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UNILATERAL PROBLEMS WITH DEGENERATE COERCIVITY

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In this note we prove some existence and regularity results for unilateral problems with degenerate coercivity.

1. Introduction.

Let Ω be a bounded, open subset of $\mathbb{R}^n$, with $N > 2$, and $a(x, s) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1 + |s|)^\theta} \leq a(x, s) \leq \beta, \tag{1}$$

for some real number θ such that

$$0 \leq \theta < 1, \tag{2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α and β are positive constants. If we define $\mathcal{A}u = -\operatorname{div}(a(x, u)Du)$, under assumption (1) the operator $\mathcal{A}$,

though well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, is not coercitive, since $\frac{1}{1+|u|}$ goes to zero when u is large.
In the papers [1], [3], [6] existence and regularity results of the Dirichlet problem associated to the operator $\mathcal{A}$ have been proved.
The objective of this note is to study the existence and regularity of the solutions of unilateral problems associated to $\mathcal{A}$ and with data f belonging to various Lebesgue space $L^m(\Omega)$, for some $m > 1$.
There are two difficulties associated with the study of unilateral problems with degenerate coercivity.
First of all, the classical method used in order to prove the existence of solutions to unilateral problems cannot be applied, even if the datum f is regular. We overcome this difficulty by considering a sequence of nondegenerate Dirichlet problems, having nonnegative solutions. This approximation has been introduced in [10] in the framework of obstacle problems associated to uniformly elliptic operators and has been already used in [5] for unilateral problems with L^1 data.
An additional advantage of this approach will be the proof of the Lewy-Stampacchia inequality.
A second difficulty appears when we weaken the summability hypotheses on f. As a matter of the fact, when $f \in L^m(\Omega)$, with $m < \frac{2(N-\theta)}{N+2-\theta}$, even in the case of the equations, the product fu does not belong to $L^1(\Omega)$. Hence, the classical definition of unilateral problem is inadeguate.
We solve this difficulty by using another formulation, already used in the framework of uniformly elliptic unilateral problems having data in $L^1(\Omega)$.

2. Statements of the results.

Up to now, we will assume that hypotheses (1) holds. The first result we state concerns with data having high summability and coincides with the classical boundedness result for the unilateral problems associated to uniformly elliptic operators.

Theorem 1. *Let $f \in L^m(\Omega)$, $m > \frac{N}{2}$.
Then there exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which is solution of the following unilateral problem*

$$\begin{cases} u \geq 0 \; a.e. \; in \; \Omega \\ \langle \mathcal{A}u, u-v \rangle \leq \displaystyle\int_\Omega f(u-v) \\ \forall v \in H_0^1(\Omega), \; v \geq 0, \; a.e. \; in \; \Omega. \end{cases} \tag{3}$$

Moreover u satisfies the inequality

$$f \leq \mathcal{A}u \leq f^{+}. \tag{4}$$

The next result deals with data f which give unbounded solutions in $H_0^1(\Omega)$.

Theorem 2. *Let $f \in L^m(\Omega)$, with m such that*

$$\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}. \tag{5}$$

Then there exists a function $u \in H_0^1(\Omega) \cap L^r(\Omega)$, with

$$r = \frac{Nm(1-\theta)}{N-2m} \tag{6}$$

which is a solution of problem (3).
Moreover u satisfies the inequality (4).

Remark 1. We observe that the hypotheses on m imply that f belongs to $L^{2^{\star\prime}}(\Omega)$, where $2^{\star} = \frac{2N}{N-2}$ is the Sobolev embedding exponent for $H_0^1(\Omega)$. Then, the second term of (3) is well defined, as well as the first one, since a is bounded. When $\theta = 0$ the previous theorem gives the classical regularity result for uniformly elliptic unilateral problems.

If we weaken the summability hypotheses on f we obtain solutions not belonging to $H_0^1(\Omega)$, even if f belongs to the space $H^{-1}(\Omega)$, as the following theorem states.

