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A FINITE ELEMENT APPROXIMATION AND UNIFORM ERROR ESTIMATES FOR DEGENERATE ELLIPTIC EQUATIONS

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Let Ω denote the bounded subset of $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$ defined by $\Omega =]-1, 1[\times]-1, 1[$, and let Γ be its boundary. We consider the second order differential operator in divergence form in Ω defined by

$$\mathcal{L} = \partial_x^2 + \lambda^2(x)\partial_y^2, \tag{1}$$

where λ is a bounded non negative Lipschitz continuous function in $\mathbb{R}$ belonging to RH_∞, i.e. such that for any compact interval $I \subset \mathbb{R}$

$$0 < c_1 \max_I \lambda \leq \frac{1}{|I|}\int_I \lambda(x)\,dx \leq \max_I \lambda \tag{2}$$

where $|I|$ denotes the Lebesgue measure of I and c_1 is a positive constant independent of I (see [10], [2] for related examples and a geometric interpretation of condition (2)). In particular, the following crucial inequality follows from (2) (see, e.g., [10], Proposition 2.2): there exists $\gamma \geq 0$ such that

$$\int_0^t \lambda(x + s\xi)\,ds \geq ct^{1+\gamma} \tag{3}$$

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for any x in a neighborhood of $]-1, 1[$, ξ in a small interval that does not contain the origin and t small. To introduce simpler notations, let us denote by X and Y the two vector fields ∂_x and $\lambda(x)\partial_y$. In [10] the last two authors developed a finite element approximation scheme for homogeneous the Dirichlet problem in Ω associated with $\mathcal{L}$, together with a sharp error estimates in a class of intrinsic Sobolev spaces (the spaces $W^{1,2}_\lambda(\Omega)$ we shall introduce below). In this Note we shall apply the same scheme to the Dirichlet problem

$$\begin{cases} -\mathcal{L}u = f_0 + Xf_1 + Yf_2 \quad \text{in} \quad \Omega \\ u = u_0 \quad \text{on} \quad \Gamma, \end{cases} \tag{4}$$

where u_0 belongs to a suitable Sobolev space, and $f_0, f_1, f_2 \in L^p(\Omega)$ with $p > 2 + \gamma$, and we shall prove L^∞ error estimates via a discrete maximum principle, by adapting to our scheme a technique introduced by Ciarlet ([4], [5]). In fact, the possibility of this approach relies on the validity of Stampacchia's maximum principle for our class of degenerate elliptic operators. We stress that maximum principles for operators like $\mathcal{L}$ go back to Bony's pioneering work [1]. Moreover we notice that the condition $p > 2+\gamma$ has an intrinsic geometric meaning related to Stampacchia's maximum principle through Sobolev imbedding theorem (see [7]), since the quantity $2 + \gamma$ plays the role of a dimension (the homogeneous dimension), and thus naturally replaces the condition $p > n$ for elliptic operators. This fact, together with its influence on error estimates, is widley analyzed in [10].

Let us start by introducing a class of intrinsic anisotropic Sobolev spaces that we shall use throughout this paper, and by formulating precise assumptions on u_0. We set

$$W^{1,p}_\lambda(\Omega) = \{u \in L^p(\Omega);\, Xu, Yu \in L^p(\Omega)\},$$

endowed with its natural norm, and we choose $u_0 \in W^{1,p}_\lambda(\Omega) \subset W^{1,2}_\lambda(\Omega)$. In fact, if in particular $\lambda(x) = |x|^\gamma$, due to the particular structure of the vector fields X and Y and to the geometry of Ω, then we could start from a function u_0 defined only on Γ and belonging to suitable trace spaces, thanks to the existence of sharp trace theorems for $W^{1,p}_\lambda(\Omega)$: see [6]. Moreover, we point out that u_0 is continuous in $\overline{\Omega}$. Indeed, this follows basically from [12], where it is proved in particular that, if $p > \gamma + 2$, then a function $u \in W^{1,p}_\lambda(\Omega)$ is continuous in Ω, so that we have only to show that u_0 can be continued outside Ω is an open neighborhood Ω_0 of $\overline{\Omega}$. Extension theorems for Sobolev spaces associated with a general family of vector fields are still an open problem, but here such an extension can be easily provided, because of the structure of Ω,

X and Y. Indeed, we have only to build an extension in a neighborhood of the points $(0, \pm 1)$, since away from $x = 0$ our intrinsic Sobolev space $W_\lambda^{1,p}(\Omega)$ is nothing but an usual Sobolev space of order one, in a Lipschitz domain for which extension theorems are well known. Now an easy computation shows that a continuation by reflextion across $y = \pm 1$ is compatible with our spaces, and we are done.

