LE MATEMATICHE
Vol. LIV (1999) – Supplemento, pp. 95–109

THE TRACE INEQUALITY AND SOME APPLICATIONS

MIROSLAV KRBEC

1. Introduction.

As is well known the trace inequality

$$\left(\int_{\mathbb{R}^n} u^2(x)V(x)\,dx\right)^{1/2} \le c\left(\int_{\mathbb{R}^N} (\nabla u(x))^2\,dx\right)^{1/2}, \quad u \in W^{1,2}(\mathbb{R}^N), \tag{1.1}$$

and its various clones have turned out to be a very powerful tool to handle many topical problems in analysis, in particular, in PDEs theory.

We shall give a (by no means complete) survey of relevant results about (1.1) and its local variant, namely, of conditions for a weight function V in order that

$$\left(\int_{B} u^2(x)V(x)\,dx\right)^{1/2} \le c\left(\int_{B} (\nabla u(x))^2\,dx\right)^{1/2}, \quad u \in W_0^{1,2}(B), \tag{1.2}$$

where B is a ball in $\mathbb{R}^N$. Applications of these and similar inequalities have been the reason for a strong effort to obtain various conditions, either sufficient or necessary and sufficient. Our concern will be efficient and manageable conditions for the function V, guaranteeing validity of (1.2) and the so called *size condition*. We shall use a natural idea of a decomposition of the imbedding in (1.2) into an imbedding of $W_0^{1,2}$ into a suitable target space and an imbedding from this target into $L^2(V)$; we invoke imbedding theorems for the Sobolev space $W_0^{1,2}$– the classical Sobolev theorem and a refinement in terms of Lorentz spaces in the role of target spaces in the dimension $N \geq 3$, and the limiting imbedding theorem due to Brézis and Wainger [4] (see also [35], Lemma 2.10.5) in the dimension $N = 2$, which can be viewed as an analogous refinement of Trudinger's celebrated limiting imbedding [31]. The method suggested for proving (1.2) is a kind of a generator of n-dimensional Hardy inequalities or, alternatively, of weighted imbeddings $W_0^{1,2} \hookrightarrow L^2(V)$.

It is rather surprising that working with superpositions of imbeddings we do not lose much. Next, we shall combine our conditions for validity of (1.2) with the conditions for the SUCP due to Chanillo and Sawyer [7] and we recover or generalize some of known results about the strong unique continuation property for $|\Delta u| \leq V|u|$ in dimensions 2 and 3. This text is based on a joint work with Thomas Schott (see [19]).

2. Recent history – a partial survey.

The natural idea is to study the behaviour of Riesz potential between acting from a(n unweighted) Lebesgue space into a weighted Lebesgue space.

Necessary and sufficient conditions have been found for the case of imbeddings of $W^{1,p}$ into $L_q(V)$, see Adams' inequality in [1] and Maz'ya [23], when $p < q$. If $p = q = 2$ and $N \geq 3$, then a necessary and sufficient condition is due to Kerman and Sawyer [17]; it reads

$$\int_{\mathbb{R}^N} \Big(\int_Q \frac{V(y)}{|x-y|^{N-1}} dy \Big)^2 dx \leq K \int_Q V(x)\, dx \tag{2.1}$$

for all dyadic cubes $Q \subset \mathbb{R}^N$, with a constant K independent of Q. This condition uses local potentials in an intrinsic way since it hangs on Sawyer's theorem on two weight maximal inequality (see [29]) and on the good-λ-inequality due to Muckenhoupt and Wheeden [25]; the latter gives a link between an inequality for the corresponding Riesz potential and for the associated fractional maximal function. The condition (2.1) can sometimes be difficult to verify since it involves the local potential of V, or, alternatively, the fractional integral of V. Hence various sufficient conditions, including those preceding [17] are of importance.

Inequalities of this type have been studied in pioneering works by Maz'ya, [21], [22] already in the early 1960s. Necessary and sufficient conditions are, however, formulated in the language of capacities and were also published only in Russian. The conditions in terms of Marcinkiewicz or Morrey spaces can be derived from them.

Let I_α be the Riesz potential of order α. Then

$$\operatorname{cap}_\alpha(E) = \inf\left\{\int_\Omega g^p\,dx : g \in L^p_+,\ I_\alpha g \geq \chi_E\right\}$$

is the *Riesz capacity* (of order α) of a Borel set $E \subset \Omega$.

