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AN EMBEDDING THEOREM FOR NON SMOOTH VECTOR FIELDS OF STEP TWO

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1. Introduction.

In this note we prove the following embedding result.

Theorem 1.1. *Let us consider in* $\mathbb{R}^N$, $N \geq 3$, *the vector fields*

$$X_k = \sum_{j=1}^{N} a_k^{(j)} \partial_{x_j}, \quad k = 1, \dots, p, \tag{1}$$

where the coefficients $a_k^{(j)}$ *belong to the classical Sobolev space* $W^{2,2N/3}(\mathbb{R}^N)$. *Let us assume that for every* $j = 1, \dots, N$

$$\partial_{x_j} = \sum_{k=1}^{p} b_j^{(k)} X_k + \sum_{1 \leq k < h \leq p} \alpha_j^{(k,h)} [X_k, X_h] \tag{2}$$

with $b_j^{(k)}, \alpha_j^{(k,h)} \in L^\infty(\mathbb{R}^N)$ *and* $X_k \alpha_j^{(k,h)}, X_h \alpha_j^{(k,h)} \in L^{2N}(\mathbb{R}^N)$ *for all* j, k, h. *Then there exists a positive constant* C, *only depending on*

$$M := \sum_{j=1}^{N} \Big(\sum_{k=1}^{p} \big(||a_k^{(j)}||_{2,2N/3} + ||b_j^{(k)}||_\infty \big) + \tag{3}$$

$$+ \sum_{1 \leq k < h \leq p} \big(||X_k(\alpha_j^{(k,h)})||_{2N} + ||X_h(\alpha_j^{(k,h)})||_{2N} + ||\alpha_j^{(k,h)})||_\infty \big) \Big)$$

such that

$$||u||_{1/2,2} \leq C\Big(||u||_2 + \sum_{k=1}^{p} ||X_k u||_2\Big) \tag{4}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$.

In (3) and (4) $||\cdot||_{s,q}$ and $||\cdot||_q$ denote the norms in the classical Sobolev space $W^{s,q}(\mathbb{R}^N)$ and in the space $L^q(\mathbb{R}^N)$, respectively.

When the coefficients $a_k^{(j)}$ are smooth the condition (2) implies that the vector fields X_k, together with their commutator of step two $[X_k, X_h]$, generate the space $\mathbb{R}^N$ at every point. Then, inequality (4) with a rougher positive constant C is a consequence of a general result on smooth vector fields whose Lie Algebra has maximum rank at every point: see for example [4], [5]. The main novelty of our Theorem 1.1 lies in showing that the constant C in (4) only depends on the quantity M in (3). We will prove Theorem 1.1 in Section 3. Section 2 is devoted to the proof of a preliminary result that seems to have an independent interest.

The question answered by Theorem 1.1 naturally arise in studying regularity properties of viscosity solutions to the prescribed Levi-curvature equation in $\mathbb{R}^3$. Given a C^2 function u on a open set $\Omega \subseteq \mathbb{R}^3$, the Levi-curvature of the graph of u is the real number

$$H = -\frac{(1+(\partial_{x_3}u)^2)^{1/2}}{(1+(a_1)^2+(a_2)^2)^{3/2}}(X_1^2 u + X_2^2 u), \tag{5}$$

where

$$X_k = \partial_{x_k} + a_k \partial_{x_3}, \quad k = 1, 2 \tag{6}$$

and the coefficients a_k depend on the derivatives of u in the following way:

$$\begin{aligned} a_1 &= a_1(Du) = \frac{\partial_{x_2}u - \partial_{x_1}u\,\partial_{x_3}u}{1+(\partial_{x_3}u)^2}, \\ a_2 &= a_2(Du) = -\frac{\partial_{x_1}u + \partial_{x_2}u\,\partial_{x_3}u}{1+(\partial_{x_3}u)^2}. \end{aligned} \tag{7}$$

