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AN EXTENSION OF A THEOREM BY M.I. FREIDLIN TO GOOD SOLUTIONS TO ELLIPTIC NONDIVERGENCE EQUATIONS

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In the context of second order linear uniformly elliptic equations with measurable coefficients, a result of Freidlin [9], which deals with homogenization type properties for elliptic equations with smooth periodic coefficients, is extended to generalized solutions for equations with measurable coefficients.

Introduction.

Let L be an elliptic operator defined on $\mathbb{R}^d$ as:

$$L = \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1}$$

where $a^{ij} = a^{ji}$ are measurable functions satisfying the uniform ellipticity condition:

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}\xi_i\xi_j \leq \lambda|\xi|^2, \quad \forall\ \xi \in \mathbb{R}^d \tag{2}$$

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with $\lambda \geq 1$.

In what follows, for convenience, the short notation: $L = a^{ij}\partial_{ij}$ will be used.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial\Omega$; let us consider the Dirichlet problem:

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where g is a given continuous function. We will deal with the so called good solutions to the problem (3) (see [5] and [14]).

A function $u \in C^0(\overline{\Omega})$ is a good solution to the problem (3) if there exist a sequence $\{L_n\}$ of elliptic operators $L_n = a_n^{ij}\partial_{ij}$, with smooth coefficients in $\overline{\Omega}$, satisfying for any n the ellipticity condition (2), such that $a_n^{ij} \to a^{ij}$ a.e. in Ω, and functions $u_n \in C^0(\overline{\Omega}) \cap W^{2,d}_{loc}(\Omega)$, solutions to the problems

$$\begin{cases} L_n u_n = 0 & \text{in } \Omega \\ u_n = g & \text{on } \partial\Omega, \end{cases}$$

such that $\{u_n\}$ converges uniformly to u in $\overline{\Omega}$.

A relevant question, within the class of operators and the class of solutions we are dealing with, is that of the uniqueness for the Dirichlet problem associated to the operator L of the form (1)-(2).

For a detailed analysis on this matter, let us refer to [16], [19] or [1]. At any rate, let us remind that, in the dimension $d = 2$, uniqueness in $W^{2,2}$ holds without any smoothness of the coefficients a_{ij} (see [20]). In the dimension $d \geq 3$, the uniqueness problem for discontinuous a_{ij} was investigated by many mathematicians; nevertheless, positive results were obtained only under additional restrictions on a_{ij}, (see [5], [6], [14], [18] and references therein). Related results are those of [2], [7].

Recently, N. Nadirashvili [16] has been able to prove that there exists an elliptic operator of the form (1)-(2) in the unit ball $B_1 \subset \mathbb{R}^d$, $d \geq 3$, and there is a function $g \in C^2(\partial B_1)$, such that the Dirichlet problem (3) has at least two good solutions.

A constructive proof and a slight generalization of Nadirashvili's result has been given by M.V. Safonov [19].

To prove his result N. Nadirashvili does assume, by contradiction, that the good solution of the Dirichlet problem (3) is always unique and, under this assumption, then, he proves an extension to good solutions of a theorem due to M.I. Freidlin [9]. The theorem of Freidlin reads as follows:

Theorem. (M.I. Freidlin [9]). *Let L be an elliptic operator of the type (1)-(2), defined in $\mathbb{R}^d$, with smooth coefficients a^{ij}, which are periodic functions of each variable with period 1.*
Denote:

$$L_t = a^{ij}(t\cdot)\partial_{ij}.$$

Let Ω be a bounded smooth domain of $\mathbb{R}^d$, $g \in C^0(\partial\Omega)$ and

$$L_t u_{(t)} = 0 \qquad in \quad \Omega, \quad u_{(t)}|_{\partial\Omega} = g.$$

Then

$$\lim_{t\to+\infty} u_{(t)} = \overline{u} \qquad in \quad \Omega,$$

where $\overline{u}$ is solution to the problem:

$$\widehat{L}\overline{u} = 0 \qquad in \quad \Omega, \quad \overline{u}|_{\partial\Omega} = g,$$

with

$$\widehat{L} = \widehat{a}^{ij}\partial_{ij}$$

an elliptic operator with constant coefficients.

Actually, it turns out ([9]; see also [4]) that:

$$\widehat{a}^{ij} = \int_{\mathbb{T}^d} a^{ij} m\, dx,$$

$\mathbb{T}^d$ being the d-dimensional torus of measure 1. Moreover, m is the unique equilibrium probability measure on $\mathbb{T}^d$, defined as:

$$L^* m = 0, \quad m > 0 \qquad \text{in} \quad \mathbb{T}^d \quad (L^* = \text{adjoint operator})$$

and

$$\int_{\mathbb{T}^d} m\, dx = 1.$$

It seems to be of some interest the following question: once we do know ([16]) that uniqueness of good solutions to the Dirichlet problem (3), generally, does not hold, is a version of Freidlin theorem for good solutions still valid?

As it will be shown, the answer to this question is positive.

To get our target, in Section 1, elliptic operators on $\mathbb{T}^d$ will be studied. In Section 2 (Theorem A), it will be proved that the convergence in Freidlin theorem is uniform with respect to the coefficients (smooth but arbitrary) of the operator L.

Next, in Section 3, periodic good solutions will be used and (Theorem B) the extension to good solutions of Freidlin's theorem will be proved.

1. Elliptic operators on $\mathbb{T}^d$.

Let $\mathbb{T}^d$ denote the d-dimensional torus of measure 1, naturally imbedded in $\mathbb{R}^d$. Let us identify $\mathbb{T}^d$ with $[-1/2, 1/2)^d$ and the functions on $\mathbb{T}^d$ with periodic functions of period 1 in each of the variables.