Theorem 2.1. ([23], [2], [32]). *The following conditions are equivalent*

(i) $\|I_\alpha u\|_{L^p(V)} \leq c\|u\|_{L^p}, \qquad u \in L^p;$

(ii) $\|I_\alpha u\|_{L^{p,\infty}(V)} \leq c\|u\|_{L^p}, \quad u \in L^{p,\infty};$

(iii) $\left(\int\limits_B (I_\alpha * V)(y)\,dy\right)^p dx \leq cV(B)$ *where* $B = B_r(x)$ *is an arbitrary ball;*

(iv) $V(E) \leq c\operatorname{cap}_\alpha(E)$ *for all Borel sets* E.

Remark 2.2. The conditions from the above theorem are more general than the two weight condition

$$(|B(x,r)|)^{1/p}(V(B(x,r))^{1/p} \leq cr,$$

which can be shown to coincide with

$$V(B(x,r)) \leq cr^{n-\alpha p}$$

for all balls $B(x,r)$. (Indeed, it is $\operatorname{cap}_\alpha(B(x,r)) \sim r^{n-\alpha p}$).

Fefferman's paper [12] gave the following sufficient condition: Let us recall that the *Fefferman-Phong class* F_p, $1 \leq p \leq N/2$ consists of functions V such that

$$\|V\|_{F_p} = \sup_{\substack{x\in\mathbb{R}^N\\ r>0}} r^2\left(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |V(y)|^p\,dy\right)^{1/p} < \infty.$$

Theorem 2.3. (Fefferman [12]). *Let* $N \geq 3$, $1 < p \leq N/2$, *and* $V \in F_p$. *Then* (1.1) *holds*.

A particularly fine and elegant proof of (1.1) was given by Chiarenza and Frasca [9].

It is worth observing that $F_{p_2} \subset F_{p_1}$ for $1 \le p_1 \le p_2 \le N/2$, and plainly $F_{N/2} = L^{N/2}$. Provided that we restrict ourselves to balls $B(x,r)$ with radius smaller than some $\varepsilon_0 > 0$ in the above definition one can talk about the *Morrey space* $\mathcal{L}^{p,N-2p}$. Let us recall that, for $0 < \lambda \le N$ and $1 \le p < \infty$, the Morrey space $\mathcal{L}^{p,\lambda}$ is the collection of all $V \in L^p_{\mathrm{loc}}$ such that

$$\|V\|_{\mathcal{L}^{p,\lambda}} = \sup_{\substack{x\in\mathbb{R}^N \\ 0<r\le r_0}} r^{-\lambda/p}\left(\int_{B(x,r)} |V(y)|^p\,dy\right)^{1/p} < \infty.$$

Inserting a 'hat function', that is,

$$u(x) = (r - |x|)\chi_{B(0,r)}, \qquad x \in \mathbb{R}^N,$$

into (1.1) shows that the weight V must belong to $\mathcal{L}^{1,1}$ in order that (1.1) holds. Nevertheless, as is well known this is not sufficient. Further investigation shows that the situation near $\mathcal{L}^{1,N-2}$ is of rather delicate nature. Observe also that when passing to various refined conditions, then the constant C in (1.1) can depend on $\operatorname{supp} u$; this is quite sufficient for relevant applications.

For $f \in L^1_{\mathrm{loc}}$, let us denote

$$\eta(f,\varepsilon) = \sup_{x\in\mathbb{R}^N} \int_{|x-y|\le\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}}\,dy.$$

The *Stummel-Kato* class is defined by

$$S = \{f;\ \eta(f,\varepsilon) < \infty \text{ for all } \varepsilon \text{ and } \eta(f,\varepsilon) \searrow 0 \text{ as } \varepsilon \searrow 0\}.$$

A *variant of the Stummel-Kato class*, sometimes denoted by $\widetilde{S}$ is defined as

$$\widetilde{S} = \{f;\ \eta(f,\varepsilon) < \infty \text{ for all } \varepsilon > 0\}.$$

Restriction of these spaces to a domain in $\mathbb{R}^N$, say, Ω can be done in an obvious way, namely, by considering $\chi_\Omega V$ instead of V.

Relations between the spaces considered up to now are discussed e.g. in Zamboni [34], Di Fazio [10], Piccinini [27] and Kurata [20]; the last quoted author considers also other variants of the Stummel-Kato class to get a background tailored for more general elliptic operators.

Proposition 2.4. *The following statements are true:*

(i) $\mathcal{L}^{1,\lambda} \subset S \subset \widetilde{S} \subset \mathcal{L}^{1,N-2}$, $\lambda > N-2$;

(ii) $L^{N/2,\infty} \subset F_p$ *for every* $1 \le p < N/2$, *where the former space denotes the weak* $L^{N/2}$ *space (the Marcinkiewicz space);*

(iii) *For each* $p \ge 2$ *and each* $0 < \lambda < n$, *there exists a function* $f \in \mathcal{L}^{p,\lambda} \setminus L^q$ *for every* $q > p$;

(iv) *For every sufficiently small* $p > 1$ *there exists a function* $f \in F_p \setminus L^{N/2,\infty}$;

(v) $S(\Omega) \subset F_1(\Omega)$, *and* $L^{N/2}(\Omega)$ *is incomparable with* $S(\Omega)$.