Due to (5) the operator

$$X_1^2 u + X_2^2 u = -H(\cdot, u)\frac{(1+(a_1)^2+(a_2)^2)^{3/2}}{(1+(\partial_{x_3}u)^2)^{1/2}}, \tag{8}$$

where H is a given real function defined on $\Omega \times \mathbb{R}$, is called the equation of the prescribed Levi-curvature H. We explicitly remark that the left hand side of (8) is a second order quasilinear operator whose characteristic form is

$$\begin{aligned}\mathcal{A}(x, Du; \xi) &:= \langle X_1, \xi\rangle^2 + \langle X_2, \xi\rangle^2 = \\ &= (\xi_1 + a_1(Du)\xi_3)^2 + (\xi_2 + a_2(Du)\xi_3)^2.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{A}(x, Du; \xi) \geq 0, \quad &\text{for any} \quad (x, Du; \xi), \\ \mathcal{A}(x, Du; \xi) = 0, \quad &\text{when} \quad \xi = (-a_1, -a_2, 1).\end{aligned}$$

As a consequence the operator in (8) has to be considered a quasilinear degenerate elliptic operator which is non-elliptic at any point (x, Du). For this reason to viscosity solutions of (8) cannot be applied the well known regularity results for viscosity solutions to (fully nonlinear) elliptic operator (see e.g. [1]).

Very recently Citti and the authors have introduced in [3] a new iteration technique for studying regularity properties of Lipschitz-continuous viscosity solutions to (8). When the prescribed curvature H is different from zero at every point a crucial step of the previous mentioned iteration scheme is the embedding inequality (4) in $\mathbb{R}^3$ related to the vector fields X_k in (6) with coefficients a_k given by (7). On the other hand, as it has been first noticed in [2], if X_k, $k = 1, 2$, are these vector fields, the following identity formally holds:

$$[X_1, X_2] = Q\partial_{x_3},$$

with

$$Q = H\frac{(1 + (a_1)^2 + (a_2)^2)^{3/2}}{(1 + (\partial_{x_3}u)^2)^{1/2}}.$$

Then, if u is a Lipschitz-continuous solution to (8) with prescribed curvature H bounded away from zero, then

$$\begin{aligned}\partial_{x_1} &= X_1 - \frac{a_1}{Q}[X_1, X_2], \\ \partial_{x_2} &= X_2 - \frac{a_2}{Q}[X_1, X_2], \\ \partial_{x_3} &= \frac{1}{Q}[X_1, X_2]\end{aligned}$$

and condition (2) is (formally) satisfied. This remark makes Theorem 1.1 applicable to the iteration procedure in [3].

This application is the main motivation of the present note. We would like to stress that embedding theorems for vector fields with non regular coefficients seem to be crucial tools in studying regularity properties of weak solutions to many classes of non linear degenerate elliptic equations. Our paper is a first contribution in this direction.

2. A commutator theorem.

Let $m \in C^{\infty}(\mathbb{R}^N)$ be such that

$$|D^{\alpha} m(\xi)| \leq c_{\alpha} |\xi|^{-|\alpha|} \tag{9}$$

for any multi-index α. Let us denote by $\mathcal{F}$ the Fourier transform and by T the convolution operator related to

$$K = \mathcal{F}^{-1}(m),$$

$$T : \mathcal{S}' \to \mathcal{S}', \quad Tv = K * v.$$

From condition (9) it follows that $m \in L^{\infty}$, $K \in C^{\infty}(\mathbb{R}^N \setminus \{0\})$ and

$$|D^{\alpha} K(x)| \leq c'_{\alpha} |x|^{-N-|\alpha|} \tag{10}$$

for any multi-index α (see [6], p. 26). Then K is a Calderon Zygmund kernel and T is a continuous linear operator from $L^r(\mathbb{R}^N)$ to itself, for every $r \in]1, \infty[$.

Given a smooth function $a : \mathbb{R}^N \to \mathbb{R}$, let us consider the commutator

$$[a, T]v := a(Tv) - T(av).$$

The main result of this section is the following theorem.

Theorem 2.1. *Let $1 < p < \infty$ and $1 < q < N$ be such that $\frac{1}{N} < \frac{1}{p} + \frac{1}{q} < 1$. Define*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - \frac{1}{N}. \tag{11}$$

Then, for any function $a \in C^{\infty}(\mathbb{R}^N)$,

$$||D[a, T]v||_r \leq c||D^2 a||_q ||v||_p, \quad \forall v \in C_0^{\infty}(\mathbb{R}^N) \tag{12}$$

where c is a positive constant independent on a and v. In (12) D denotes the gradient operator in $\mathbb{R}^N$, while $D^2 a$ stands for the hessian matrix of a.