Problem (4) can be formulated now in the weak form

$$\begin{cases} a(v,u) = \ell(v) \quad \text{for all} \quad v \in \overset{\circ}{W}{}_\lambda^{1,2}(\Omega), \\ u - u_0 \in \overset{\circ}{W}{}_\lambda^{1,2}(\Omega) \end{cases} \tag{5}$$

where $\overset{\circ}{W}{}_\lambda^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_\lambda^{1,2}(\Omega)$,

$$a(u,v) = a(v,u) = \int_\Omega (\partial_x u \partial_x v + \lambda^2(x) \partial_y u \partial_y v)\, dx dy$$

and

$$\ell(v) = \int_\Omega (f_0 v - f_1 \partial_x v - \lambda(x) f_2 \partial_y v)\, dx dy.$$

Theorem 1. *Suppose $p > \gamma + 2$; then problem (5) has a unique variational solution that il Hölder continuous up to the boundary.*

Proof. The existence of a solution follows from Lax-Milgram theorem. The regularity can be derived from the general result stated in Theorem 3.

If $n \in \mathbb{N}$, consider now the triangulation $\mathcal{T}_n$ introduced in [10], Theorem 3.1. For sake of simplicity, from now on we restrict ourserves to the model situation $\lambda(x) = |x|^\gamma$, so that $\mathcal{T}_n$ is defined by the family of nodes $(\pm\delta_\ell, \pm\frac{k}{n})$, $j, k = 0, \dots, n$, where

$$\delta_\ell = \left(\frac{\ell}{n}\right)^{1/(\gamma+1)} \quad , \quad \ell = 0, \dots, n.$$

Let $\{P_j, j = 1, \dots, N\}$ be the set of nodes lying in Ω ($N \approx n^2$), and $\{P_j, j = N+1, \dots, N+M\}$ the set of those lying on Γ ($M \approx n$), and let ϕ_j, $j = 1, \dots, N+M$ be piecewise linear functions such that $\phi_i(P_j) = \delta_{ij}$, $i, j = 1, \dots, N+M$. Denote by X_n (respectively V_n) the linear space generated by $\{\phi_1, \dots, \phi_{N+M}\}$ (respectively by $\{\phi_1, \dots, \phi_N\}$). Note that $V_n \subseteq \overset{\circ}{W}{}_\lambda^{1,2}(\Omega)$ and $X_n \subseteq W_\lambda^{1,2}(\Omega)$, by Rademacher's theorem. If we put

$$u_{0n} = \sum_{i=N+1}^{N+M} u_0(P_i)\phi_i \in X_n$$

(u_{0n} is well defined since u_0 is continuous up to the boundary), then the discrete problem corresponding to (5) is: find $u_n \in X_n$ such that

$$\begin{cases} a(v_n, u_n) = \ell(v_n) \quad \text{for all} \quad v_n \in V_n, \\ u_n - u_{0n} \in V_n. \end{cases} \tag{6}$$

If we write $u_n - u_{0n} = \sum_{i=1}^{N} \xi_i \phi_i$ and we choose $v_n = \phi_j$, $1 \le j \le N$, problem (6) is equivalent to the system

$$\sum_{i=1}^{N} a_{ij}\xi_i = \ell(\phi_j) - \sum_{i=N+1}^{N+M} a_{ij} u_0(P_i), \tag{7}$$

$1 \le j \le N$, where $a_{ij} = a(\phi_i, \phi_j)$, $1 \le i \le N$, $1 \le j \le N + M$. Notice that the stiffness matrix $\mathcal{A} = (a_{ij})_{i,j=1,\dots,N}$ is invertible, by Poincaré inequality ([10], Theorem 2.4).