Let us observe that (ii) gives a sufficient condition for the validity of (1.1) in terms of another scale of function spaces, namely, of weak Lebesgue spaces. We shall come to use of more general Lorentz spaces later.

For instance, employing the class $\widetilde{S}$, it is possible to prove ([34]):

Theorem 2.5. *Let* $V \in \widetilde{S}$. *Then for every* $r > 0$ *there is* C_r *depending only on* $\eta(V, r)$ *and* N *such that*

$$\int_{\mathbb{R}^N} u^2(x)V(x)\,dx \le C_r \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx$$

holds for every $u \in C_0^\infty$ *supported in* $B(0, r)$.

Further interesting results can be found e.g. in Chang, Wilson and Wolff [6], who consider a certain Orlicz variant of Morrey spaces. An Orlicz type refinement of the well-known Adams' inequality [1], has recently appeared in Ragusa and Zamboni [28].

3. Rotation-invariants weights.

Rotation invariant weights admit a particularly simple approach. Let us consider

$$\int_B |f(x)|^2 V(|x|)\,dx \le c \int_B |\nabla f(x)|^2\,dx, \quad f \in C_0^\infty(B), \tag{3.1}$$

where $B = B(0,1) \subset \mathbb{R}^n$, $n \ge 1$ (and f is *not necessarily* rotation-invariant).

Proposition 3.1. *The inequality* (3.1) *holds for all* $f \in C_0^\infty(B)$ *iff*

$$\int_1^0 |\varphi(r)|^2 V(r) r^{n-1}\,dr \le c \int_0^1 |\varphi'(r)|^2 r^{n-1}\,dr, \tag{3.2}$$

$$\textit{for all } \varphi \in C^\infty,\ \varphi(1) = 0.$$

Proof. Assume (3.1), let f be rotation-invariant, i.e. $f(x) = \varphi(|x|)$ with $\varphi \in C^\infty$, and φ vanishes near 1. Then

$$\int_B |f(x)|^2 V(|x|)\,dx = |S_{n-1}| \int_0^1 |f(r)|^2 V(r) r^{n-1}\,dr$$

and

$$\int_B |\nabla f(x)|^2\,dx = \int_B \left|\frac{x_j}{|x|}\varphi'(|x|)\right|^2 dx = |S_{n-1}| \int_0^1 |\varphi'(r)|^2 r^{n-1}\,dr.$$

This together with (3.1) yields (3.2).

Assume (3.2), let $f \in C_0^\infty(B)$, not necessarily rotation-invariant. Then, for any $\sigma \in S_{n-1}$,

$$\int_0^1 |f(r,\sigma)|^2 V(r) r^{n-1}\,dr \le c \int_0^1 |\partial f(r,\sigma)|^2 r^{n-1}\,dr.$$

We have, for every $x = |x| \cdot x/|x| = r\sigma$,

$$|\partial f(r,\sigma)| \le |(\nabla f)(x)|,$$

hence

$$\int_0^1 |f(r,\sigma)|^2 V(r) r^{n-1}\,dr \le c \int_0^1 |\nabla f(r,\sigma)|^2 r^{n-1}\,dr.$$

Now integrate over S_{n-1}. $\square$

Recall (Muckenhoupt 1972) that, for $1 < p < \infty$, and weight function v and w,

$$\int_0^1 |\varphi(r)|^p v(r)\,dr \le c \int_0^1 |\varphi'(r)|^p w(r)\,dr$$

$$\text{for all } \varphi \in C^\infty,\ \varphi(1) = 0$$

iff

$$\sup_{0<r<1} \left(\int_0^1 v\,dt\right)\left(\int_r^1 w^{1/(1-p)} dt\right) < \infty.$$

Put $p = 2$, $v(r) = V(r) r^{n-1}$, $w(r) = r^{n-1}$, then Proposition (3.1) yields:

Theorem 3.2. *The inequality* (3.1) *holds iff*

$$B = \sup_{0<r<1} B(r) =$$
$$= \sup_{0<r<1} \left(\int_0^r V(t) t^{n-1} dt \right) \left(\int_r^1 t^{1-n} dt \right) < \infty.$$

Remark 3.3. We have

$$B(r) = (1 - R) \int_0^r V(t) dt \quad \text{for} \quad n = 1,$$
$$B(r) = \left(\log \frac{1}{r} \right) \int_0^r V(t) t dt \quad \text{for} \quad n = 2,$$
$$B(r) = \frac{r^{2-n} - 1}{n - 2} \int_0^r V(t) dt \quad \text{for} \quad n = 3.$$

Example 3.4. Let $V(r) = r^\alpha$. Then

(i) if $n = 1$, then (3.1) holds iff $\alpha > -1$;
(ii) if $n = 2$, then (3.1) holds iff $\alpha > -2$;
(iii) if $n = 3$, then (3.1) holds iff $\alpha \geq -2$.