Proof. First of all, by using standard devices, we recognize that for every function $v \in C_0^\infty(\mathbb{R}^N)$

$$([a, T]v)(x) = \int_{\mathbb{R}^N} K(x-y)(a(x)-a(y))v(y)\,dy, \quad x \in \mathbb{R}^N.$$

Then, chosen a cut-off function $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$, such that $\eta(x) = 0$ if $|x| < 1/2$ and $\eta(x) = 1$ if $|x| > 1$,

$$[a, T]v = \lim_{\epsilon \to 0} R_\epsilon v \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^N),$$

where

$$(R_\epsilon v)(x) := \int_{\mathbb{R}^N} K(x-y)\eta\Big(\frac{x-y}{\epsilon}\Big)(a(x)-a(y))v(y)\,dy, \quad x \in \mathbb{R}^N.$$

As a consequence

$$D[a, T]v = \lim_{\epsilon \to 0} DR_\epsilon v \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^N).$$

By computing the gradient of $R_\epsilon v$ we obtain

$$\begin{aligned}(DR_\epsilon v)(x) = &\int_{\mathbb{R}^N} DK(x-y)\eta\Big(\frac{x-y}{\epsilon}\Big)(a(x)-a(y))v(y)\,dy+\\ &+\frac{1}{\epsilon}\int_{\mathbb{R}^N} K(x-y)(D\eta)\Big(\frac{x-y}{\epsilon}\Big)(a(x)-a(y))v(y)\,dy+\\ &+(Da)(x)\int_{\mathbb{R}^N} K(x-y)\eta\Big(\frac{x-y}{\epsilon}\Big)v(y)\,dy=\\ &=(R_\epsilon^{(1)}v + R_\epsilon^{(2)}v + R_\epsilon^{(3)}v)(x).\end{aligned}$$

Let $\eta_\epsilon(\cdot) = \eta\Big(\frac{\cdot}{\epsilon}\Big)$. From Calderon-Zygmund theory it follows that, as $\epsilon \to 0$, $K\eta_\epsilon * v$ converges in $L^p(\mathbb{R}^N)$ to a function w such that

$$||w||_p \le c||v||_p \tag{13}$$

where c is a positive constant only dependent on K and p. Then, as $\epsilon \to 0$, $R_\epsilon^{(3)}v$ converges in $\mathcal{S}'(\mathbb{R}^N)$ to

$$R^{(3)}v := Da \cdot w.$$

Moreover, thanks to (11) and (13),

$$||R^{(3)}v||_r \leq c||Da||_{\frac{Nq}{N-q}}||v||_p \tag{14}$$

(by the classical Sobolev's Theorem)

$$\leq c||D^2a||_q||v||_p.$$

Here and in the sequel we always denote by c a positive constant independent on a and v.

We now study the behaviour of $R_\epsilon^{(2)}v$. A change of variable gives

$$R_\epsilon^{(2)}v(x) = \epsilon^{N-1}\int_{\mathbb{R}^N} K(\epsilon z)(D\eta)(z)(a(x) - a(x - \epsilon z))v(x - \epsilon z)\, dz.$$

As ϵ goes to zero the function $(x, z) \longmapsto \epsilon^{-1}(a(x) - a(x - \epsilon z))$ converges to $(x, z) \longmapsto \langle Da(x), z\rangle$ uniformly on every compact subset of $\mathbb{R}^N \times \mathbb{R}^N$. Moreover, by estimate (10),

$$|(DK)(\epsilon z)| \leq \frac{c}{|\epsilon z|^{N+1}},$$

and

$$|\epsilon^N D(K(\epsilon z))| = \epsilon^{N+1}|(DK)(\epsilon z)| \leq \frac{c_1}{|z|^{N+1}},$$