In what follows and throughout the paper, $L^p_\# = L^p(\mathbb{T}^d)$ and $W^{2,p}_\# = W^{2,p}(\mathbb{T}^d)$, $1 < p < +\infty$, will stand for the spaces of functions belonging respectively to $L^p_{\text{loc}}(\mathbb{R}^d)$ and $W^{2,p}_{\text{loc}}(\mathbb{R}^d)$, periodic of period 1 in each of the variables.

$C()$ will be positive constants (not always the same) depending on the quantities in parentheses, only.

Let $\mathcal{L}^\lambda_\#$ denote the family of linear elliptic operators:

$$L = a^{ij}(\cdot)\partial_{ij},$$

of the type (1)-(2), with the coefficients $a^{ij} \in L^\infty_\# = L^\infty(\mathbb{T}^d)$.

Let us recall the Alexandrov-Bakel'man-Pucci's estimate (see e.g. [3]), called ABP for short.

Theorem 1.1. (ABP estimate). *Let G be a bounded domain, $u \in W^{2,d}_{\text{loc}}(G) \cap C^0(\overline{G})$ be a solution to $Lu = f \in L^d(G)$ in G, $u = 0$ on ∂G. Then*

$$\| u \|_{L^\infty(G)} \leq C(d, \lambda) \operatorname{diam} G \, \| f \|_{L^d(G)}, \tag{1.1}$$

where $C(d, \lambda)$ is a constant depending only on d and λ.

A version of the ABP estimate on $\mathbb{T}^d$ will be needed. The argument is a rearrangement of one in Krylov [12].

Theorem 1.2. *Let $u \in W^{2,d}_\#$ and $Lu - u = f \in L^d_\#$. Then, there exists $C = C(d, \lambda)$ such that:*

$$\| u \|_{L^\infty(\mathbb{T}^d)} \leq C(d, \lambda) \| f \|_{L^d(\mathbb{T}^d)}. \tag{1.2}$$

Proof. Let us start by making the following remark. Let $N \in \mathbf{N}$, $Q_N \subset \mathbb{R}^d$ be the cube $|x_j| \leq N/2$, $1 \leq j \leq d$, and let in Q_N:

$$b(x) = [\cosh(\lambda^{-1/2}N)]^{-1} \sum_{j=1}^{d} \cosh(\alpha_j^{-1} x_j),$$

where

$$\alpha_j^2 = \sup_{Q_N} a^{jj} = \sup_{Q_1} a^{jj}, \quad \alpha_j \leq \lambda^{1/2}.$$

Then, it turns out that

$$Lb - b \leq 0 \qquad \text{in} \quad Q_N, \quad b|_{\partial Q_N} \geq 1$$
$$b(0) = d[\cosh(\lambda^{-1/2}N)]^{-1}.$$

With no loss of generality one may assume that L has continuous coefficients and $u(0) = \max u > 0$.

Choose the cube Q_N in the above remark, with $N = N(d, \lambda)$ so large that $b(0) \leq 1/2$ and let $v \in W^{2,d}_{loc}(Q_N) \cap C^0(\overline{Q}_N)$ be the solution to

$$Lv - v = f \qquad \text{in} \quad Q_N, \quad v|_{\partial Q_N} = 0.$$

By ABP estimate, one has:

$$\| v \|_{L^\infty(Q_N)} \leq NC(d, \lambda)\| f \|_{L^d(Q_N)}$$

and then the periodicity of f yields:

(1.3) $$\| v \|_{L^\infty(Q_N)} \leq N^{d+1}C(d, \lambda)\| f \|_{L^d(Q_1)}.$$

Define:

$$w = u - v - u(0)b.$$

We have

$$Lw - w \geq 0 \qquad \text{in} \quad Q_N, \quad w|_{\partial Q_N} = [u - u(0)b]_{\partial Q_N} \leq 0.$$

Then it follows that $w \leq 0$ in Q_N, so

$$u - u(0)b \leq v \qquad \text{in} \quad Q_N.$$

Thus, by using (1.3):

$$u - u(0)b \leq N^{d+1}C(d, \lambda)\| f \|_{L^d(Q_1)}.$$

Since $\frac{1}{2} \leq 1 - b(0)$, one gets:

$$\frac{1}{2}u(0) \leq u(0) - u(0)b(0) \leq N^{d+1}C(d, \lambda)\| f \|_{L^d(Q_1)}.$$

As $u(0) = \max u$, we have

$$\max u \leq 2C(d, \lambda)N^{d+1}\| f \|_{L^d(Q_1)}$$

and therefore the theorem is proved. $\square$

Next theorem will provide a further useful estimate.

Theorem 1.3. *Let $u \in W^{2,d}_{\#}$ and $L \in \mathcal{L}^{\lambda}_{\#}$. Then, there exists a constant C, depending on d and λ only, such that*

$$|u(x) - u(0)| \leq C(d, \lambda) \| Lu \|_{L^d_{\#}}, \quad x \in \mathbb{R}^d. \tag{1.4}$$

Proof. Assume that (1.4) is not true; then there exist a sequence of operators $L_\nu \in \mathcal{L}^{\lambda}_{\#}$ and a sequence of functions $u_\nu \in W^{2,d}_{\#}$, satisfying:

$$\max_{x \in \mathbb{R}^d} |u_\nu(x) - u_\nu(0)| = 1, \ \| L_\nu u_\nu \|_{L^d_{\#}} \leq 1, \ \lim_\nu \| L_\nu u_\nu \|_{L^d_{\#}} = 0.$$