Let us show now that Problem (6) satisfies a discrete maximum principle ([4], [5]), i.e., if $\ell(\phi_j) \le 0$ for $1 \le j \le N$, then

$$\max_{\bar{\Omega}} u_n \le \max \left\{0, \max_{\Gamma} u_{0n}\right\}. \tag{8}$$

To this end, we can apply Theorem 3 in [4] (see also Chapter 4, Section 20 in [5]), by showing that

Lemma 1. *We have:*

(i) The matrix $\mathcal{A}$ is irreducibly diagonal dominant ([14], p. 23)*;*

(ii) $a_{ij} \le 0$ for $i \ne j$, $1 \le i \le N$, $1 \le j \le N + M$;

(iii) $\sum_{j=1}^{N+M} a_{ij} \ge 0$ for $1 \le i \le N$.

Proof. Let us start by proving (ii): if $i \ne j$, $a_{ij} \ne 0$, then a_{ij} is a sum of integrals on all triangles $K \in \mathcal{T}_n$ that have P_i and P_j as vertices. Let us prove that each of these integrals is nonnegative. Let $P_i = (x_i, y_i)$, $P_j = (x_j, y_j)$, $Q = (\xi, \eta)$ be the vertices of K; then ϕ_i and ϕ_j coincide on K respectively with λ_i and λ_j, where

$$\lambda_i(x, y) = \det \begin{bmatrix} x & y & 1 \\ x_j & y_j & 1 \\ \xi & \eta & 1 \end{bmatrix} \cdot \left(\det \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ \xi & \eta & 1 \end{bmatrix} \right)^{-1},$$

$$\lambda_j(x, y) = \det \begin{bmatrix} x & y & 1 \\ x_i & y_i & 1 \\ \xi & \eta & 1 \end{bmatrix} \cdot \left(\det \begin{bmatrix} x_j & y_j & 1 \\ x_i & y_i & 1 \\ \xi & \eta & 1 \end{bmatrix} \right)^{-1}.$$

Suppose first $x_i = x_j$; then, for instance, $\eta = y_i$, so that

$$\frac{\partial \lambda_j}{\partial x}(x,y) = \det \begin{bmatrix} y_i & 1 \\ \eta & 1 \end{bmatrix} \left(\det \begin{bmatrix} x_j & y_j & 1 \\ x_i & y_i & 1 \\ \xi & \eta & 1 \end{bmatrix} \right)^{-1} \equiv 0.$$

On the other hand, $\dfrac{\partial \lambda_j}{\partial x} = -\dfrac{\partial \lambda_i}{\partial x}$, so that

$$\begin{aligned} I_{ij} &= -\int_K |x|^{2\gamma} \det \begin{bmatrix} x_j & 1 \\ \xi & 1 \end{bmatrix}^2 \left(\det \begin{bmatrix} x_j & y_j & 1 \\ x_i & y_i & 1 \\ \xi & \eta & 1 \end{bmatrix} \right)^{-2} dxdy \\ &= (x_j - \xi)^2 (y_i - y_j)^{-2} \int_K |x|^{2\gamma}\, dxdy \le 0. \end{aligned}$$

Similar arguments can be carried on when $y_i = y_j$. The last case to consider is (for instance) $\xi = x_i$, $\eta = y_j$, where $\frac{\partial \lambda_y}{\partial y} \equiv 0 \equiv \frac{\partial \lambda_i}{\partial x}$, so that $I_{ij} = 0$. Thus, (ii) is proved.

Notice now that $I_{ij} = 0$ if and only if P_i and P_j have both different coordinates (when they belong to the same triangle), so that, if P_i and P_j are consecutive on the same line (horizontal or vertical), then $a_{ij} \neq 0$, and hence P_i and P_j are connected in the graph $G(\mathcal{A})$ associated with $\mathcal{A}$ (see [14], p. 19) whenever P_i and P_j are continguous points on the same line of the mesh $\mathcal{T}$. Hence $G(\mathcal{A})$ is strongly connected and $\mathcal{A}$ is irreducible ([14], Theorem 1.6). Thus, to prove (i) holds we have only to show that