Example 3.5. The limit case for $n = 2$. Let

$$V(r) = r^{-2} \log^\beta \left(\frac{e}{r} \right),$$

then

$$B(r) = \left(\log \frac{1}{r} \right) \int_0^r \frac{1}{t} \log^\beta \left(\frac{e}{t} \right) dt.$$

The last integral is finite iff $\beta + 1 < 0$. Now if $\beta < -1$, then

$$B(r) = \frac{1}{|\beta + 1|} \left(\log \frac{1}{r} \right) \log^{\beta+1} \left(\frac{e}{r} \right).$$

Hence (3.1) holds iff $\beta \leq -2$.

4. The size condition and some applications.

For the sake of applications we shall pay a special attention the so called '*smallness condition*' or the '*size condition*' (see (4.1) below), playing a important role in the study of the strong unique continuation property. We shall restrict ourselves to a differential inequality arising from the Schrödinger operator, namely, $|\Delta u| \leq V|u|$.

As in the case of the trace inequality there is a very rich literature on the subject of the unique continuation property and the references included here cover just a part of those relevant for what we pursue here.

A locally integrable function u is said to have a *zero of infinite order* at x_0 if

$$\lim_{r\to 0_+} r^{-k} \int\limits_{|x-x_0|<r} |u(x)|^2\,dx = 0$$

for all $k = 1, 2, \ldots$. If every solution of a given differential equation, with a zero of infinite order, vanishes identically, then the corresponding operator is said to satisfy the *strong unique continuation property* (the SUCP). As to non-analytic setting of the problem let us recall that in 1939 Carleman [5] proved that the operator $-\Delta + V$ has the strong unique continuation property provided $V \in L^\infty_{\mathrm{loc}}$, that is, he showed that under this assumption a solution of the equation $-\Delta u + V(x)u = 0$ with a zero of infinite order vanishes identically. There is a lot of results concerning the SUCP, with various assumptions on the potential V and also on coefficients in the case of a more general elliptic operator in question. Here we shall go along the lines of sufficient conditions in terms of integrability of the potential with no apriori assumptions on its pointwise behaviour.

Recall Jerison and Kenig [16], Stein [30], where the SUCP is proved for $V \in L^{N/2}_{\mathrm{loc}}$ or for V locally small in the Marcinkiewicz space $L^{N/2,\infty}$, $N \geq 3$, Garofalo and Lin [14], and Pan [26] with the pointwise growth condition $V(x) \leq M/|x|^2$ (and variations of that), $N \geq 2$, and without the size conditions for V.

Wolff [33] has constructed counterexamples for $N = 3$ and $N = 2$, showing that the assumption about the local smallness of the imbedding norm in (1.1) cannot be removed in general. For $N = 2$ there is the result due to Gossez and Loulit [15] with the sufficient condition $V \in L^1 \log L$ for the SUCP.

Theorem 4.1. (Wolff [33]). *The following statements are true:*

(i) *There exists a function $u : \mathbb{R}^3 \to \mathbb{R}^1$, smooth and not identically zero, vanishing at infinite order at the origin and such that $|\Delta u| \leq V|u|$ with $V \in L^{3/2,\infty}$.*

(ii) *There exists a function* $u : \mathbb{R}^2 \to \mathbb{R}^1$, *smooth and not identically zero, vanishing at infinite order at the origin and such that* $|\Delta u| \leq V|u|$ *with* $V \in L^1$.

Chanillo and Sawyer [7] considered the classes F_p for $p > (N-1)/2$ and proved the SUCP for potentials V which have locally small F_p-norm in the sense that

$$\limsup_{r\to 0} \|V\chi_{B(y,r)}\|_{F_p} \leq \varepsilon(p, N)$$
$$\text{for all } y \in \mathbb{R}^N,$$

where $\varepsilon(p, N)$ is a sufficiently small constant. Since $L^{N/2,\infty} \subset F_p$ for all $p < N/2$ (see Proposition 2.4) this gives a result for V in a larger class than in [16], [30], however, with the size constraint, this time in the F_p class; again the value of the constant appearing in the size condition is not specified.

