisfies the boundary value problem:

$$\begin{cases} P_{m-1}^{(m)}[x](t) = 0 \\ P_{m-1}[x](a) = \beta_0 \\ \Delta P_{m-1}^{(k)} \equiv P_{m-1}^{(k)}(b) - P_{m-1}^{(k)}(a) = \beta_{k+1}, \qquad k=0,...,m-2 \end{cases}$$

Therefore, the boundary value problem $(1a)-(1b)$ is equivalent to the following nonlinear Fredholm integral equation:

$$x(t) = P_{m-1}[x](t) + \int_a^b G(t,s)\cdot f(s,\overline{x}(s))\, ds \tag{15}$$

Now, we have the following results:
Theorem 5.[6] Let us suppose that

(i) $k_i > 0,\quad 0 \le i \le m-1$ are given real numbers and let Q be the maximum of $|f(t,x_0,...,x_{m-1})|$ on the compact set $[a,b]\times D_0$, where $D_0 = \{(x_0,...,x_{m-1}) : |x_i| \le 2k_i,\quad 0 \le i \le m-1\}$;

(ii) $\max\left|P_{m-1}^{(i)}[x](t)\right| \le k_i \quad 0 \le i \le m-1$, where $P_{m-1}[x](t)$ is the polynomial relative to x as in (4);

(iii) $(b-a) \le \left(\dfrac{k_i}{Q\cdot C_{m,i}}\right)^{\frac{1}{(m-i)}} \qquad 0 \le i \le m-1.$

Then, the Bernoulli boundary value problem has a solution in D_0.
Proof. The set

$$B[a,b] = \left\{x(t) \in C^{(m-1)}[a,b] : \left\|x^{(i)}\right\|_\infty \le 2\cdot k_i, \quad 0 \le i \le m-1\right\}$$

is a closed convex subset of the Banach space $C^{(m-1)}[a,b]$.
Now we define an operator $T : C^{(m-1)}[a,b] \to C^{(m)}[a,b]$ as follows:

$$(T[x](t)) = P_{m-1}[x](t) + \int_a^b G(t,s)\cdot f(s,\overline{x}(s))ds \tag{16}$$

It is clear, after (15), that any fixed point of (16) is a solution of the boundary value problem $(1a)$ and $(1b)$.
Let $x(t) \in B[a,b]$, then from (16), lemma 1, hypotheses **(i)**,**(ii)**,**(iii)** we find:

(a) $TB[a,b] \subseteq B[a,b]$;

(b) the sets $\left\{T[x]^{(i)}(t) : x(t) \in B[a,b]\right\}$, $0 \le i \le m-1$ are uniformly bounded and equicontinuous in $[a,b]$;On Bernoulli boundary value problem

(c) $\overline{TB[a,b]}$ is compact from the *Ascoli - Arzela theorem*;

(d) from the *Schauder fixed point theorem* a fixed point of T exists in D_0.

Corollary 1. Suppose that the function $f(t,x_0,x_1,...,x_{m-1})$ on $[a,b] \times \mathbf{R}^m$ satisfies the following condition

$$|f(t,x_0,x_1...,x_{m-1})| \le L + \sum_{i=0}^{m-1} L_i |x_i|^{\alpha_i}$$

where $L, L_i \quad 0 \le i \le m-1$ are non negative constants, and $0 \le \alpha_i \le 1$. Then the boundary value problem $(1a)-(1b)$ has a solution.

Lemma 2 [6] For the *Green* function defined by (6), for $\nu = m-1$ the following inequalities hold:

$$|G(t,s)| \le g$$

withOn Bernoulli boundary value problem

$$g = \frac{1}{\nu!}(b-a)^m \left(1 + \frac{2\pi^2 m!}{3(2\pi - 1)}\right).$$

Proof. The proof follows from the known inequalities of Bernoulli polynomials and from simple calculations.