where c and c_1 are independent on ϵ. Then, on every fixed compact $C \subseteq \mathbb{R}^N \setminus \{0\}$ $\{z \mapsto \epsilon^N K(\epsilon z),\ \epsilon > 0\}$ is a family of equibounded and equicontinuous functions. As a consequence, taking a sequence $\epsilon_j \downarrow 0$ if necessary, $K_\epsilon(z) := \epsilon^N K(\epsilon z)$ pointwise converges in $\mathbb{R}^N \setminus \{0\}$ to a continuous function K_0. Moreover, the convergence of K_ϵ to K_0 is uniform on every compact subset of $\mathbb{R}^N \setminus \{0\}$. Then, since $D\eta$ has compact support and $D\eta(z) = 0$ for $|z| < 1/2$,

$$\lim_{\epsilon\to 0} R_\epsilon^{(2)}v(x) = \Big(\int_{\mathbb{R}^N} K_0(z)D\eta(z)\langle Da(x), z\rangle\, dz\Big)v(x) =: R^{(2)}v(x)$$

uniformly on every compact subset of $\mathbb{R}^N$. Since

$$|R^{(2)}v| \leq c|Da||v|,$$

we get

$$||R^{(2)}v||_r \leq c||Da||_{\frac{Nq}{N-q}}||v||_p \leq c||D^2a||_q||v||_p. \tag{15}$$

We next estimate $R_\epsilon^{(1)}$, the crucial part of DR_ϵ. First of all, we use the Taylor's formula to write

$$a(x)-a(y)=\langle Da(y),x-y\rangle+\int_0^1(1-t)\langle D^2a(y+t(x-y))(x-y),x-y\rangle\,dt.$$

Then, denoting by ∂_j and ∂_{ij} the derivatives $\frac{\partial}{\partial x_j}$ and $\frac{\partial^2}{\partial x_i\partial x_j}$, respectively, we have

$$\text{(16)}\qquad R_\epsilon^{(1)}v(x)=\int_{\mathbb{R}^N}DK(x-y)\langle Da(y),x-y\rangle\eta\Big(\frac{x-y}{\epsilon}\Big)v(y)\,dy+$$

$$+\int_{\mathbb{R}^N}DK(x-y)\eta\Big(\frac{x-y}{\epsilon}\Big)v(y)\Big(\int_0^1(1-t)\langle D^2a(y)(x-y),x-y\rangle dt\Big)\,dy=$$

$$=\sum_{j=1}^N\int_{\mathbb{R}^N}DK(x-y)(x_j-y_j)\eta\Big(\frac{x-y}{\epsilon}\Big)\partial_ja(y)v(y)\,dy+$$

$$+\sum_{i,j=1}^N\int_0^1(1-t)\Big(\int_{\mathbb{R}^N}DK(x-y)(x_i-y_i)(x_j-y_j)\cdot$$

$$\cdot\,\eta\Big(\frac{x-y}{\epsilon}\Big)\partial_{ij}a(y+t(x-y))v(y)\,dy\Big)dt.$$

The kernels

$$K_{h,j}(z):=\partial_hK(z)z_j\,,\quad h,j=1,\dots,N,$$

are Calderon-Zygmund kernels since

$$K_{h,j}=\mathcal{F}^{-1}(m_{h,j}),$$

being

$$m_{h,j}(\xi):=\frac{\partial}{\partial\xi_j}(\xi_hm(\xi))$$

C^∞ functions satisfying the condition

$$|D^\alpha m_{h,j}(\xi)|\le c_\alpha|\xi|^{-\alpha}$$

for any multi-index α. Then, as $\epsilon\to 0$, the first summation to the right hand side of (16) converges to

$$\sum_{j=1}^N R^{(1,j)}(\partial_jav),$$

where $R^{(1,j)}$ is a linear continuous operator in $L^r(\mathbb{R}^N)$. As a consequence

$$(17)\qquad ||\sum_{j=1}^{N} R^{(1,j)}(\partial_j a v)||_r \leq c||(Da)v||_r \leq c||Da||_{\frac{Nq}{N-q}}||v||_p \leq$$

$$\leq c||D^2 a||_q||v||_p.$$

It remains to estimate the integrals in the second summation in the right hand side of (16). The kernels

$$K_{i,h,j}(z) := \partial_h K(z) z_i z_j$$

appearing in those integrals satisfy the estimates

$$|K_{h,i,j}(z)| \leq c|z|^{-N+1}.$$

Indeed, keeping in mind that $|K_{h,j}(z)| \leq c|z|^{-N}$ with $K_{h,j}$ a Calderon-Zigmund kernel,

$$|K_{h,i,j}(z)| = |z_i||K_{h,j}(z)| \leq c|z|^{-N+1}.$$

The second summation to the right hand side of (16) can be then estimated from above by a positive constant, independent on ϵ, times the function

$$w(x) = \int_0^1 (1-t) \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} |v(y)||D^2 a(y + t(x-y))|\, dy dt.$$