$$|a_{ii}| \ge \sum_{\substack{j=1 \\ j\neq i}}^{N} |a_{ij}|. \tag{9}$$

Now, if $i \neq j$, $a_{ij} \neq 0$ and $1 \le j \le N$, then $\mathrm{supp}\phi_i \cap \mathrm{supp}\phi_j$ is given by the union of two triangles. As we showed above, $a_{ij} = 0$ if P_i and P_j are not on the same row. Moreover, if we sum up with respect to j then each triangle appears twice, first with P_i and P_j on the same horizontal row, and then with P_i and P_j on the same vertical row, so that, if $\mathrm{supp}\phi_i = \cup_{i=1}^{6} K_i$, $K_i \in \mathcal{T}_n$, we have

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} |a_{ij}| = \sum_{i=1}^{6} \left(\frac{1}{(\delta_{j+1} - \delta_j)^2} |K| + n^2 \int_K |x|^{2\gamma}\, dxdy \right) = a_{ii}.$$

Finally, to prove (iii), notice that $\sum_{j=1}^{N+M} \phi_j \equiv 1$ on $\overline{\Omega}$, and hence

$$\sum_{j=1}^{N+M} a_{ij} = a(\phi_i, \sum_{j=1}^{N+M} \phi_j) = 0.$$

Arguing as in [5], Theorems 21.3 and 21.4, we are able to prove the following uniform estimates for u_n (discrete Stampacchia's maximum principle).

Proposition 1. *There exists a geometric constant $C > 0$ such that, if $u_n \in X_n$ is the solution of* (6)*, then*

$$\sup_{\Omega} |u_n| \leq \sup_{\Gamma} |u_0| + C \sum_{k=0}^{2} \| f_k \|_{L^p(\Omega)}.$$

Proof. Put

$$a_{0n} = \max\{0, \max_{\Gamma} u_{0n}\} = \max\{0, \max\{u_n(P_i), N+1 \leq i \leq N+M\}\}.$$

Take $\alpha \geq a_{0n}$ and let $u_{n\alpha} \in X_n$ be such that

$$u_{n\alpha}(P_i) = \min\{\alpha, u_n(P_i)\},$$

so that $v_{n\alpha} = u_n - u_{n\alpha} \in V_n$. Arguing as in [5], Theorem 21.4, if $E(\alpha) = \{x \in \Omega, v_{n\alpha} > 0\}$, then

$$\| v_{n\alpha} \|_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} \leq C \Big(\sum_{k=0}^{2} \| f_k \|_{L^p(\Omega)} \Big) |E(\alpha)|^{1/2 - 1/p}.$$

By Sobolev imbedding theorem associated with the vector fields X and Y ([8]), this yields

$$\| v_{n\alpha} \|_{L^q(\Omega)} \leq C \Big(\sum_{k=0}^{2} \| f_k \|_{L^p(\Omega)} \Big) |E(\alpha)|^{1/2 - 1/p},$$

where $q = 2(\gamma + 2)/\gamma$. Again as in [5], if $\beta > \alpha \geq a_{0n}$, then

$$|E(\beta)| \leq C \frac{\sum_{k=0}^{3} \| f_k \|_{L^p(\Omega)}}{(\beta - \alpha)^q} |E(\beta)|^{\nu},$$

where $\nu = q(1/2 - 1/p) > 1$, by our choice of p. Thus we can conclude as in [5], following Stampacchia's classical argument ([13]).

We can state now our main result. By $W^{2,p}_{\lambda}(\Omega)$ we shall denote the function space of all $u \in W^{1,p}_{\lambda}(\Omega)$ such that $X^2 u$, XYu, YXu, $Y^2 u \in L^p(\Omega)$, endowed with its natural norm. Thus we have:

Theorem 2. *Let u be the solution of the variational Dirichlet problem* (5) *and let u_n be the solution of the discrete Dirichlet problem* (6)*; moreover denote by $\Pi_n : C^0(\Omega) \mapsto X_n$ the usual interpolation operator associated with $\mathcal{T}$ and defined by*

$$\Pi_n(v) = \sum_{i=1}^{N+M} v(P_i)\phi_i .$$

(i) If $p > \gamma + 2$ and $u \in W_\lambda^{1,p}(\Omega)$, then

$$\|u - u_n\|_{L^\infty(\Omega)} \le C\, \|u - \Pi_n(u)\|_{W_\lambda^{1,p}(\Omega)} \to 0,$$

as $n \to \infty$.