The functions $\{u_\nu(x) - u_\nu(0)\}$ are equibounded and equicontinuous; in fact, in every ball B, by Krylov–Safonov results, if $\beta = \beta(d, \lambda, B)$:

$$\begin{aligned} \| u_\nu - u_\nu(0) \|_{C^{0,\beta}(B)} &\leq C(d, \lambda, B)(\| L_\nu u_\nu \|_{L^d_{\#}} + 1) \\ &\leq 2C(d, \lambda, B). \end{aligned}$$

Because of the periodicity, there exists a subsequence still called $\{u_\nu\}$, such that $\{u_\nu - u_\nu(0)\} \to u$ uniformly in $\mathbb{R}^d$. The function u is periodic, Hölder continuous, $u(0) = 0$ and $\max |u| = 1$. By using again Krylov-Safonov results (see [10]), if B_R and B_{R_0}, $B_R \subset B_{R_0}$ are balls, there exist $C(d, \lambda)$ and $\beta(d, \lambda) > 0$ such that:

$$\text{osc}_{B_R}(u_\nu) \leq C(d, \lambda) \left(\frac{R}{R_0} \right)^{\beta} [\text{osc}_{B_{R_0}}(u_\nu) + R_0 \| L_\nu u_\nu \|_{L^d(B_{R_0})}];$$

as $\text{osc}_{B_{R_0}}(u_\nu) \leq 2$, $\| L_\nu u_\nu \|_{L^d(B_{R_0})} \to 0$ as $\nu \to \infty$, we have:

$$\text{osc}_{B_R} u \leq 2C(d, \lambda) \left(\frac{R}{R_0} \right)^{\beta}.$$

Since R/R_0 can be made arbitrarily small, then u is constant and, being $u(0) = 0$, it follows $u \equiv 0$ contradicting $\max u = 1$. □

A version of Fredholm alternative for equations with periodic coefficients is true.

Theorem 1.4. *Let $L \in \mathcal{L}^{\lambda}_{\#}$, L with smooth coefficients a^{ij}. Then*

(i) *there exists a positive smooth function m on $\mathbb{T}^d$, such that $\int_{\mathbb{T}^d} m \, dx = 1$ and $L^* m = 0$, L^* being the adjoint of L;*

(ii) *the equation*

$$Lu = f \in L^p(\mathbb{T}^d), \quad p \geq d,$$

has a solution $u \in W^{2,p}(\mathbb{T}^d)$ if and only if

$$\int_{\mathbb{T}^d} f m \, dx = 0.$$

A proof of (i) and (ii), with $f \in C^\infty(\mathbb{T}^d)$ is in [19]. The present result follows.

Remark 1.1. The above function m (unique) is referred as *the equilibrium probability measure of L*.

It is not difficult to get the following result (proved in $\mathbb{R}^d$ by Krylov [12]).

Theorem 1.5. *The equilibrium probability measure m of L, in theorem 1.4, is such that:*

$$\| m \|_{L^{d/(d-1)}(\mathbb{T}^d)} \leq K = K(d, \lambda). \tag{1.5}$$

Proof. If $\varphi \in W^{2,d}_{\#}$, we have that:

$$\int_{\mathbb{T}^d} m(L\varphi - \varphi)\, dx = -\int_{\mathbb{T}^d} m\varphi\, dx.$$

Let now $f \in L^d_{\#}$. By classical arguments it turns out that there exists a unique $\varphi \in W^{2,d}_{\#}$ such that $L\varphi - \varphi = f$. Thus

$$\left| \int_{\mathbb{T}^d} mf\, dx \right| = \left| \int_{\mathbb{T}^d} m\varphi\, dx \right| \leq \| \varphi \|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} m\, dx.$$

Taking into account Theorem 1.2, we get:

$$\left| \int_{\mathbb{T}^d} mf\, dx \right| \leq C(d, \lambda) \| f \|_{L^d_{\#}}$$

and, therefore, the estimate (1.5) follows. $\square$

2. The uniform convergence theorem for smooth operators and a priori bounds.

A revised form of the convergence theorem by Freidlin, recalled in the introduction, is studied.

In [9], for the proof of his result, M.I. Freidlin used probabilistic arguments. An alternative proof has been given, recently, by M. Safonov in [19]. By revisiting Safonov proof, an a priori bound is proved which will yield a convergence result, uniform also with respect to the operators.

Let $L \in \mathcal{L}_{\#}^{\lambda}$ a smooth operator. Let

$$L_t = a^{ij}(t\cdot)\partial_{ij} \quad \text{and} \quad \widehat{L} = \widehat{a}^{ij}\partial_{ij} \tag{2.1}$$

with $\widehat{a}^{ij} = \int_{\mathbb{T}^d} a^{ij} m\, dx$, m being the equilibrium probability measure for L.

Let g be a uniformly continuous function, bounded in $\mathbb{R}^d$ and let $\omega_g(\delta)$ be the modulus of continuity of g, $\omega_g(\delta) = \sup_{x,y\in\mathbb{R}^d,\,|x-y|\le\delta} |g(x) - g(y)|$.

The following result holds.

Theorem A. *Let Ω be a bounded smooth domain of $\mathbb{R}^d$. Let the operators $L, L_t, \widehat{L}$ and the function g as above. Let:*

$$L_t u_{(t)} = 0 \quad in \quad \Omega, \quad u_{(t)}|_{\partial\Omega} = g,$$

and

$$\widehat{L}\,\overline{u} = 0 \quad in \quad \Omega, \quad \overline{u}|_{\partial\Omega} = g.$$

Then, if $t > 1$:

$$\| u_{(t)} - \overline{u} \|_{L^\infty(\Omega)} \le C(d, \lambda, \Omega)\frac{1}{t^{1/11}}[1 + \| g \|_{C^0}^2] + 2\omega_g(\frac{1}{t^{1/11}}). \tag{2.2}$$