Let us define

$$s = 1 + \frac{p}{q}, \quad s' = 1 + \frac{q}{p}.$$

Since

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{1+p/q} + \frac{1}{1+q/p} = \frac{q}{p+q} + \frac{p}{p+q} = 1,$$

we have

$$w(x) \leq \int_0^1 (1-t) \Big(\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} |v(y)|^s\, dy \Big)^{1/s} \cdot$$

$$\cdot \Big(\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} |D^2 a(y + t(x-y))|^{s'}\, dy \Big)^{1/s'} dt$$

(putting $y + t(x-y) = z$ in the last integral)

$$= \int_0^1 (1-t)\Big(T(|v|^s)(x)\Big)^{1/s}\Big(T(|D^2a|^{s'})(x)\Big)^{1/s'}(1-t)^{-N/s'+(N-1)/s'}dt =$$

$$= \int_0^1 (1-t)^{1-1/s'}dt\Big(T(|v|^s)(x)\Big)^{1/s}\Big(T(|D^2a|^{s'})(x)\Big)^{1/s'},$$

where

$$Tf(x) = \int_{\mathbb{R}^N} |x-y|^{-N+1} f(y)\,dy.$$

Reminding that T is a continuous linear operator from $L^m(\mathbb{R}^N)$ to $L^{m*}(\mathbb{R}^N)$ with $\frac{1}{m*} = \frac{1}{m} - \frac{1}{N}$ for any $1 < m < N$, we have

$$||T(|v|^s)||_r \le c\Big(\int |v(x)|^{sp/s}dx\Big)^{s/p} = c||v||_p^s,$$

since

$$\frac{1}{r} = \frac{s}{p} - \frac{1}{N}.$$

Analogously

$$||T(|D^2a|^{s'})||_r \le c\Big(\int_{\mathbb{R}^N} |D^2a(x)|^{s'q/s'}dx\Big)^{s'/q} = c||D^2a||_q^{s'}$$

since

$$\frac{1}{r} = \frac{s'}{q} - \frac{1}{N}.$$

As a consequence, since

$$||w||_r^r \le c\int_{\mathbb{R}^N} (T(|v|^s))^{r/s}(T(|D^2a|^{s'}))^{r/s'}dx \le$$

$$\le c||T(|v|^s)||_r^{r/s}||T(|D^2a|^{s'})||_r^{r/s'},$$

we finally obtain

$$||w||_r \le c||v||_p||D^2a||_q.$$

This ends the proof of our Theorem.

Corollary 2.1. *Suppose the hypotheses of Theorem 2.1 are satisfied. Then, for any* $j \in \{1, \ldots, N\}$,

$$||[a\partial_j, T]v||_r \le c||a||_{2,q}||v||_p.$$

Proof. We first note that

$$
\begin{aligned}
[a\partial_j, T]v = a\partial_j(Tv) - T(a\partial_j v) = \\
&= \partial_j(aTv) - (\partial_j a)Tv - T(\partial_j(av)) + T((\partial_j a)v)
\end{aligned}
$$

(since $[T, \partial_j] = 0$)

$$
= \partial_j\Big((aTv) - T(av)\Big) + [T, \partial_j a]v = \partial_j[a, T]v + [T, \partial_j a]v.
$$

From Theorem 2.1 we get

$$
||\partial_j[a, T]v||_r \le C||D^2 a||_q||v||_p.
$$

On the other hand, since T is a Calderon-Zigmund operator,

$$
\begin{aligned}
||[T, \partial_j a]v||_r &\le ||T((\partial_j a)v)||_r + ||(\partial_j a)Tv||_r \\
&\le C||(\partial_j a)v||_r + ||\partial_j a||_{\frac{Nq}{N-q}}||Tv||_p \\
&\le C||\partial_j a||_{\frac{Nq}{N-q}}||v||_p \le C||a||_{2,q}||v||_p.
\end{aligned}
$$

The thesis follows.