Teorema 3. *Let f be a function in $L^m(\Omega)$, with*

$$\frac{N(2-\theta)}{N+2-N\theta} \leq m < \frac{2N}{N+2-\theta(N-2)}. \tag{7}$$

Then, there exists a function $u \in W_0^{1,q}(\Omega)$, with

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)} \tag{8}$$

such that

$$a(x,u)|Du|^2 \in L^1(\Omega). \tag{9}$$

Moreover u is a solution of the unilateral problem (3) and satisfies the inequality (4).

Remark 2. We notice that every term in (3) is meaningfull. This is clear for the left hand side, since (9) holds. As to the right hand side we note that by Sobolev's embedding the solution u given by the previous theorem belongs to $L^r(\Omega)$, with r as in (6) and hypotheses on m implies $f \in L^{r'}(\Omega)$.

As already mentioned in the introduction, if we decrease the summability of f, the classical formulation of unilateral problem fails since the product fu, even in the case of equations (see [6]), does not belong to $L^1(\Omega)$. In order to introduce the new formulation of unilateral problem let us recall the definition of the truncature function.

Given a constant $k > 0$ let $T_k : \mathbb{R} \to \mathbb{R}$ the function defined by

$$T_k(s) = \max\{-k, \min\{k, s\}\}.$$

We will prove the following result

Teorema 4. *Let f be a function in $L^m(\Omega)$, with $m > 1$ such that*

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{N(2-\theta)}{N+2-N\theta}. \tag{10}$$

Then, there exists a function $u \in W_0^{1,q}(\Omega)$, with q as in (8) such that

$$\begin{cases} u(x) \geq 0 \quad a.e.\ x \in \Omega \\ T_k(u) \in H_0^1 \quad \forall k > 0 \\ \langle \mathcal{A}u, T_k(u-v)\rangle \leq \displaystyle\int_\Omega f T_k(u-v) \\ \forall v \in H_0^1 \cap L^\infty(\Omega), \quad v \geq 0 \quad a.e.\ in\ \Omega. \end{cases} \tag{11}$$

Moreover u satisfies the inequality (4).

Remark 3. We point out that the previous definition of solution of unilateral problem has been used in [5] in the framework of unilateral problems associated to uniformly elliptic operators and having L^1- functions as data. Moreover, this definition is quite similar to the definition of "entropy solution" of the Dirichlet problem, already used in [6].

Remark 4. Notice that both terms in (11) are well defined. The second term offers no difficulty, since $f \in L^1(\Omega)$. As to the first member, we observe that, for all $k > 0$,

$$\langle \mathcal{A}u, T_k(u-v)\rangle = \int_\Omega a(x,u) D T_h u D T_k(u-v)$$

where $h = k + \|v\|_\infty$.

Remark 5. As in Remark 2, we observe that u belongs to $L^r(\Omega)$, with r defined by (6).
In the case $m = 1$ the previous result is not true in general. As a matter of the fact, if $\theta = 0$ the operator $\mathcal{A}$ is uniformly elliptic and the solution of unilateral problem with L^1- data does not belong to $W_0^{1,\frac{N}{N-1}}(\Omega)$, but to $W_0^{1,s}(\Omega)$, for every $s < \frac{N}{N-1}$ (see [5]).
The lower bound for m guarantees that q is greater than 1.
For reasons of coincision we have chosen not to include the study of the case $1 \le m \le \max\left\{\frac{N}{N+1-\theta(N-1)}, 1\right\}$. In this case we have to use another functional setting since the gradient of u may no longer be in $L^1(\Omega)$. However, following the method used in [6] for the Dirichlet problem, our approach allows us to prove that the unilateral problem (11) has a solution belonging (as well as its *weak gradient*, see [2]) to a suitable Marcinkiewicz space.