Proof. We have that

$$\int_{\mathbb{T}^d} (a^{ij} - \widehat{a}^{ij})m\, dx = 0.$$

Then, by Theorem 1.4 there exist A^{ij} smooth and periodic, satisfying $A^{ij}(0) = 0$ and

$$LA^{ij} = a^{ij} - \widehat{a}^{ij}, \quad 1 \le i, j \le d.$$

Moreover, by Theorem 1.3, one gets:

$$\sup_{x\in\mathbb{R}^d} |A^{ij}(x)| = \sup_{x\in\mathbb{T}^d} |A^{ij}(x)| \le C(d, \lambda)\| a^{ij} - \widehat{a}^{ij} \|_{L^d_{\#}} \le C(d, \lambda). \tag{2.3}$$

Note that:

$$\frac{1}{t^2}L_t[A^{ij}(t\cdot)] = (LA^{ij})(t\cdot) = a_t^{ij} - \widehat{a}^{ij}. \tag{2.4}$$

Let us assume, for a moment, the boundary datum g to be smooth. Define:

$$v_{(t)}(x) = \overline{u}(x) - \frac{1}{t^2}A^{ij}(tx)\partial_{ij}\overline{u}(x).$$

One has:

$$v_{(t)}|_{\partial\Omega} = g - \frac{1}{t^2} A^{ij}(t\cdot)\partial_{ij}\overline{u}|_{\partial\Omega}$$

and moreover:

$$L_t v_{(t)} = L_t \overline{u} - \frac{1}{t^2} L_t[A^{ij}(t\cdot)]\partial_{ij}\overline{u} +$$
$$- \frac{1}{t^2} A^{ij}(t\cdot) L_t(\partial_{ij}\overline{u}) - \frac{2}{t} a_t^{hk}(\partial_h A^{ij})(t\cdot)\partial_k(\partial_{ij}\overline{u}).$$

Taking into account (2.4), it follows that in Ω :

$$L_t(v_{(t)} - u_{(t)}) = -\frac{1}{t^2} A^{ij}(t\cdot) L_t(\partial_{ij}\overline{u}) +$$
$$- \frac{2}{t} a_t^{hk}(\partial_h A^{ij})(t\cdot)\partial_k(\partial_{ij}\overline{u}).$$

We have also:

$$(v_{(t)} - u_{(t)})|_{\partial\Omega} = -\frac{1}{t^2} A^{ij}(t\cdot)\partial_{ij}\overline{u}|_{\partial\Omega}.$$

Let $w_0 \in C^3(\overline{\Omega})$ be the solution to:

$$L_t w_0 = -\frac{1}{t^2} A^{ij}(t\cdot) L_t(\partial_{ij}\overline{u}) \qquad \text{in } \Omega,$$
$$w_0|_{\partial\Omega} = -\frac{1}{t^2} A^{ij}(t\cdot)\partial_{ij}\overline{u}|_{\partial\Omega}.$$

Then, by maximum principle and (2.3):

$$\| w_0 \|_{L^\infty(\Omega)} \le \frac{1}{t^2} \sup_{\partial\Omega} |A^{ij}(t\cdot)\partial_{ij}\overline{u}| + \tag{2.5}$$
$$+ \frac{1}{t^2} C(d, \lambda, \Omega) \left\| D^4 \overline{u} \right\|_{L^\infty(\Omega)} \Sigma_{ij} \left\| A^{ij}(t\cdot) \right\|_{L^\infty(\Omega)} \le$$
$$\le \frac{1}{t^2} C(d, \lambda, \Omega) \| \overline{u} \|_{C^4(\overline{\Omega})}.$$

On the other hand:

$$\left| - \frac{2}{t} a_t^{hk}(\partial_h A^{ij})(t\cdot)\partial_k(\partial_{ij}\overline{u}) \right| \le$$
$$\le \frac{1}{t} a_t^{hk} \Sigma_{ij} (\partial_h \partial_{ij}\overline{u})(\partial_k \partial_{ij}\overline{u}) +$$
$$+ \frac{1}{t} a_t^{hk} \Sigma_{ij} (\partial_h A^{ij})(t\cdot)(\partial_k A^{ij})(t\cdot).$$

Let w_1, w_2 be the solutions to the problems:

$$L_t w_1 = -\frac{1}{t} a_t^{hk} \Sigma_{ij} (\partial_h \partial_{ij} \overline{u})(\partial_k \partial_{ij} \overline{u}) \quad \text{in } \Omega, \quad w_1|_{\partial\Omega} = 0,$$

and

$$L_t w_2 = -\frac{1}{t} a_t^{hk} \Sigma_{ij} (\partial_h A^{ij})(t\cdot)(\partial_k A^{ij})(t\cdot) \quad \text{in } \Omega, \quad w_2|_{\partial\Omega} = 0.$$

By maximum principle:

$$\| v_{(t)} - u_{(t)} - w_0 \|_{L^\infty(\Omega)} \le \| w_1 \|_{L^\infty(\Omega)} + \| w_2 \|_{L^\infty(\Omega)} \tag{2.6}$$

and

$$\| w_1 \|_{L^\infty(\Omega)} \le \frac{1}{t} C(d, \lambda, \Omega) \Big(\| D^3 \overline{u} \|_{L^\infty(\Omega)} \Big)^2. \tag{2.7}$$

