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# ELLIPTIC EQUATIONS AND BMO-FUNCTIONS

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*Dedicated to the memory of Filippo Chiarenza*

## 1. Introduction.

The main object of the present work is the linear operator

$$\mathcal{A}u = \operatorname{div}(A(x)\nabla u) \tag{1.1}$$

where $A : \Omega \to \mathbb{R}^{n\times n}$ is a measurable function in open set $\Omega \subset \mathbb{R}^n$ with values in the space of all $n \times n$ symmetric matrices, satisfying the usual ellipticity condition at almost every point of the domain $\Omega$

$$\frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2. \tag{1.2}$$

In Section 2 we treat the case

$$K(x) \equiv K$$

and

$$A(x) \in VMO(\Omega; \mathbb{R}^{n\times n}) \tag{1.3}$$

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and report briefly on some results obtained in [12].
In Section 3 we assume only

$$K(x) \in BMO(\Omega) \tag{1.4}$$

and we describe some recent results contained in joint papers with T. Iwaniec (see [13], [14]).
Recall that (1.4) means

$$||K||_{BMO} = \sup\{ \fint_Q |K(x) - K_Q| dx,\ Q \text{ cube},\ Q \subset \Omega\} < \infty$$

and that $VMO$ is the closure of $C_0^\infty$ in the BMO norm.

## 2. The coefficient matrix in VMO.

The domain of the operator $\mathcal{A}$ will be the Sobolev space $W_0^{1,p}(\Omega)$, $1 < p < \infty$, which is the completion of $C_0^\infty$ with respect to the norm

$$|u|_p = ||\nabla u||_{L^p}.$$

For some unbounded regions, such as $\Omega = \mathbb{R}^n$, functions from $W_0^{1,p}(\Omega)$ which differ by a constant are indistinguishable. We use the notation $\mathcal{W}^{1,p}(\mathbb{R}^n)$ for the space $W_0^{1,p}(\mathbb{R}^n)$.
In the present section we assume that the operator $\mathcal{A}$ is uniformly elliptic, that is

$$\frac{|\xi|^2}{K} \le \langle A(x)\xi, \xi\rangle \le K|\xi|^2 \tag{2.1}$$

for a certain constant $K \ge 1$, for almost every $x \in \Omega$ and all vectors $\xi \in \mathbb{R}^n$.
Thus

$$\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \tag{2.2}$$

and the problem whether or not this operator has an inverse is of interest for us.
In other words, given $f \in L^p(\Omega; \mathbb{R}^n)$, does the differential equation

$$\operatorname{div}(A(x)\nabla u) = \operatorname{div} f \tag{2.3}$$

have a unique solution $u \in W_0^{1,p}(\Omega)$?

This is obviously true for $p = 2$ with uniform bound

$$|u|_2 \leq K||f||_{L^2}.$$

When $p$ is "close" to 2, the same is true if we require some regularity on the domain $\Omega$. According to Meyers [16], there is infact an $\varepsilon = \varepsilon(n, K) \in (0, 1]$ such that

$$|u|_p \leq c_p(n, K, \Omega)||f||_{L^p} \tag{2.4}$$

for $2-\varepsilon \leq p \leq 2+\varepsilon$. Similar results concerning nonlinear variational equations such as

$$\mathrm{div} A(x, \nabla u) = \mathrm{div} F \tag{2.5}$$

where $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ verifies the Leray-Lions usual assumptions, have been recently establish in [11].

We want to emphasize that, without further assumptions on the coefficients, inequality (2.4) fails if $p$ is too far from 2 (see [12] e.g.).

As far as we are aware the fact that continuity of $A(x)$ is sufficient to obtain (2.4) for any $1 < p < \infty$ goes back to Simader [19].

Let us now recall that $VMO$ is the space of functions with vanishing mean oscillation introduced by Sarason. The idea to relax continuity of $A$ into the assumption $A \in VMO(\Omega; \mathbb{R}^{n\times n})$ is due to [6] and can be traced back to the important work of Filippo with M. Frasca and P. Longo [4], [5] dedicated to $W_0^{2,p}$ estimates for nondivergence elliptic equations with $VMO$ coefficients.

In [12] similar results based on estimates for the Riesz transforms, were established for the equation (2.3) in $\mathbb{R}^n$. Other related papers are [7], [8], [9], [15], [1], [2].

Let us mention a recent result concerning nonlinear equations in divergence form

$$\mathrm{div}\left(\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{p}} A(x)\nabla u\right) = 0 \tag{2.6}$$

where $A(x)$ is a symmetric matrix verifying (2.2), due to Greco-Verde [10].

**Theorem 2.1.** *Assume $A \in VMO(\Omega; \mathbb{R}^{n\times n})$. For any $1 < r_1 < 2$ there exists $\delta > 0$ such that, if $|p - 2| < \delta$ and $u \in W^{1,r_1}_{\mathrm{loc}}(\Omega)$ satisfies (2.6), then $u \in W^{1,r_2}_{\mathrm{loc}}(\Omega)$ for all exponents $1 < r_2 < \infty$.*

## 3. The coefficient matrix with upper bound in BMO.

In the following we wish to illustrate some recent theorems obtained with T. Iwaniec for equation (2.3) under the non-uniform ellipticity condition

$$\frac{|\xi|^2}{K(x)} \leq \langle A(x)\xi, \xi\rangle \leq K(x)|\xi|^2. \tag{3.1}$$