(ii) If $u \in W_\lambda^{2,p}(\Omega)$, then

$$\|u - u_n\|_{L^\infty(\Omega)} \le C\, n^{-1/(\gamma+1)} \|u\|_{W_\lambda^{2,p}(\Omega)}.$$

Proof. The first statement follows from Proposition 1 by repeating the arguments of [5], Theorem 21.5. The second one can be derived from (i) because

$$\|u - \Pi_n(u)\|_{W_\lambda^{1,p}(\Omega)} \le C\, n^{-1/(\gamma+1)} \|u - u_n\|_{W_\lambda^{2,p}(\Omega)}. \tag{10}$$

Indeed, (10) is Lemma 4.2 of [10], that can be straightforwardly generalized to the case $p \neq 2$.

Corollary 1. *Suppose $f_1 \equiv f_2 \equiv 0$, $f \in L^p(\Omega)$, $u_0 \in W_\lambda^{2,p}(\Omega)$ with $p > 2+\gamma$, then*

$$\|u - u_n\|_{L^\infty(\Omega)} \le C\, n^{-1/(\gamma+1)} \Big(\|f\|_{L^p(\Omega)} + \|u_0\|_{W_\lambda^{2,p}(\Omega)} \Big). \tag{11}$$

Proof. The error estimate (11) can be proved by estimating the $W_\lambda^{2,p}(\Omega)$-norm of $u - u_0$ in terms of the L^p-norm of f and the $W_\lambda^{2,p}$-norm of u_0 as in [10], Theorem 4.3 and Corollary 4.4.

Let us state a general Hölder regularity result that implies in paricular the second part of Theorem 1.

Let $X = (X_1, \dots, X_m)$ be a family of smooth vector fields defined in an open neighborhood Ω_0 of Ω, where Ω is a bounded open subset of $\mathbb{R}^n$. For all definitions in the sequel, see e.g. [9]. Suppose X satisfies the so-called Hörmander's rank condition; we shall denote by d the canonical Carnot-Carathédory metric associated with X, by $B(x, r)$ its metric balls, and by Q

is homogeneous dimension. Let $\mathcal{L}$ be the second order operator defined by $\mathcal{L} = \sum_{i,j} X_j^*(a_{ij} X_j)$, where

(a) the a_{ij}'s are bounded measurable functions,

(b) $\sum_{ij} a_{ij}(x)\xi_i\xi_j \approx |\xi|^2$.

If we denote by $W_X^{1,2}(\Omega)$ the Banach space of all functions $u \in L^2(\Omega)$ such that $X_j u \in L^2(\Omega)$ for $j = 1, \dots, m$, and by $\overset{\circ}{W}{}_\lambda^{1,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_X^{1,2}(\Omega)$, then the following regularity result holds.

Theorem 3. *Suppose Ω satisfies the following condition: for any $x \in \partial\Omega$, for any $r \in (0, r_0)$ the Lebesgue measure of $B(x,r) \setminus \Omega$ is comparable to the Lebesgue measure of $B(x,r)$. Let $u \in \overset{\circ}{W}{}_\lambda^{1,2}(\Omega)$ be a variational solution of*

$$\mathcal{L}u = f + \sum_j X_j^* f_j, \tag{12}$$

where $f \in L^{p/2}(\Omega)$, $f_1, \dots, f_m \in L^p(\Omega)$, with $p > Q$. Then u is Hölder continuous on $\overline{\Omega}$.

Sketch of the proof. To claim that u is Hölder continuous in $\overline{\Omega}$ is equivalent to say that there exist two geometric constants K_1 and σ $(0 < \sigma < 1)$ such that:

$$\omega(r) \le K_1 \left\{ \sum_{j=1}^m \| f_j \|_{L^p(\Omega)} + \| f \|_{L^{p/2}(\Omega)} \right\} r^\sigma, \tag{13}$$

where $\omega(r) := \sup_{\Omega \cap B(x_0,r)} u - \inf_{\Omega \cap B(x_0,r)} u$. Let us prove (13).