To evaluate w_2, let us use the technique of [8]. Let $G_{(t)}$ be the Green function, in Ω, to the Dirichlet problem for L_t. Then:

$$\begin{aligned}
w_2(x) &= \frac{1}{t} \int_\Omega G_{(t)}(x, y) a_t^{hk}(y) \Sigma_{ij} (\partial_h A^{ij})(ty)(\partial_k A^{ij})(ty)\, dy = \\
&= \frac{1}{t^3} \int_\Omega G_{(t)}(x, y) a_t^{hk}(y) \Sigma_{ij} \frac{\partial}{\partial y_h}[A^{ij}(ty)] \frac{\partial}{\partial y_k}[A^{ij}(ty)]\, dy = \\
&= \frac{1}{t^3} \int_\Omega G_{(t)}(x, y) \{L_t[\frac{1}{2} \Sigma_{ij} (A^{ij})^2(t\cdot)]\}_y\, dy + \\
&- \frac{1}{t^3} \int_\Omega G_{(t)}(x, y) \Sigma_{ij} A^{ij}(ty) \{L_t[A^{ij}(t\cdot)]\}_y\, dy.
\end{aligned}$$

By maximum principle and (2.3):

$$\begin{aligned}
\Big| \frac{1}{2t^3} \int_\Omega G_{(t)}(x, y) \{L_t[\Sigma_{ij} (A^{ij})^2(t\cdot)]\}_y\, dy \Big| &\le \frac{1}{t^3} \sup_\Omega \Sigma_{ij} ((A^{ij})(t\cdot))^2 \le \\
&\le \frac{C(d, \lambda)}{t^3}.
\end{aligned}$$

The equation (2.4) and the bound (2.3) give us:

$$\begin{aligned}
&\Big| \frac{1}{t^3} \int_\Omega G_{(t)}(x, y) \Sigma_{ij} A^{ij}(ty) \{L_t[A^{ij}(t\cdot)]\}_y\, dy \Big| = \\
&= \Big| \frac{1}{t} \int_\Omega G_{(t)}(x, y) \Sigma_{ij} A^{ij}(ty)(a^{ij}(ty) - \hat{a}^{ij})\, dy \Big| \le \\
&\le \frac{1}{t} C(d, \lambda, \Omega).
\end{aligned}$$

Then:

$$\| w_2 \|_{L^\infty(\Omega)} \le C(d, \lambda, \Omega)\frac{1}{t}. \tag{2.8}$$

The bounds (2.6), (2.5), (2.7), (2.8), give us:

$$\left\| u_{(t)} - v_{(t)} \right\|_{L^\infty(\Omega)} \le C(d, \lambda, \Omega)\frac{1}{t}[1 + (\| \overline{u} \|_{C^4(\overline{\Omega})})^2]. \tag{2.9}$$

On the other hand the definition of $v_{(t)}$ and (2.3) give us:

$$\left\| v_{(t)} - \overline{u} \right\|_{L^\infty(\Omega)} \le \frac{1}{t^2}C(d, \lambda)\| \overline{u} \|_{C^2(\overline{\Omega})}.$$

This inequality and (2.9), give:

$$\begin{aligned} \left\| u_{(t)} - \overline{u} \right\|_{L^\infty(\Omega)} &\le \\ &\le \frac{1}{t}C(d, \lambda, \Omega)\Big(1 + (\| \overline{u} \|_{C^4(\overline{\Omega})})^2\Big) \le \\ &\le \frac{1}{t}C(d, \lambda, \Omega)\Big(1 + (\| g \|_{C^5(\partial\Omega)})^2\Big). \end{aligned} \tag{2.10}$$

Assume, now, $g \in C^0$. For every $\epsilon > 0$, there exists a function $g^{(\epsilon)} \in C^\infty$, satisfying:

$$\left\| g - g^{(\epsilon)} \right\|_{C^0(\partial\Omega)} \le \omega_g(\epsilon)$$

and:

$$\left\| g^{(\epsilon)} \right\|_{C^5(\partial\Omega)} \le \frac{C(d, \Omega)}{\epsilon^5}\| g \|_{C^0}.$$

Let us consider the Dirichlet problems:

$$L_t u^\epsilon_{(t)} = 0 \quad \text{in} \quad \Omega, \quad u^\epsilon_{(t)}|_{\partial\Omega} = g^\epsilon,$$

and

$$\hat{L}\overline{u}^\epsilon = 0 \quad \text{in} \quad \Omega, \quad \overline{u}^\epsilon|_{\partial\Omega} = g^\epsilon.$$

The maximum principle, the properties of $g^{(\epsilon)}$ and (2.10) give:

$$\left\| u_{(t)} - u^\epsilon_{(t)} \right\|_{C^0(\overline{\Omega})} \le \omega_g(\epsilon),$$

$$\| \overline{u}^\epsilon - \overline{u} \|_{C^0(\overline{\Omega})} \le \omega_g(\epsilon),$$

$$\left\| \overline{u}^\epsilon - u^\epsilon_{(t)} \right\|_{C^0(\overline{\Omega})} \le \frac{1}{t}C(d, \lambda, \Omega)[1 + (\frac{C(d, \Omega)}{\epsilon^5}\| g \|_{C^0})^2],$$

i.e.:

$$\left\| u_{(t)} - \overline{u} \right\|_{C^0(\overline{\Omega})} \le 2\omega_g(\epsilon) + \frac{1}{t}C(d, \lambda, \Omega)[1 + (\frac{C(d, \Omega)}{\epsilon^5}\| g \|_{C^0})^2];$$

choosing $\epsilon = t^{-1/11}$, the thesis follows. $\square$

3. Periodic good solutions and the main theorem.

Let $L \in \mathcal{L}^{\lambda}_{\#}$ and $f \in L^{d}_{\#}$.

Definition. A function $u \in C^{0,\alpha}_{\#}$ (for some $0 < \alpha < 1$) is a $\{L_n\}$-good solution to $Lu = f$, if there exist sequences $\{L_n\}$ and $\{u_n\}$, $L_n \in \mathcal{L}^{\lambda}_{\#}$ smooth, $u_n \in W^{2,d}_{\#}$, such that:

$\{L_n\} \to L$ a.e., $\{u_n\} \to u$ uniformly on every compact subset of $\mathbb{R}^d$, and $\{L_n u_n\} \to Lu$ in $L^{d}_{\#}$.