Theorem 6.[6] Suppose that the function $f(t,x_0,x_1...,x_{m-1})$ on $[a,b] \times D_1$ satisfies the following condition

$$|f(t,x_0,x_1...,x_{m-1})| \le L + \sum_{i=0}^{m-1} L_i |x_i| \tag{17}$$

where

$$D_1 = \{(x_0,x_1...,x_{m-1}) : |x_i| \le \max_{a \le t \le b} \left|P_{m-1}^{(i)}[x](t)\right| +$$

$$+ C_{m,i}(b-a)^m g \cdot h \cdot \left(\frac{L+C}{1-\theta}\right), \quad 0 \le i \le m-1 \Big\}$$

$$C = \max_{a \le t \le b} \sum_{i=0}^{m-1} L_i \left|P_{m-1}^{(i)}[x](t)\right|$$

$$\theta = h\cdot g\cdot\left(\sum_{i=0}^{m-1} C_{m,i}L_i(b-a)^{m-i}\right) < 1, \quad h = b-a \tag{18}$$

Then, the boundary value problem $(1a)-(1b)$ has a solution in D_1.
Proof. Let $y(t) = x(t) - P_{m-1}[x](t)$, so that $(1a)$ and $(1b)$ is the same as

$$\begin{cases} y^{(m)}(t) = f(t,\bar{y}(t)) \\ y(a) = y(b) = 0 \\ \Delta y_a^{(k)} = 0 \qquad\qquad 1 \le k \le m-2 \end{cases} \tag{19}$$

whereOn Bernoulli boundary value problem

$$\bar{y}(t) = y(t) + P_{m-1}[x](t), \qquad y^{'}(t) + P_{m-1}^{'}[x](t), \ldots\ldots, y^{(m-1)}(t) + P_{m-1}^{(m-1)}[x](t).$$

Define $M[a,b]$ as the space of m times continuously differentiable functions satisfying the boundary conditions of (19). If we introduce in $M[a,b]$ the norm:

$$\|y(t)\|_\infty = \max_{a\le t\le b}\left|y^{(m)}(t)\right|$$

then it becomes a Banach space. As in theorem 5, it suffices to show that the operator $T: M[a,b]\to M[a,b]$ defined by

$$T[y](t) = \int_a^b G(t,s)\cdot f(s,\bar{y}(s))ds$$

maps the set

$$S = \left\{y(t)\in M[a,b] : \|y\|_\infty \le hg\left(\frac{L+C}{1-\theta}\right)\right\}$$

into itself. In order to demonstrate this, it is sufficient to utilise the conditions (17), lemma 1 and lemma 2.
The thesis follows from the application of the *Schauder fixed point* theorem to the operator T.
Definition 1. A function $\bar{x}(t)\in C^{(m)}[a,b]$ is called an approximate solution of $(1a)-(1b)$ if there exist non-negative constants δ and ε such that:

$$\max_{a\le t\le b}\left|\bar{x}_{(m)}(t) - f(t,\bar{x}(t))\right| \le \delta$$

$$\max_{a\le t\le b}\left|P_{m-1}^{(i)}[x](t) - \overline{P}_{m-1}^{(i)}[x](t)\right| \le \varepsilon\cdot C_{m,i}\cdot(b-a)^{m-i}, \quad 0\le i\le m-1 \tag{20}$$

where $\overline{P}_{m-1}^{(i)}[x](t)$ and $P_{m-1}^{(i)}[x](t)$ are the polynomials defined by (5).
The inequality (20) means that there exists a continuous function $\eta(t)$ such that:

$$\bar{x}^{(m)}(t) = f(t,\bar{x}(t)) + \eta(t)$$

and

$$\max_{a\le t\le b}|\eta(t)|\le\delta$$

Thus the approximate solution $\bar{x}(t)$ can be expressed as:

$$\bar{x}(t)=\overline{P}_{m-1}[x](t)+\int_a^b G(t,s)\cdot[f(s,\bar{x}(s))+\eta(s)]\,ds$$

In the following we shall consider the Banach space $C^{(m-1)}[a,b]$ and for $y(t)\in C^{(m-1)}[a,b]$ the norm $\|y\|$ is defined by:

$$\|y\|=\max_{0\le j\le m-1}\left\{\frac{C_{m,0}(b-a)^j}{C_{m,j}}\cdot\max_{a\le t\le b}\left|y^j(t)\right|\right\}$$

Now we have:

Theorem 7.(Picard's iteration)[2]

With respect to the boundary value problem $(1a)-(1b)$ we assume the existence of an approximate solution $\bar{x}(t)$ and:

(i) the function $f(t,x_0,\ldots,x_{m-1})$ satisfies the Lipschitz condition:

$$|f(t,x_0,\ldots,x_{m-1})-f(t,\bar{x}_0,\ldots,\bar{x}_{m-1})|\le\sum_{i=0}^{m-i}L_i\,|x_i-\bar{x}_i|\quad on\,[a,b]\times D_2$$

where $D_2=\left\{(x_0,\ldots,x_i):\left|x_j-\bar{x}^{(j)}(t)\right|\le N\cdot\frac{C_{m,j}}{C_{m,0}(b-a)^j},\ 0\le j\le\ m-1\right\}$

(ii) $\theta<1$

(iii) $N_0=(1-\theta)^{-1}\cdot(\varepsilon+\delta)\cdot C_{m,0}(b-a)^m\le N$

Then, the following results hold:

(21_a) there exists a solution $x^*(t)$ of $(1a)$ and $(1b)$ in

$$\overline{S}(\bar{x},N_0)=\left\{x\in C^{(m-1)}[a,b]:\|x-\bar{x}\|\le N_0\right\}$$

(21_b) $x^*(t)$ is, the, unique solution of $(1a)$ and $(1b)$ in $\overline{S}(x,N)$

(21_c) the Picard iterative sequence $x_n(t)$ defined by:

$$\begin{cases} x_0(t)=\bar{x}(t) \\ x_{n+1}(t)=P_{m-1}(t)+\int_a^b G(t,s)\cdot f(s,\bar{x}_n(s))ds \qquad n=0,1,\ldots \end{cases}$$

converges to $x^*(t)$ with: $\|x^*-x_0\|\le\theta^n\cdot N_0$
and

$$\|x^*-x_n\|\le\theta(1-\theta)^{-1}\cdot\|x_0-x_{n-1}\|.$$

Proof. It suffices to show that the operator $T:\overline{S}(\overline{x},N)\to C^{(m)}[a,b]$ defined by

$$T[x](t)=P_{m-1}[x](t)+\int_a^b G(t,s)\cdot f(s,X(s))ds$$

where $X(s)=(x(s),x^{'}(s),....,x^{(m-1)}(s))$, satisfies the conditions of the *contraction mapping theorem.*

4. Newton's iteration

For an efficient numerical calculation of the solution of problem $(1a)-(1b)$ we can consider Newton's iteration method. For our problem $(1a)-(1b)$ the quasilinear iterative scheme is defined as:

$$(22_a)\qquad x_{n+1}^{(m)}(t)=f(t,\overline{x}_n(t))+\sum_{i=0}^{m-1}\left(x_{n+1}^{(i)}(t)-x_n^{(i)}(t)\right)\cdot\frac{\partial f(t,\overline{x}_n(t))}{\partial x_n^{(i)}(t)}$$

$$(22_b)\qquad \begin{cases} x_{n+1}(a)=\beta_0 \\ x_{n+1}^{(h)}(b)-x_{n+1}^{(h)}(a)=\beta_{h+1}, \qquad h=0,\dots,m-2, \quad n=0,1,\dots \end{cases}$$

where $x_0(t)=\overline{x}(t)$ is an approximate solution of $(1a)-(1b)$.

Theorem 8.(Newton's iteration)
With respect to the boundary value problem $(1a)-(1b)$ we assume that there exists an approximate solution $\overline{x}(t)$, and:

(i) the function $f(t,x_0,x_1,\dots,x_{m-1})$ is continuously differentiable with respect to all $x_i \quad 0\le i\le m-1$ on $[a,b]\times D_2$;

(ii) there exist non-negative constants L_i, $0\le i\le m-1$ such that for all $(t,x_0,\dots,x_{m-1})\in[a,b]\times D_2$ we have:

$$\left|\frac{\partial f(t,x_0,\dots,x_{m-1})}{\partial x_i}\right|\le L_i$$

(iii) $3\theta<1$

(iv) $N_3=(1-3\theta)^{-1}(\varepsilon+\delta)\cdot C_{m,0}(b-a)^m\le N$

Then, the following results hold:

(23_a) the sequence $x_n(t)$ generated by the iterative scheme $(22_a)-(22_b)$ remains in $\overline{S}(\overline{x},N_3)$.

(23_b) the sequence $x_n(t)$ converges to the unique solution $x^*(t)$ of the boundary value problem $(1a)-(1b)$.

Proof. The proof requires the equalities and the inequalities that we have previously determined and is based on inductive arguments.

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