3. A priori estimates and proofs of the Theorems.

Let f be a function of $L^m(\Omega)$, with m as in the statement of the Theorems and let $\{f_n\}$ be a sequence of regular functions such that

$$f_n \in L^{\frac{2N}{N+2}}(\Omega) \ f_n \to f \text{ strongly in } L^m(\Omega) \tag{12}$$

and

$$\|f_n\|_{L^m(\Omega)} \le \|f\|_{L^m(\Omega)}, \ \forall n \in \mathbb{N}. \tag{13}$$

Let us define the following sequence of Dirichlet problems

$$\begin{cases} \mathcal{A}_n u_n + f_n^- \frac{u_n}{\frac{1}{n}+|u_n|} = f_n^+ & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

where

$$\mathcal{A}_n u_n = -\operatorname{div}(a(x, T_n(u_n))Du_n).$$

We remark that, for every $n \in \mathbb{N}$, the function $a(x, T_n(s))$ satisfies the condition (1). Moreover, since

$$a(x, T_n(s)) \ge \frac{\alpha}{(1+n)^\theta}, \text{ for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \tag{15}$$

and since f_n belongs to H^{-1}, by well-known results (see [11]) there exists at least a solution u_n of problem (14) in the sense that

$$(16) \qquad \begin{cases} u_n \in H_0^1(\Omega) \\ \displaystyle\int_\Omega a(x, T_n(u_n)) Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n} + |u_n|} v = \int_\Omega f_n^+ v \\ \forall v \in H_0^1(\Omega). \end{cases}$$

Note that, $\forall n \in \mathbb{N}$

$$(17) \qquad u_n(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

As a matter of the fact, taking as test function in (16) $v = u_n^-$ we obtain

$$-\int_\Omega a\big(x, T_n(u_n)\big)|Du_n^-|^2 = \int_\Omega f_n^+ u_n^- + \int_\Omega f_n^- \frac{(u_n^-)^2}{\frac{1}{n} + |u_n|}.$$

Since the right hand side is non negative and using condition (15) we have

$$\frac{\alpha}{(1+n)^\theta} \int_\Omega |Du_n^-|^2 dx \leq 0,$$

which implies (17). Consequently u_n is a solution of the problem

$$(18) \qquad \begin{cases} u_n \in H_0^1(\Omega) \\ \displaystyle\int_\Omega a(x, T_n(u_n)) Du_n Dv + \int_\Omega f_n^- \frac{u_n}{\frac{1}{n} + u_n} v = \int_\Omega f_n^+ v \\ \forall v \in H_0^1(\Omega). \end{cases}$$

The main tool of the proofs of our results will be some a priori estimates on the solutions of the approximate problems (14). Once this has been accomplished, thanks to the linearity of the operator with respect to the gradient and the boundedness and continuity of a, we will pass to the limit, thus finding a solution of the unilateral problem.

In order to prove Theorem 1 we need the following L^∞ a priori estimate

Lemma 1. *Let $f \in L^m(\Omega)$, with $m > \frac{N}{2}$ and let u_n be a solution of (14). Then, there exist two positive constants c_1, c_2, depending on N, m, α, θ, $|\Omega|$, $\|f\|_{L^m(\Omega)}$, such that, for any $n \in \mathbb{N}$,*

$$(19) \qquad \|u_n\|_{L^\infty(\Omega)} \leq c_1,$$

$$(20) \qquad \|u_n\|_{H_0^1(\Omega)} \leq c_2.$$

Proof. Let us define, for s in $\mathbb{R}$ and $k > 0$,

$$G_k(s) = s - T_k(s),$$

and set, for n in $\mathbb{N}$

$$A_k = \{x \in \Omega : u_n(x) > k\}. \tag{21}$$

If we take $G_k(u_n)$ as test function in (18), and use assumption (1) and condition (17), we obtain

$$\begin{aligned} \alpha \int_{A_k} \frac{|Du_n|^2}{(1+u_n)^\theta} &\le \int_\Omega |f| G_k(u_n) \le \\ &\le \|f\|_{L^m(\Omega)} \Big(\int_\Omega (G_k(u_n))^{m'} \Big)^{\frac{1}{m'}}, \end{aligned} \tag{22}$$