We have to study the behavior of solutions of (12) near a portion of $\partial\Omega$ where the solution vanishes. The corresponding problem for the non degenerate case, was studied by Stampacchia ([13]); in this note we shall follow proofs and notations used in [13] (see also [11]). For sake of simplicity, we shall omit parts of the proof that do not need any substantial change from the corresponding ones in the elliptic case. In order to establish inequality (13), we need now some estimates for solutions relative to $\mathcal{L}$.

Step 1. The main result we need to know is that u is bounded. The boundedness of u is a consequence of Stampacchia's Maximum Principle which states that a function $u \in \overset{\circ}{W}{}_\lambda^{1,2}(\Omega)$, the solution of the equation (12), vanishing on $\partial\Omega$, satisfies:

$$\sup_\Omega u \le K \left\{ \sum_{j=1}^m \| f_j \|_{L^p(\Omega)} + \| f \|_{L^{p/2}(\Omega)} \right\} |\Omega|^{\frac{1}{Q} - \frac{1}{p}}. \tag{14}$$

Step 2. We shall show that $\sup_{\Omega\cap B(x_0,r)} u$ and $\inf_{\Omega\cap B(x_0,r)} u$ are both finite. This is basically a consequence of the following Propositions (that correspond respectively to Theorem 5.3 and Theorem 5.4 of [13]; there are no differences in their proofs except for the exponent n replaced by Q and the usual gradient replaced by the intrinsic gradient).

Proposition 2. *Let $u \in W_X^{1,2}(\Omega \cap B(x_0, R))$ be a subsolution of $\mathcal{L}u = 0$ such that $u \leq 0$ on $\partial\Omega \cap B(x_0, R)$. Let k_0 be a non negative real number. If $r < R$ is small enough, then there exists a geometric constant K such that*

$$\sup_{\Omega\cap B(x_0,r/2)} \leq k_0 + K\left\{\frac{1}{|B(x_0,r)|}\int_{A(k_0,r)} (u-k_0)^2 dx\right\}^{\frac{1}{2}}\left(\frac{|A(k_0,r)|}{|B(x_0,r)|}\right)^{\frac{\theta-1}{2}}, \tag{15}$$

where

$$A(k_0,r) := \{x \in \Omega \cap B(x_0,r) : u(x) \geq k_0\}, \ \textit{and}\, \theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{Q}}.$$

Remark 1. If u is a solution of $\mathcal{L}u = 0$ vanishing on $\partial\Omega \cap B(x_0, R)$, and if $k_0 = 0$, from Proposition 2 we get

$$\sup_{\Omega\cap B(x_0,r/2)} \leq K\left\{\frac{1}{|B(x_0,r)|}\int_{\Omega\cap B(x_0,R)} u^2 dx\right\}^{\frac{1}{2}}.$$

By previous Proposition 2 and by (14), we get the following:

Proposition 3. *Let $x_0 \in \partial\Omega$ and let u be a solution of (12), vanishing on $\partial\Omega \cap B(x_0, R)$, then there exists a geometric constant K such that, for $p > Q$ we have:*

$$\sup_{\Omega\cap B(x_0,R/2)} |u| \leq K\left[\left\{\frac{1}{|B(x_0,R)|}\int_{\Omega\cap B(x_0,R/2)} u^2 dx\right\}^{\frac{1}{2}} + \left\{\sum_{j=1}^m \|f_j\|_{L^p(\Omega)} + \|f\|_{L^{p/2}(\Omega)}\right\}|B(x_0,R)|^{\frac{1}{Q}-\frac{1}{p}}\right]. \tag{16}$$