If $N \leq 3$, then a condition for the SUCP in terms of the local smallness of the constant C in (1.1) appears; more specifically:

Theorem 4.2. (Chanillo and Sawyer [7]). *Let us assume that* $N = 2$ *or* $N = 3$ *and that* Ω *is a bounded open and connected subset of* $\mathbb{R}^N$. *Let* $T(V)$ *denote the imbedding in* (1.1). *If*

$$\limsup_{r\to 0_+} \|T(V\chi_{B(x,r)})\| \leq \varepsilon \tag{4.1}$$

with a sufficiently small $\varepsilon > 0$ *for all* $x \in \Omega$, *then any solution* $u \in W^{2,2}_{\text{loc}}$ *of the inequality* $|\Delta u| \leq V|u|$ *in* Ω *has the SUCP.*

It turns out that the size condition can be effectively verified in some cases. We shall consider the scale of Lorentz spaces in the dimension 3, and for $N = 2$ we present a general theorem, including [15] as a special case. Proofs can be found in [19].

We shall need some basic facts from the Orlicz, Lorentz-Zygmund and Orlicz-Lorentz spaces theory. Let us agree that all the spaces in the sequel will be considered on a ball $B \subset \mathbb{R}^N$ with the unit measure, $N \geq 2$, or on the interval (0, 1); we shall usually omit the appropriate symbol for the domain since it will be clear from the context.

We shall also need a finer scale of spaces, which includes Orlicz spaces in a rather same manner as Lorentz spaces include Lebesgue spaces (see, e.g. Montgomery-Smith [24]).

Let us recall that an even and convex function $\Phi : \mathbb{R} \to [0, \infty)$ such that $\lim_{t\to 0} \Phi(t) = \lim_{t\to\infty} 1/\Phi(t) = 0$ is called a *Young function*.

Let Φ and Ψ be Young functions. For a function g even on $\mathbb{R}^1$ and positive on $(0, \infty)$ put

$$\widetilde{g}(t) = \begin{cases} 1/g(1/t), & t > 0, \\ \widetilde{g}(-t), & t < 0, \\ g(0), & t = 0. \end{cases}$$

Let V be a weight in B and let f_V^* denote the non-increasing rearrangement of f with respect to the measure $V(x)\,dx$. An *Orlicz-Lorentz space* $L^{\Phi\Psi}(V)$ is the set of all measurable f on B for which the *Orlicz-Lorentz functional*

$$\|f\|_{\Phi,\Psi;V} = \|f_V^* \circ \widetilde{\Phi} \circ \widetilde{\Psi}^{-1}\|_{\Psi} = \tag{4.2}$$

$$= \inf\left\{\lambda > 0; \int_0^\infty \Psi\left(\frac{f_V^*(\widetilde{\Phi}(\widetilde{\Psi}^{-1}(t)))}{\lambda}\right) dt \le 1\right\}$$

is finite. A measurable function f defined on B belongs to a *weak Orlicz* (or *Orlicz-Marcinkiewicz*) space $L^{\Phi,\infty}(V)$ if its *Orlicz-Marcinkiewicz functional*

$$\|f\|_{\Phi,\infty;V} = \sup_{\xi>0} \widetilde{\Phi}^1(\xi) f_V^*(\xi) \tag{4.3}$$

is finite. If $V \equiv 1$, we shall simply write $L^{\Phi,\Psi}$ and $L^{\Phi,\infty}$ instead of $L^{\Phi,\Psi}(1)$ and $L^{\Phi,\infty}(1)$, resp.

For brevity and in accordance with a general usage we shall often use only the major part of a Young function (that is, functions equivalent to the Young function in question in a neighbourhood of infinity) in symbols for spaces.

The quantities in (4.2) and (4.3) are not generally norms. Nevertheless, they are quasinorms in many relevant cases; cf. Montgomery-Smith [24], and Krbec and Lang [18].

Let us observe that $L^{\Phi,\Phi} = L^{\Phi}$, the *Orlicz space*. If $\Phi(t) = |t|^p$ and $\Psi(t) = |t|^q$, then $L^{\Phi,\Psi} = L^{p,q}$, the *Lorentz space*, $L^{\Phi,\infty} = L^{p,\infty}$, the Marcinkiewicz space; analogously for the weighted variants.