where $m' = \frac{m}{m-1}$. Thanks to estimate (22) we get the L^∞-estimate as in the proof of Lemma 2.2 of [6].
In order to prove estimate (20), let us take u_n as test function in (18). Using hypothesis (1) we get

$$\alpha \int_\Omega \frac{|Du_n|^2}{(1+u_n)^\theta} \le \int_\Omega f^+ u_n - \int_\Omega f^- \frac{u_n}{\frac{1}{n} + u_n} u_n \le \int_\Omega |f| u_n.$$

From this estimate, using (19) we obtain

$$\int_\Omega |Du_n|^2 \le \frac{(1+c_1)^{\theta+1}}{\alpha} \|f\|_{L^m(\Omega)}.$$

The next result will be used in the proof of Theorem 2.

Lemma 2. *Let $f \in L^m(\Omega)$, with m satisfying hypothesis (5), and let u_n be a solution of problem (14).*
Then, there exist two positive constants c_3, c_4, depending on N, m, α, θ, $|\Omega|$, $\|f\|_{L^m(\Omega)}$, such that, for any $n \in \mathbb{N}$,

$$\|u_n\|_{L^r(\Omega)} \le c_3, \tag{23}$$

$$\|u_n\|_{H_0^1(\Omega)} \le c_4, \tag{24}$$

where r is defined by (6).

Proof. Let $k > 0$. Following the outline of the proof of Lemma 2.3 of [6] we have to prove the following estimate

$$\alpha \int_{B_k} |Du_n|^2 \leq (2+k)^\theta \int_{A_k} |f|, \tag{25}$$

where A_k is the set defined in (21) and

$$B_k = \{x \in \Omega : k \leq u_n < k+1\}. \tag{26}$$

If we take in (18) $v = T_1(G_k(u_n))$, thanks to hypothesis (1) and condition (17) we obtain

$$\alpha \int_{B_k} \frac{|Du_n|^2}{(1+u_n)^\theta} \leq \int_{A_k} |f| T_1(G_k(u_n)),$$

which implies (25).

The next lemma deals with the case in which the sequence $\{u_n\}$ is not bounded in H_0^1 and will be used in the proof of Theorems 3, 4.

Lemma 3. *Assume* $f \in L^m(\Omega)$ *with*

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}. \tag{27}$$

Let $\{f_n\}$ *be a sequence of functions satisfying (12) and (13), and let* u_n *be a solution of (14).*
Then, for any $n \in \mathbb{N}$ *and* $k > 0$ *we have*

$$\int_\Omega |DT_k(u_n)|^2 dx \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha}(1+k)^{\theta+1}. \tag{28}$$

Moreover

$$\|u_n\|_{W_0^{1,q}} \leq c_5, \quad \forall n \in \mathbb{N}, \tag{29}$$

where c_5 *depends on* $N, m, \theta, \alpha, |\Omega|, \|f\|_{L^m(\Omega)}$ *and* q *is defined by (8).*

Proof. Let us take $T_k(u_n)$ as test function in (18); using (1) and condition (17) we obtain

$$\frac{\alpha}{(1+k)^\theta} \int_\Omega |DT_k(u_n)|^2 \leq \int_\Omega f_n^+ T_k(u_n) +$$
$$+ \int_\Omega f_n^- \frac{u_n}{\frac{1}{n}+u_n} T_k(u_n) \leq \int_\Omega |f_n| T_k(u_n) dx$$

which implies (28).
The estimate (29) follows working as in the proof of Lemma 2.5 of [6].

Before proving the theorems we state the following "weak lower semicontinuity" result (for the proof see Lemma 2.8, [6]).