Step 3. In this part we shall prove the following Proposition

Proposition 4. *Let $u(x)$ be a solution of the equation $\mathcal{L}u = 0$, vanishing on $\partial\Omega \cap B(x_0, R)$, where $x_0 \in \partial\Omega$. Then there exist η with $0 < \eta < 1$ and $\rho_0 < r_0$ such that for $r \leq \rho_0$*

$$\omega(r) \leq \eta\omega(4r). \tag{17}$$

Proof. The proof is again similar to the one of Lemma 7.4 in [13]. Put $k_0 = \dfrac{\sup_{\Omega\cap B(x_0,2R)} - \inf_{\Omega\cap B(x_0,2R)}}{2} \geq 0$; without loss of generality we can assume that

$$|A(k_0, R)| \leq \frac{1}{2}|\Omega \cap B(x_0, R)|.$$

Let $h > k > k_0$, and let $v = \min\{u, h\} - \min\{u, k\} \in W^{1,2}_X(\Omega \cap B(x_0, R))$. By the previous inequality and since, by the hypothesis on Ω, the measure of $B(x_0, R) \setminus \Omega$ is comparable to the Lebesgue measure of $B(x_0, R)$, we have:

$$|\{x \in B(x_0, R) : v = 0\}| > \frac{1}{2}|B(x_0, R)|.$$

By Poincaré inequality ([9], [3]), we get:

$$\|v\|_{L^{\frac{Q}{Q-1}}(B(x_0,R))} \leq c\|Xv\|_{L^1(B(x_0,R))}.$$

Therefore, by previous inequality, we have:

$$|\{x \in B(x_0, R) : |v| > \sigma\}| \leq \Big(\frac{1}{\sigma}\Big)^{\frac{Q}{Q-1}}\Big(\int_{B(x_0,R)} |Xv|dx\Big)^{\frac{Q}{Q-1}},$$

for any $\sigma > 0$. By definition of v, this imply in particular that

$$(h-k)^2|A(h, R)|^{\frac{2Q-2}{Q}} \leq c\int_{A(k,R)} |Xu|^2dx\big\{|A(k, R)| - |A(h, R)|\big\}. \tag{18}$$

Using (18), the proof proceeds with almost no changes from the corresponding one of [13].

Step 4. From Proposition 4, the next result follows easily.

Lemma 2. *Let $x_0 \in \partial\Omega$. If for any $\eta < 1$, $H > 0$, $\alpha > 0$, we have*

$$\omega(r) \leq \eta\omega(4r) + H|B(x_0, r)|^{\alpha}, \tag{19}$$

for $r < \rho_0$, then there exist λ with $0 < \lambda < 1$, and $K > 0$, such that

$$\omega(r) \leq KH|B(x_0, r)|^{\lambda}. \tag{20}$$

Proof. Once more the proof can imitate the one of Lemma 7.5 in [13], because of the doubling condition satisfied by the metric balls $|B(x, 2r)| \leq c|B(x, r)|$.

Step 5. We are able to prove (13) and then Theorem 3. Indeed, let u be a variational solution of (12), vanishing on $\partial\Omega$, and denote by v the weak solution of the equation $\mathcal{L}u = f + \sum_j X_j^* f_j$ in $\Omega \cap B(x_0, 8r)$, $v = 0$ on $\partial(\Omega \cap B(x_0, 8r))$, and set $u = v + w$; then w is a solution of the equation $\mathcal{L}w = 0$ vanishing on $\partial\Omega \cap B(x_0, 8r)$. We use Stampacchia's Maximum Principle and Proposition 4. Since

$$\sup_{x,y \in \Omega \cap B(x_0,r)} |u(x) - u(y)| \leq 2 \sup_{\Omega \cap B(x_0,r)} |v| + \sup_{x,y \in (\Omega \cap B(x_0,r))} |w(x) - w(y)|,$$

there exist costants $K > 0$, $\rho_0 > 0$ and $\eta \in]0, 1[$ such that

$$\omega(r) \leq \eta\omega(4r) + K \left\{ \sum_{j=1}^{m} \|f_j\|_{L^p(\Omega)} + \|f\|_{L^{p/2}(\Omega)} \right\} |B(x_0, r)|^{\frac{1}{Q} - \frac{1}{p}}\, r \; < \rho_0.$$

By Lemma 2, we deduce that there exist constant $\sigma \in]0, 1[$ and $K' > 0$ such that

$$\omega(r) \leq K' \left\{ \sum_{j=1}^{m} \|f_j\|_{L^p(\Omega)} + \|h\|_{L^{p/2}(\Omega)} \right\} |B(x_0, r)|^{\sigma},$$

and we are done.

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