The following theorem holds.

Theorem 3.1. *Let $L \in \mathcal{L}^{\lambda}_{\#}$ and $\{L^{(0)}_{\nu}\}$ be a sequence of smooth operators from $\mathcal{L}^{\lambda}_{\#}$, $\{L^{(0)}_{\nu}\} \to L$ a.e.. Let $f \in L^{d}_{\#}$. There exist:*

(i) *$\{L_{\nu}\}$, a subsequence to $\{L^{(0)}_{\nu}\}$,*

(ii) *$m > 0$ a.e., $m \in L^{d/(d-1)}_{\#}$, $\int_{\mathbb{T}^d} m\, dx = 1$, solution to $L^* m = 0$ (i.e. $\int_{\mathbb{T}^d} m \cdot L\phi\, dx = 0$, for every $\phi \in C^{\infty}_{\#}$),*

such that if $\int_{\mathbb{T}^d} f \cdot m\, dx = 0$ then $Lu = f$ has a $\{L_{\nu}\}$-good solution.

The function $m = m(\{L_{\nu}\})$ above can be referred as the $\{L_{\nu}\}$-equilibrium probability measure for L. Such a measure m is $\{L_{\nu}\}$-unique.

Proof. Let $\{m^{(0)}_{\nu}\}$ be the sequence of equilibrium probability measures corresponding to $\{L^{(0)}_{\nu}\}$, given by (i) of theorem 1.4. Then:

$$1 = \int_{\mathbb{T}^d} m^{(0)}_{\nu}\, dx \leq 1 \cdot \| m^{(0)}_{\nu} \|_{L^{d/(d-1)}_{\#}} \leq C(d, \lambda);$$

thus, there exists a subsequence $\{m_{\nu}\}$ weakly convergent in $L^{d/(d-1)}_{\#}$ to a function $m \geq 0$ a.e. solution to $L^* m = 0$; moreover

$$1 = \int_{\mathbb{T}^d} m_{\nu}\, dx \to \int_{\mathbb{T}^d} m\, dx.$$

Let $\{L_{\nu}\}$ the corresponding subsequence of operators; let f in a countable dense subset of $\{f : f \in L^{d}_{\#}, \int_{\mathbb{T}^d} f \cdot m\, dx = 0\}$.

Set:

$$f_{\nu} = f - m^{1/(d-1)}_{\nu} \Big(\int_{\mathbb{T}^d} f m_{\nu}\, dx \Big) \Big(\int_{\mathbb{T}^d} m^{d/(d-1)}_{\nu} \Big)^{-1}.$$

One has that: $f_\nu \in L^d_\#$ and

$$\| f_\nu - f \|_{L^d_\#} \le C(d,\lambda) \left| \int_{\mathbb{T}^d} f m_\nu \, dx \right| \to 0, \quad \text{as} \quad \nu \to +\infty;$$

moreover:

$$\int_{\mathbb{T}^d} f_\nu m_\nu \, dx = 0.$$

Now, let $u_\nu \in W^{2,d}_\#$ be the unique solution to $L_\nu u_\nu = f_\nu$, $u_\nu(0) = 0$. By Theorem 1.3 and Krylov-Safonov results, there exists a subsequence, that can be called again $\{u_\nu\}$, converging uniformly to a function u_0 periodic, $u_0(0) = 0$, satisfying for $x \in \mathbb{R}^d$:

$$|u_0(x)| \le C \| f \|_{L^d_\#},$$

$\{L_\nu\}$-good solution to $Lu_0 = f$. Using a diagonalization process and an approximation technique, the existence theorem in (ii) is proved.

To end the proof of the theorem, one has to show that m is actually a.e. positive. Assume, indeed, by contradiction that $m = 0$ on a set $E \subset \mathbb{T}^d$, with positive measure.

Let $\overline{f} = -\mathbf{1}_E \le 0$. Since

$$\int_{\mathbb{T}^d} \overline{f} m \, dx = 0,$$

then the equation $L\overline{u} = \overline{f} \le 0$ has a periodic non constant $\{L_\nu\}$-good solution $\overline{u}$, which cannot have interior minima. Contradiction. $\square$

Let $\mathcal{L}^\lambda$ denote the family of linear elliptic operators of the type (1)-(2) with coefficients $a^{ij} \in L^\infty(\mathbb{R}^d)$. Clearly $\mathcal{L}^\lambda \supset \mathcal{L}^\lambda_\#$.

Theorem 3.2. *Let $L \in \mathcal{L}^\lambda$ and $\{L_k\}$ be a sequence of smooth operators $L_k \in \mathcal{L}^\lambda$, such that $L_k \to L$ almost everywhere in $\mathbb{R}^d$. Then there exists $\{L_{[s]}\}$, subsequence of $\{L_k\}$, with the property:*

for any smooth bounded subdomain Ω of $\mathbb{R}^d$ and for any $f \in L^d(\Omega)$, the problem:

$$Lu = f \qquad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

has a unique $\{L_{[s]}\}$-good solution.

To prove this theorem the following Lemma is needed, which provides a bound uniform, with respect to the operators from $\mathcal{L}^\lambda$, for solutions to Dirichlet problems.

Lemma. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^d$, $\eta > 0$ *and let* $D \subset \Omega$ *a smooth subdomain, such that* $\partial D \in \{x \in \mathbb{R}^d : dist(x, \partial\Omega) \leq \eta\}$. *Let* $L \in \mathcal{L}^\lambda$ *a smooth operator. Let* $f_0 \in L^d(D)$, $f \in L^d(\Omega)$.