Lemma 4. *Let $\{v_n\}$ be a sequence of functions which is weakly convergent to v in $H_0^1(\Omega)$, and let u_n be a sequence of functions which is almost everywhere convergent to some function u in Ω. Then*

$$\int_\Omega a(x,u)|Dv|^2 \le \liminf_{n\to+\infty} \int_\Omega a(x, T_n(u_n))|Dv_n|^2 \le c.$$

We are now in position to prove the Theorems.

Proof of Theorems 1 and 2.

Let $f \in L^m(\Omega)$, with m as in the statements of the theorems and let $\{u_n\}$ be a sequence of solutions of (14). Using the results of Lemmas 1 and 2 we obtain that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and in the Lebesgue spaces as in the statements of the theorems.
Then, there exists a subsequence, still denoted by $\{u_n\}$, which is weakly convergent to some function u in $H_0^1(\Omega)$. Moreover, u_n converges to u almost everywhere in Ω as a consequence of the Rellich theorem.
Let us prove that u is a solution of the unilateral problem (3).
Since $u_n(x) \ge 0$ a.e. $x \in \Omega$ for any $n \in \mathbb{N}$ we have

$$u(x) \ge 0 \text{ a.e. } x \in \Omega.$$

Let $w \in H_0^1(\Omega)$, $w \ge 0$, and take $u_n - w$ as test function in (18). We obtain

$$\int_\Omega a(x, T_n(u_n)) Du_n D(u_n - w) = \tag{30}$$
$$= \int_\Omega f_n(u_n - w) + \frac{1}{n}\int_\Omega f_n^-(u_n - w).$$

Applying Lemma 4 with $v_n = u_n$ we have

$$\int_\Omega a(x,u)|Du|^2 \le \liminf_{n\to+\infty} \int_\Omega a(x, T_n(u_n))|Du_n|^2.$$

Thanks to the boundeness and the continuity of $a(x,s)$, and since u_n converges to u weakly in $H_0^1(\Omega)$ and almost everywhere in Ω we have

$$\lim_{n\to+\infty} \int_\Omega a(x, T_n(u_n)) Du_n Dw = \int_\Omega a(x,u) Du Dw.$$

Hence, taking the limit as $n \to +\infty$ in (30), since the right hand side converges to

$$\int_\Omega f(u - w)\, dx,$$

u is a solution of (3).
In order to prove the inequality (4) we note that, since u_n is nonnegative, from (14) we derive

$$f \leq \mathcal{A}_n u_n \leq f^+.$$

Thanks to the linearity of $\mathcal{A}_n u_n$ with respect to Du_n, letting $n \to +\infty$ in the previous inequality, we obtain inequality (4).

Proof of Theorem 3.

Let $\{f_n\}$ be a sequence of functions satisfying (12) and (13), with m as in the statement of Theorem 3, and let $\{u_n\}$ be a sequence of solutions of problem (14). By Lemma 3 the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$. Moreover the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ and in $L^r(\Omega)$, with q and r defined by (8), (6), respectively. Thus, there exists a subsequence, denoted by $\{u_n\}$ such that

$$\begin{cases} u_n \to u \text{ weakly- } W_0^{1,q}(\Omega) \\ u_n \to u \text{ strongly- } L^q, \text{ and a.e. } x \in \Omega, \\ T_k u_n \to T_k u \text{ weakly- } H_0^1(\Omega). \end{cases} \tag{31}$$

Let us prove that u satisfies (9).
Taking $T_k(u_n)$ as test function in (18), we have

$$\int_\Omega a(x, T_n(u_n))|DT_k(u_n)|^2 \leq \int_\Omega f_n T_k(u_n) + \frac{1}{n}\|f_n^-\|_1. \tag{32}$$

Applying Lemma 4 with $v_n = T_k(u_n)$, we thus have

$$\int_\Omega a(x, u)|DT_k(u)|^2 \leq \liminf_{n\to+\infty} \int_\Omega a(x, T_n u_n)|DT_k(u_n)|^2. \tag{33}$$

Passing to the limit as $n \to +\infty$ in (32) we obtain

$$\int_\Omega a(x, u)|DT_k(u)|^2 \leq \int_\Omega f T_k(u).$$

Letting k tend to infinity, we obtain

$$\int_\Omega a(x, u)|Du|^2 \leq \int_\Omega f u \leq c. \tag{34}$$

Now we can prove that u is a solution of the unilateral problem (3). First of all we note that $u(x) \geq 0$ almost everywhere $x \in \Omega$.