3. Embedding theorem.

In this section we prove our main result.

Proof of Theorem 1.1. We first remark that the $W^{1/2,2}$ norm of a real function $u \in C_0^\infty(\mathbb{R}^N)$ is equivalent to

$$
||u||_2 + \sum_{j=1}^{N} ||\mathcal{F}^{-1}(J\mathcal{F}(\partial_j u))||_2
$$

where

$$
J(\xi) := (1 + |\xi|^2)^{-1/4}, \quad \xi \in \mathbb{R}^N.
$$

On the other hand, defining

$$
m_j(\xi) := \xi_j J^2(\xi) = \frac{\xi_j}{(1 + |\xi|^2)^{1/2}}, \tag{18}
$$

by Parseval's identity we have

$$
\begin{aligned}
(19)\qquad ||\mathcal{F}^{-1}(J\mathcal{F}(\partial_j u))||_2^2 &= \Big(\frac{1}{2\pi}\Big)^N \int_{\mathbb{R}^N} J^2 \overline{\mathcal{F}(\partial_j u)}\mathcal{F}(\partial_j u) d\xi = \\
&= \Big(\frac{1}{2\pi}\Big)^N \int_{\mathbb{R}^N} -i\, m_j \overline{\mathcal{F}(u)}\mathcal{F}(\partial_j u) d\xi = \\
&= \Big(\frac{1}{2\pi}\Big)^N \int_{\mathbb{R}^N} \mathcal{F}\Big(-i\, m_j \overline{\mathcal{F}(u)}\Big)\partial_j u d\xi = \\
&= \int_{\mathbb{R}^N} \overline{\mathcal{F}^{-1}\Big(i\, m_j \mathcal{F}(u)\Big)}\partial_j u d\xi = \\
&= \int_{\mathbb{R}^N} (T_j u)\partial_j u d\xi.
\end{aligned}
$$

In the last term we have set

$$
T_j u := \mathcal{F}^{-1}\Big(i\, m_j \mathcal{F}(u)\Big).
$$

By using identity (2) in the last term of (19) we obtain

$$
\begin{aligned}
(20)\qquad \int_{\mathbb{R}^N} (T_j u)\partial_j u &= \sum_{k=1}^{p}\Big(\int_{\mathbb{R}^N}(T_j u) b_j^{(k)} X_k u\Big) + \\
&+ \sum_{1\le k<h\le p}\Big(\int_{\mathbb{R}^N}(T_j u)\alpha_j^{(k,h)}(X_k X_h - X_h X_k)u\Big)
\end{aligned}
$$

(integrating by parts the second term in the right hand side)

$$
\begin{aligned}
= \sum_{k=1}^{p}\Big(\int_{\mathbb{R}^N}(T_j u) b_j^{(k)} X_k u\Big) + \sum_{1\le k<h\le p}\Big(\int_{\mathbb{R}^N} X_k^*((T_j u)\alpha_j^{(k,h)})X_h u - \\
- \int_{\mathbb{R}^N} X_h^*((T_j u)\alpha_j^{(k,h)})X_k u\Big).
\end{aligned}
$$

The function m_j in (18) satisfies condition (9) so that T_j is a Calderon Zygmund operator. As a consequence

$$
\begin{aligned}
(21)\qquad \Big|\int_{\mathbb{R}^N}(T_j u) b_j^{(k)} X_k u\Big| &\le ||T_j u||_2 ||b_j^{(k)}||_\infty ||X_k u||_2 \le \\
&\le C||u||_2 ||b_j^{(k)}||_\infty ||X_j u||_2.
\end{aligned}
$$

Corollary 2.1 will enable us to estimate

$$\int_{\mathbb{R}^N} X_k^*(\alpha_j^{(k,h)} T_j u) X_h u.$$

We first note that

$$X_k^*(\alpha_j^{(k,h)} T_j u) = X_k^*(\alpha_j^{(k,h)}) T_j u - \alpha_j^{(k,h)} X_k(T_j u) =$$
$$= X_k^*(\alpha_j^{(k,h)}) T_j u - \alpha_j^{(k,h)} T_j(X_k u) - \alpha_j^{(k,h)} [X_k, T_j] u =$$