Special cases of the Orlicz-Lorentz spaces are the Lorentz-Zygmund spaces, that is, logarithmic Lorentz spaces, investigated by Bennett and Rudnick [3]. For $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}^1$, the Lorentz-Zygmund space $L^{pq}(\log L)^\alpha$ consists of functions f with the finite functional

$$\|f\|_{L^{p,q}(\log L)^\alpha} = \left(\int_0^1 [t^{1/p}(\log(e/t))^\alpha f^*(t)]^q \frac{dt}{t}\right)^{1/q} \qquad \text{for } q < \infty,$$

$$\|f\|_{L^{p,q}(\log L)^\alpha} = \sup_{0<t<1} t^{1/p} (\log(e/t))^\alpha f^*(t) \qquad \text{for } q = \infty$$

(we put $t^{1/\infty} = 1$). It is easy to check that these spaces increase with decreasing p, increasing q and decreasing α.

Later we shall also need the spaces of the form $L^{\exp t^{r'}, t^r}$, where $1/r + 1/r' = 1$. It turns out that they coincide (see [11]) with spaces characterized by the integral condition (used e.g. in [4] and in [35], Lemma 2.10.5)

$$\int_0^1 \left(\frac{f^*(t)}{\log(e/t)} \right)^r dt < \infty,$$

which equal to $L^{\infty,r}(\log L)^{-1}$ in the [3] notation. Also, the Zygmund space $L \log L$ equals to $L^{1,1} \log L$ and it is nothing but $L^{t \log t, t \log t}$.

Remark 4.3. We recall that $L^{p_1,q_1}(\log L)^{\alpha_1} \subset L^{p_2,q_2}(\log L)^{\alpha_2}$ if any of the following conditions holds:
(i) $p_1 > p_2$;
(ii) $p_1 = p_2$, $q_1 > q_2$, and $\alpha_1 + 1/q_1 > \alpha_2 + 1/q_2$;
(iii) $p_1 = p_2 < \infty$, $q_1 \leq q_2$, and $\alpha_1 \geq \alpha_2$;
(iv) $p_1 = p_2 = \infty$, $q_1 \leq q_2$, and $\alpha_1 + 1/q_1 \geq \alpha_2 + 1/q_2$
(see [3], Theorems 9.1 and 9.3 and 9.5).

Remark 4.4. According to the limiting imbedding theorem due to Brézis and Wainger [3] we have, for $N = 2$,

$$W_0^{1,2} \hookrightarrow L^{\infty,2}(\log L)^{-1}. \tag{4.4}$$

The latter space, as was observed above, is the Orlicz-Zygmund space $L^{\exp t^2, t^2}$, a space smaller than $L^{\exp t^2} = L^{\exp t^2, \exp t^2}$, and this interpretation of the target space in (4.4) gives a natural analogue to the (sublimiting) imbeddings of Sobolev spaces into Lebesgue spaces and their Lorentz refinements.

5. Decomposition of imbeddings.

For the sake of simplicity we suppose again that the domain B is a ball, $|B| = 1$ and we shall usually omit the symbol for it in notation. We are seeking for sufficient conditions for (1.2) and (4.1). We shall even find a condition stronger than (4.1), namely,

$$\lim_{\delta \to 0} \sup_{\substack{A \subset B \\ |A| < \delta}} \|T(V\chi_A)\| = 0. \tag{5.1}$$

First consider separately the scale of Lorentz spaces.

Theorem 5.1. ([19]). *Let* $N \geq 3$ *and* $V \in L^{N/2,r}$, $N/2 \leq r < \infty$. *Then (*1.2) *and (*5.1) *hold.*

We shall pass to Lorentz-Zygmund spaces to get a more general sufficient condition for (1.2) and various sufficient conditions for (5.1) The situation is not straightforward since three parameters can change. The first parameter will be kept fixed, equal to 1 since its changes lead to changes too big for the fine tuning we need.

Theorem 5.2. ([19]). *Let* $N = 2$.
(i) *The inequality* (1.2) *holds if* $V \in L^{1,\infty}(\log L)^2$.
(ii) *Let* $V \in L^{1,s}(\log L)^\beta$, *where either*

$$0 < s \leq 1, \qquad \beta \geq 1, \tag{5.2}$$

or

$$1 < s < \infty, \qquad \beta \geq 2 - 1/s, \tag{5.3}$$

or

$$s = \infty, \qquad \beta > 2. \tag{5.4}$$

Then (1.2) *and* (5.1) *hold.*

Remark 5.3. The *proofs* of Theorems 5.1 and 5.2 can be carried out making use of the refined Sobolev imbedding $W^{1,2} \hookrightarrow L^{2N/(N-2),s}$ for $N \geq 3$ and of the refined limiting imbedding in (4.4) for $N = 2$ together with conditions (necessary and sufficient) for the imbeddings of weighted Orlicz-Lorentz spaces, taking, moreover, care about the quantitative behaviour of norms of the imbeddings. The details can be found in [19].