Let φ be a function in $C_0^\infty(\Omega)$, $\varphi(x) \geq 0$ a.e. $x \in \Omega$ and $k > 0$. Taking $T_k(u_n) - \varphi$ as test function in (18), we obtain

$$\int_\Omega a(x, T_n(u_n))Du_n DT_k(u_n) - \int_\Omega a(x, T_n(u_n))Du_n D\varphi \leq \tag{35}$$

$$\leq \int_\Omega f_n(T_k(u_n) - \varphi)) + \frac{1}{n}\|f_n^-\|_1.$$

The right hand side easily passes to the limit as n tends to infinity. As for the left hand side, we note that condition (33) holds; moreover $a(x, T_n u_n)D\varphi$ converges to $a(x, u)D\varphi$ in any L^p. Thus, it is possible to pass to the limit in (35) to obtain

$$\int_\Omega a(x, u)|DT_k u|^2 - \int_\Omega a(x, u)DuD\varphi \leq \int_\Omega f(T_k u - \varphi).$$

A further limits on $k \to +\infty$ yelds

$$\int_\Omega a(x, u)Du(Du - D\varphi) \leq \int_\Omega f(u - \varphi), \tag{36}$$

$\forall \varphi \in C_0^\infty(\Omega)$, $\varphi(x) \geq 0$ a.e. $x \in \Omega$. At least, by standard density argument we can prove that (36) holds also for nonnegative test functions in $H_0^1(\Omega)$.
The proof of the inequality (4) follows as in Theorems 1 and 2.

Proof of Theorem 4.
Let $\{f_n\}$ be a sequence of functions satisfying (12) and (13), with m as in the statement of Theorem 4, and let $\{u_n\}$ be a sequence of solutions of problem (14). As in the proof of Theorem 3, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$ satisfying (31). Moreover, $u(x) \geq 0$ a.e. $x \in \Omega$.
Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v(x) \geq 0$ a.e. $x \in \Omega$.
Taking $T_k(u_n - v)$ as test function in (18) we obtain

$$\int_\Omega a(x, T_n u_n)Du_n DT_k(u_n - v) \leq \int_\Omega f_n T_k(u_n - v) + \frac{1}{n}\|f_n^-\|_1. \tag{37}$$

The left hand side of the previous inequality can be rewritten as follows

$$\int_\Omega a(x, T_n(u_n))|DT_k(u_n - v)|^2 - \int_\Omega a(x, T_n(u_n))DvDT_k(u_n - v).$$

Since the sequence $\{T_k(u_n - v)\}$ is weakly convergent to $T_k(u - v)$ in H_0^1 by Lemma 4 with $v_n = T_k(u_n - v)$ we have

$$\int_\Omega a(x, u)|DT_k(u - v)|^2 \leq \liminf_{n\to+\infty} \int_\Omega a(x, T_n(u_n))|DT_k(u_n - v)|^2.$$

Moreover, due to the boundedness and continuity of a we get

$$\int_\Omega a(x, u)DvDT_k(u - v) = \lim_{n\to+\infty} \int_\Omega a(x, T_n(u_n))DvDT_k(u_n - v).$$

Then the first member of (37) passes to the limit, as well as the second member. Hence u satisfies

$$\int_\Omega a(x, u)DuDT_k(u - v) \leq \int_\Omega f T_k(u - v),$$

for every v in $H_0^1 \cap L^\infty(\Omega)$, $v(x) \geq 0$ a.e. $x \in \Omega$, that is u is a solution of the unilateral problem (11).
As for as the inequality (4) is concerned, we can prove it as in Theorems 1 and 2.

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