Assume that:

$$u \in W^{2,d}_{\gamma_0}(D), \quad Lu = f_0 \quad in\ D,$$

$$v \in W^{2,d}_{\gamma_0}(\Omega), \quad Lv = f \quad in\ \Omega.$$

Then, there exist $\beta = \beta(\lambda, d) \in (0, 1)$ *and* $C(d, \lambda, \Omega, f)$ *such that*

$$\| u - v \|_{L^\infty(D)} \leq C(d, \lambda, \Omega) \| f_0 - f \|_{L^d(D)} + \eta^\beta C(d, \lambda, \Omega, f).$$

Proof. Writing: $u - v = (u - z) + (z - v)$ with $z \in W^{2,d}_{\gamma_0}(D)$ solution to $Lz = f$ in D, the thesis of the Lemma follows from Alexandrov-Bakel'man-Pucci estimate and boundary estimates (see e.g. [10]). $\square$

Proof of Theorem 3.2.

Let $\{\Omega^{(h)}\}$ be a countable family of smooth bounded open sets, such that for every bounded domain Ω in $\mathbb{R}^d$ there exists $\{\Omega^{(h_\nu)}\} \uparrow \Omega$. Moreover, let $f^h_k \in C^\infty_0(\Omega^{(h)})$ be a countable family of functions such that for every $f \in L^d(\Omega^{(h)})$ there exists $f^h_{k_\mu} \to f$ in $L^d(\Omega^{(h)})$, as $\mu \to \infty$.

By a diagonalization process one can construct $\{L_{[s]}\}$, subsequence of $\{L_k\}$, such that for every h and k the problem:

$$Lu^{(h)}_k = f^h_k \quad \text{in } \Omega^{(h)}, \quad u^{(h)}_k|_{\partial\Omega^{(h)}} = 0$$

has a unique $\{L_{[s]}\}$-good solution.

Choose a smooth bounded domain $\Omega \in \mathbb{R}^d$ and take $f \in L^d(\Omega)$. Let $\Omega^{(h_\nu)} \uparrow \Omega$ and $f^{h_\nu}_{k_\mu} \to f$ in $L^d(\Omega^{(h_\nu)})$ as $\mu \to +\infty$. Solve, then, the problems:

$$L_{[s]} u_s = f \quad \text{in } \Omega, \quad u_s \in W^{2,d}_{\gamma_0}(\Omega),$$

and

$$L_{[s]} u^{\nu,\mu}_s = f^{h_\nu}_{k_\mu} \quad \text{in } \Omega^{(h_\nu)}, \quad u^{\nu,\mu}_s \in W^{2,d}_{\gamma_0}(\Omega^{(h_\nu)}).$$

By previous Lemma, for any $s \in \mathbb{N}$, $\eta > 0$, if $\partial\Omega^{(h_\nu)} \in \{x : \operatorname{dist}(x, \partial\Omega) \leq \eta\}$:

$$\| u_s - u_s^{\nu,\mu} \|_{L^\infty(\Omega^{(h_\nu)})} \leq \tag{3.1}$$

$$\leq C(d, \lambda, \Omega)\Big\| f - f_{k_\mu}^{h_\nu} \Big\|_{L^d(\Omega^{(h_\nu)})} + \eta^\beta C(d, \lambda, \Omega, f).$$

Fix $\epsilon > 0$. By choosing, first, η suitably small, ν suitably large and taking, then, μ suitably large too, one gets from (3.1) that:

$$\| u_s - u_s^{\nu,\mu} \|_{L^\infty(\Omega^{(h_\nu)})} < \epsilon, \quad \text{for any } s \in \mathbb{N}.$$

Let now $p \in \mathbb{N}$. Then:

$$\| u_s - u_{s+p} \|_{L^\infty(\Omega^{(h_\nu)})} \leq 2\epsilon + \| u_s^{\nu,\mu} - u_{s+p}^{\nu,\mu} \|_{L^\infty(\Omega^{(h_\nu)})}.$$

As $\{u_s^{\nu,\mu}\}$ is uniformly convergent in $\overline{\Omega}^{(h_\nu)}$, as $s \to \infty$, it follows that $\{u_s\}$ does converge to $u \in C^0(\Omega)$ in every compact subset of Ω. Moreover, by Alexandrov-Bakel'man-Pucci and Krylov-Safonov results, $\{u_s\}$ is equibounded and equicontinuous in $\overline{\Omega}$; therefore $u_s \to u$ uniformly in $\overline{\Omega}$ and u is a $\{L_{[s]}\}$-good solution to $Lu = f$ in Ω, $u|_{\partial\Omega} = 0$. □

With similar arguments, the following fact can be proved.

Corollary. *Let $L \in \mathcal{L}^\lambda$, $\{\overline{L}_k\}$ a sequence of smooth operators such that $\overline{L}_k \to L$ a.e. in $\mathbb{R}^d$. Then the sequence $\{L_{[s]}\}$, constructed in theorem 3.2, has also the following property:*

for any smooth bounded subdomain Ω of $\mathbb{R}^d$ and $g \in C^0(\partial\Omega)$, the problem:

$$Lu = 0 \quad in\, \Omega, \quad u|_{\partial\Omega} = g$$

admits a unique $\{L_{[s]}\}$-good solution.

Now the claimed extension to good solutions of the Freidlin's result can be proved.

Let $L \in \mathcal{L}_\#^\lambda$ and $\{L_\nu^{(0)}\}$ a sequence of smooth operators from $\mathcal{L}_\#^\lambda$ such that $\{L_\nu^{(0)}\} \to L$ almost everywhere. Moreover, let

$$L_t = a^{ij}(t\cdot)\partial_{ij} \in \mathcal{L}^\lambda$$

and

$$\widehat{L} = \widehat{a}^{ij}\partial_{ij}, \quad \text{with } \widehat{a}^{ij} = \int_{\mathbb{T}^d} a^{ij} m \, dx,$$

where m is the $\{L_\nu^{(0)}\}$-equilibrium probability measure associated to L (see Theorem 3.1).