Remark 5.4. The space $L^{1,\infty}(\log L)^2$ can be identified with the Orlicz-Marcinkiewicz space $L^{t\log^2 t,\infty}$, and $L^{1,s}(\log L)^\beta$, $0 < s < \infty$, with $L^{t\log^\beta t, t^s}$. This can be checked easily. Indeed, considering for instance $V \in L^{1,\infty}(\log L)^2$, that is, if we have $\sup_{0<t<1} t(\log(e/t))^2 V^*(t) < \infty$, then $\widetilde{F}^{-1}(t) = t(\log(e/t))^2$ near the origin, hence $F(\xi) \sim \xi(\log(e/\xi))^2$ for large values of ξ.

By way of applications we give a sufficient condition for the SUCP, relying on the SUCP theorem in [7] invoked earlier.

Corollary. ([19]). *The following statements are true:*

(i) *Let* $N = 3$. *Let* $V \in L^{3/2,r}$, $3/2 \le r < \infty$. *Then the inequality* $|\Delta u| \le V|u|$ *has the* SUCP *in* $W^{2,2}_{\mathrm{loc}} \cap W^{1,2}_0$.

(ii) *Let* $N = 2$. *Let* $V \in L^{1,s}(\log l)^{\beta}$, *where* s *and* β *satisfy any of the conditions* (5.2)–(5.4). *Then the inequality* $|\Delta u| \le V|u|$ *has the* SUCP *in* $W^{2,2}_{\mathrm{loc}} \cap W^{1,2}_0$.

Remark 5.6. The statement in Corollary 5.5 (i) actually says that the size condition from Stein [30] is fulfilled under the given conditions.

If $V \in L^{1,s}(\log L)^{\beta}$, where the parameters s and β satisfy either (5.2) or (5.4), then $V \in L^{1,1}(\log L)^1$ and we recover the SUCP theorem due to Gossez and Loulit [15]. Concerning (5.3) one can construct functions, which show that $L^{1,1}(\log L)^1$ and $L^{1,s}(\log L)^{2-(1/s)}$ are incomparable for $1 < s < \infty$.

Indeed, if $V(\alpha, .)$, $0 < \alpha \le 1$, is such that

$$V^*(\alpha, t) = \frac{1}{t}\,(\log(e/t))^{-2}\,(\log(e/t)))^{-\alpha}\,,$$

for t small,

then $V(\alpha, .) \notin L^{1,1}(\log L)^1$ and if $s > 1/\alpha$, then $V(\alpha, .) \in L^{1,s}(\log L)^{2-(1/s)}$. On the other hand, if $V(\tau, .)$, $0 < \tau < 1$, is such that $V^*(\tau, t) = \chi_{(0,\tau)}(t)$, then

$$\|V(\tau, .)\|_{L^{1,1}(\log L)^1} = \tau(2 - \log\tau), \qquad 0 < \tau < 1.$$

Going through some calculation one can check that

$$\lim_{\tau\to 0} \frac{\|V(\tau, .)\|^s_{L^{1,s}(\log L)^{2-(1/s)}}}{\|V(\tau, .)\|^s_{L^{1,1}(\log L)^1}} = \infty.$$

Therefore $L^{1,1}(\log L)^1$ is not continuously imbedded into $L^{1,s}(\log L)^{2-(1/s)}$ and by the closed graph theorem we get $L^{1,1}(\log L)^1 \not\subset L^{1,s}(\log L)^{2-(1/s)}$.

REFERENCES

[1] D. Adams, *Traces of potentials arising from translation invariant operators,* Ann. Scuola Norm. Sup. Pisa, 25 (1971), pp. 1–9.

[2] D.R. Adams - L.I. Hedberg, *Function Spaces and Potential Theory,* Springer, Berlin, 1996.

[3] C. Bennett - K. Rudnick, *On Lorentz-Zygmund spaces,* Dissertationes Math. (Rozprawy Mat.), 175 (1980), pp. 1–72.

[4] H. Brézis - S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities,* Comm. Partial Diff. Equations, 5 (1980), pp. 773–789.

[5] T. Carleman, *Sur un problème dúnicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes,* Ark. Mat. Astron. Fys., 26(B) 1939, pp. 1–9.

[6] S.Y.A. Chang - J.M. Wilson - T.H. Wolff, *Some weighted norm inequalities concerning the Schrödinger operator,* Comment. Math. Helvetici, 60 (1985), pp. 217–246.

[7] S. Chanillo - E. Sawyer, *Unique continuation for* $\Delta + V$ *and the C. Fefferman-Phong class,* Trans. Amer. Math. Soc., 318 (1990), pp. 275–300.