(by using (1) in the last term)

$$= X_k^*(\alpha_j^{(k,h)}) T_j u - \alpha_j^{(k,h)} T_j(X_k u) - \alpha_j^{(k,h)} \sum_{i=1}^{N} [a_k^{(i)} \partial_i, T_j] u.$$

Then

$$\Big| \int_{\mathbb{R}^N} X_k^*(\alpha_j^{(k,h)} T_j u) X_h u \Big| \le ||X_k^*(\alpha_j^{(k,h)})||_{2N} ||T_j u||_{\frac{2N}{N-1}} ||X_h u||_2 + \tag{22}$$

$$+ ||\alpha_j^{(k,h)}||_\infty ||T_j(X_k u)||_2 ||X_h u||_2 + ||\alpha_j^{(k,h)}||_\infty \Big(\sum_{i=1}^{N} ||[a_k^{(i)} \partial_i, T_j] u||_2 \Big) ||X_h u||_2.$$

On the other hand by (9)

$$||X_k^*(\alpha_j^{(k,h)})||_{2N} \le ||X_k(\alpha_j^{(k,h)})||_{2N} + \sum_{i=1}^{N} ||\alpha_j^{(k,h)} \partial_i (a_k^{(i)})||_{2N} \le$$
$$\le ||X_k(\alpha_j^{(k,h)})||_{2N} + ||\alpha_j^{(k,h)}||_\infty \Big(\sum_{i=1}^{N} ||\partial_i (a_k^{(j)})||_{2N} \Big).$$

By the classical embedding $W^{2,2N/3}(\mathbb{R}^N) \subseteq W^{1,2N}(\mathbb{R}^N)$ we get the following estimate:

$$||X_k^*(\alpha_j^{(k,h)})||_{2N} \le C(M), \quad 1 \le k < h \le p,\ j = 1, \dots, N. \tag{23}$$

Analogously

$$||X_h^*(\alpha_j^{(k,h)})||_{2N} \le C(M), \quad 1 \le k < h \le p,\ j = 1, \dots, N. \tag{24}$$

Since T_j is a Calderon-Zigmund operator we also have

$$(25) \qquad ||T_j u||_{\frac{2N}{N-1}} \leq C||u||_{\frac{2N}{N-1}}, \quad ||T_j(X_k u)||_2 \leq C||X_k u||_2.$$

Finally, by Corollary 2.1 with $p = \frac{2N}{N-1}$, $q = \frac{2N}{3}$, $r = 2$, we have

$$(26) \qquad ||[a_k^{(i)}\partial_i, T_j]u||_2 \leq C||a_k^{(i)}||_{2,2N/3}||u||_{\frac{2N}{N-1}}.$$

Collecting (19)–(26) we obtain

$$(27) \qquad ||u||^2_{1/2,2} \leq C(M)\Big(||u||_2 + ||u||_{\frac{2N}{N-1}}\Big)\sum_{k=1}^{p}||X_k u||_2 + \\ + C(M)\Big(\sum_{k=1}^{p}||X_k u||_2\Big)^2 + ||u||_2^2.$$

Keeping in mind the classical Sobolev embedding $W^{1/2,2}(\mathbb{R}^N) \subseteq L^{\frac{2N}{N-1}}(\mathbb{R}^N)$ and the obvious estimate $||u||_2 \leq ||u||_{1/2,2}$, from (27) we obtain

$$||u||^2_{1/2,2} \leq C(M)||u||_{1/2,2}\sum_{k=1}^{p}||X_k u||_2 + C(M)\Big(\sum_{k=1}^{p}||X_k u||_2\Big)^2 + ||u||_2^2 \leq \\ \leq \frac{1}{2}||u||^2_{1/2,2} + (2C^2(M) + C(M))\Big(\sum_{k=1}^{p}||X_k u||_2\Big)^2 + ||u||_2^2.$$

From this estimate the inequality (4) immediately follows.

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