The following result holds:

Theorem B. *Let L, $\{L_\nu^{(0)}\}$, L_t and $\widehat{L}$ as above. Then there exists $\{L_{[s]}\}$ a subsequence to $\{L_\nu^{(0)}\}$ with the following properties:*
(i) *for any $t \geq 1$, Ω smooth bounded domain of $\mathbb{R}^d$, the problem:*

$$L_t u_{(t)} = 0 \quad \textit{in } \Omega, \quad u_{(t)}|_{\partial\Omega} = g \in C^0(\partial\Omega)$$

has a unique $\{(L_{[s]})_t\}$ -good solution $u_{(t)}$. (Here $\{(L_{[s]})_t\} = a_s^{ij}(t\cdot)\partial_{ij}$);
(ii) *$u_{(t)}$ converges uniformly in $\overline{\Omega}$ to $\overline{u}$, where $\overline{u}$ is the classical solution to*

$$\widehat{L}\overline{u} = 0 \quad \textit{in } \Omega, \quad \overline{u}|_{\partial\Omega} = g.$$

Proof. From the corollary to Theorem 3.2, there exists $\{L_{[s]}\}$, subsequence to $\{L_\nu^{(0)}\}$, such that for every smooth bounded subset D of $\mathbb{R}^d$, $\gamma \in C^0(\partial D)$, there exists a unique $\{L_{[s]}\}$-good solution to $Lu = 0$ in D, $u|_{\partial D} = \gamma$.

Let $L_{[s]} = a_{[s]}^{ij}\partial_{ij}$ and $m_{[s]}$ the corresponding equilibrium probability measure. Let us assume, with no loss of generality, that $m_{[s]}$ weakly converges to m in $L_\#^{d/(d-1)}$ (see Theorem 3.1); moreover, let us also assume that $a_{[s]}^{ij} \to a^{ij}$ in $L_\#^d$. Thus:

$$\widehat{a}_{[s]}^{ij} = \int_{\mathbb{T}^d} a_{[s]}^{ij} m_{[s]}\, dx \;\to\; \int_{\mathbb{T}^d} a^{ij} m\, dx = \widehat{a}^{ij}.$$

Let us prove (i).

Let us extend g as a bounded, uniformly continuous function in $\mathbb{R}^d$ and let us use a scaling argument. Set:

$$\Omega_t = \{x' : x' = tx,\ x \in \Omega\}.$$

Then, the problem:

$$Lv_{(t)} = 0 \quad \text{in } \Omega_t, \quad v_{(t)}|_{\partial\Omega_t} = g(\frac{\cdot}{t})$$

has a unique $\{L_{[s]}\}$-good solution $v_{(t)}$.

Let $v_{[s](t)}$ such that

$$L_{[s]}v_{[s](t)} = 0 \quad \text{in } \Omega_t, \quad v_{[s](t)}|_{\partial\Omega_t} = g(\frac{\cdot}{t})$$

and

$$v_{[s](t)} \;\to\; v_{(t)} \quad \text{uniformly in } \overline{\Omega}_t, \text{ as } s \to +\infty.$$

Then

$$u_{[s](t)} := v_{[s](t)}(t\cdot)$$

solves

$$(L_{[s]})_t u_{[s](t)} = 0 \quad \text{in } \Omega, \quad u_{[s](t)}|_{\partial\Omega} = g$$

and, as $s \to +\infty$, $u_{[s](t)} \to u_{(t)}$ uniformly in $\overline{\Omega}$, with

$$u_{(t)} := v_{(t)}(t\cdot).$$

That is, $u_{(t)}$ is the unique $\{L_{[s]t}\}$-good solution to the problem:

$$L_t u_{(t)} = 0 \quad \text{in } \Omega, \quad u_{(t)}|_{\partial\Omega} = g.$$

Therefore (i) is proved.

To get (ii), now, let us start from the estimate:

$$\begin{aligned} \| u_{(t)} - \overline{u} \|_{L^\infty} \le \| u_{(t)} - u_{[s](t)} \|_{L^\infty} + \| u_{[s](t)} - \overline{u}_{[s]} \|_{L^\infty} + \\ + \| \overline{u}_{[s]} - \overline{u} \|_{L^\infty}. \end{aligned} \tag{3.2}$$

Recall, then, that:

(a) for any t, $u_{[s](t)} \to u_{(t)}$ uniformly in $\overline{\Omega}$, as $s \to +\infty$;
(b) by Theorem A:

$$\begin{aligned} \| u_{[s](t)} - \overline{u}_{[s]} \|_{L^\infty(\Omega)} \le& \\ &\le \mathcal{C}(d, \lambda, \Omega) \frac{1}{t^{1/11}} [1 + \| g \|^2_{C^0}] + \\ &+ 2\omega_g\left(\frac{1}{t^{1/11}}\right). \end{aligned}$$

uniformly with respect to s.

Moreover, as $\widehat{L}_{[s]}\overline{u}_{[s]} = \widehat{L}\overline{u} = 0$ in Ω, $\overline{u}_{[s]} = \overline{u}$ on $\partial\Omega$ and $\widehat{a}^{ij}_{[s]} \to \widehat{a}^{ij}$, using classical arguments for constant coefficients elliptic operators:

$$\| \overline{u}_{[s]} - \overline{u} \|_{L^\infty(\Omega)} \to 0. \tag{3.3}$$

Hence, $u_{(t)} \to \overline{u}$ uniformly in $\overline{\Omega}$. $\square$

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