[8] S. Chanillo - R.L. Wheeden, L^p *estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operator,* Comm. Partial Diff. Equations, 10 - 9 (1985), pp. 1077–1116.

[9] F. Chiarenza - M. Frasca, *A remark on a paper by C. Fefferman,* Proc. Amer. Math. Soc., 108 (1990), pp. 407–409.

[10] G. Di Fazio, *Hölder continuity of solutions for some Schrödinger equations,* Rend. Sem. Mat. Univ. Padova, 79 (1988), pp. 173–183.

[11] D.E. Edmunds - M. Krbec, *On decomposition in exponential Orlicz spaces,* to appear in Math. Nachr.

[12] C. Fefferman, *The uncertainty principle,* Bull. Amer. Math. Soc., 9 (1983), pp. 129–206.

[13] C. Fefferman - D.H. Phong, *Lower bounds for Schrödinger operator,* Journées "Equations aux dérivées partielles", Saint-Jean-de Monts, 7–11 juin 1982.

[14] N. Garofalo - F.-H. Lin, *Unique continuation for Elliptic Operators: A geometric-variational approach,* Comm. Pure Appl. Math., 40 (1987), pp. 347–366.

[15] J.-P. Gossez - A. Loulit, *A note on two notions of unique continuation,* Bull. Soc. Math. Belg., Sér. B 45, No. 3 (1993), pp. 257–268.

[16] D. Jerison - C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operator,* Ann. of Math., 121 (1985), pp. 463–488.

[17] R. Kerman - E. Sawyer, *The trace inequality and eigenvalue estimates for Schrödinger operators,* Ann. Inst. Fourier (Grenoble), 36 (1986), pp. 207–228.

[18] M. Krbec - J. Lang, *On imbeddings between weighted Orlicz-Lorentz spaces,* Georgian Math. J., 4 (1997), pp. 117–128.

[19] M. Krbec - T. Schott, *Superposition of imbeddings and Fefferman's inequality,* to appear in Boll. Un. Mat. Ital.

[20] K. Kurata, *A unique continuation theorem for uniformly elliptic equations with strongly singular potentials,* Comm. Partial Diff. Equations, 18 (1993), pp. 1161–1189.

[21] V.G. Maz'ya, *Classes of domains and embedding theorems for functional spaces,* Dokl. Akad. Nauk SSSR, 133 (1960), pp. 527–530.

[22] V.G. Maz'ya, *On the theory of the n-dimensional Schrödinger operator,* Izv. Akad. Nauk SSSR, Ser. Mat., 28 (1964), pp. 1145–1172.

[23] V.G. Maz'ya, *Sobolev Spaces,* Springer-Verlag, Berlin, 1985.

[24] S.J. Montgomery-Smith, *Comparison of Orlicz-Lorentz spaces,* Studia Math., 103 - 2 (1992), pp. 161–189.

[25] B. Muckenhoupt - R.L. Wheeden, *Weighted norm inequalities for fractional integrals,* Trans. Amer. Math. Soc., 192 (1974), pp. 251–275.

[26] Y. Pan, *Unique continuation for Schrödinger operators with singular potentials,* Comm. Partial Diff. Equations, 17 (1992), pp. 953–965.

[27] L. Piccinini, *Inclusioni tra spazi di Morrey,* Boll. Un. Mat. Ital., 4 - 2 (1969), pp. 95–99.

[28] A. Ragusa - P. Zamboni, *A potential theoretic inequality,* to appear in Czech. Math. J.

[29] E.T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators,* Studia Math., 75 (1982), pp. 1–11.

[30] E.M. Stein, *Appendix to "Unique continuation",* Ann. of Math., 121 (1985), pp. 489–494.

[31] N. Trudinger, *On imbeddings into Orlicz spaces and some applications,* J. Math. Mech., 17 (1967), pp. 473-483.

[32] I.E. Verbitsky - R.L. Wheeden, *Weighted inequalities for fractional integrals and applications to semilinear equations,* J. Funct. Anal., 129 (1995), pp. 221–241.

[33] T.H. Wolff, *Note on counterexamples in strong unique continuation problems,* Proc. Amer. Math. Soc., 114 (1992), pp. 351–356.

[34] P. Zamboni, *Some function spaces and elliptic partial differential equations,* Matematiche (Catania), 42 (1987) pp. 171–178.

[35] W.P. Ziemer, *Weakly differentiable functions,* Springer, New York, 1989.

Institute of Mathematics,
Academy of Sciences of the Czech Republic,
Žitná 25,
115 67 Prague 1 (CZECH REPUBLIC)
e-mail: krbecm@matsrv.math.cas.cz