The point is that the function $K(x) \geq 1$ need not to be bounded. Our basic assumption will be

$$||K||_{BMO} \leq \lambda(n) \tag{3.2}$$

for $\lambda(n)$ sufficiently small.
In particular, since the BMO-norm of a constant function $K(x) \equiv K$ is zero, we are treating a natural extension of the classical case. One of the central results is the "higher" integrability of gradients.
We cannot expect the same sort of results as for the classical theory $K(x) \equiv K$, i.e. $\nabla u \in L^{2+\varepsilon}_{\text{loc}}$. We must content ourselves with only a very slight degree of improved integrability. We have the following

**Theorem 3.1.** *If* (3.1), (3.2) *hold with* $\lambda(n)$ *sufficiently small and* $u \in W^{1,1}_{\text{loc}}(\Omega)$ *is a solution to the equation*

$$\text{div}(A(x)\nabla u) = 0,$$

*if the gradient* $\nabla u$ *is of class* $L^2 \log^{-1} L$, *then it belongs to* $L^2 \log L$ *at least locally.*

It is worthwile noting that we are also relaxing the usual requirement for the ratio of the upper and the lower bounds at (3.1) to be uniformly bounded (quasi-isotropic case). Thus, the right spaces in genuine nonisotropic situation are the Orlicz-Sobolev classes $u \in W^{1,1}_{\text{loc}}$ such that $\nabla u \in L^2 \log^{\alpha} L_{\text{loc}}$. Examples show that one cannot go far beyond these classes and in particular that $\nabla u \notin L^{2+\varepsilon}_{\text{loc}}$ for any $\varepsilon > 0$ even though the $BMO$-norm of $K(x)$ can be chosen arbitrarily small.

For our purpose it will be useful to review the $L^p$ theory of Hodge decomposition in $\mathbb{R}^n$. In this case explicit calculations are possible by means of the Riesz transform

$$\boldsymbol{R} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \mathbb{R}^n), \qquad 1 < p < \infty, \tag{3.3}$$

where

$$\boldsymbol{R}f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{(x-y)f(y)}{|x-y|^{n+1}} dy.$$

Given a vector field $F = (f^1, ..., f^n) \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, we first solve the Poisson equation

$$\Delta U = (\Delta u^1, ..., \Delta u^n) = F \tag{3.4}$$

for $U = (u^1, ..., u^n) \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^n)$, which yields the following decomposition of $F$

$$F = B + E, \qquad \mathrm{div} B = 0 \quad \text{and} \quad \mathrm{curl} E = 0$$

where

$$B = \Delta U - \nabla \mathrm{div} U \qquad \text{and} \quad E = \nabla \mathrm{div} U.$$

Then we define a n-dimensional version $\boldsymbol{S}$ of the Hilbert transform by

$$\boldsymbol{S}(F) = E - B.$$

Thus $\boldsymbol{S}$ acts as identity on gradient fields and as minus identity on divergence free vector fields. In terms of the projection operators one can write

$$-\boldsymbol{S} = B - E = I + 2\boldsymbol{R} \otimes \boldsymbol{R}.$$

Let us list basic properties of the operator $\boldsymbol{S}$:

i) $\boldsymbol{S}$ is an involution, that is $\boldsymbol{S} \circ \boldsymbol{S} = I$

ii) $\boldsymbol{S}$ is self adjoint, that is

$$\int_{\mathbb{R}^n} \langle \boldsymbol{S}F, G \rangle = \int_{\mathbb{R}^n} \langle F, \boldsymbol{S}G \rangle$$

for $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ and $G \in L^q(\mathbb{R}^n; \mathbb{R}^n)$ with $1 < p, q < \infty$, $p + q = pq$. Thus, in particular

iii) $\boldsymbol{S}$ is an isometry in $L^2(\mathbb{R}^n; \mathbb{R}^n)$.

By a duality argument it can be proved that $||\boldsymbol{S}||_q = ||\boldsymbol{S}||_p$.

A device for the integral estimates for the solutions of PDE's is the Beltrami operator

$$I - \mu \boldsymbol{S} : L^\phi(\mathbb{R}^n; \mathbb{R}^n) \to L^\phi(\mathbb{R}^n; \mathbb{R}^n)$$

where $\Phi$ is a suitable Orlicz function and the matrix $\mu$ satisfies

$$|\mu(x)| \leq \frac{K(x)-1}{K(x)+1}.$$

In fact, one way to express the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x)\nabla u = \operatorname{div} A(x) f \\ u \in W_0^{1,1}(\Omega) \end{cases} \tag{3.5}$$

where $A(x)$ is a symmetric matrix with measurable coefficients satisfying the ellipticity bounds

$$\frac{|\xi|^2}{K(x)} \leq \langle A(x)\xi, \xi\rangle \leq K(x)|\xi|^2 \tag{3.6}$$

under the assumption (3.2), is the following.
By making the positions:

$$E = \nabla u,$$

$$B = A(x)(\nabla u - f), \qquad \operatorname{div} B = 0,$$

$$F^- = \frac{E-B}{2}, \qquad F^+ = \frac{E+B}{2}$$

we see that the equation (3.5) can be rewritten as

$$F^- = \mu(x)F^+ + g$$

where

$$\mu(x) = \frac{I - A(x)}{I + A(x)},$$

$$g(x) = \left[\frac{-A(x)}{I + A(x)}\right] f(x), \qquad |g| \leq |f|$$

and the solvability of (3.5) relies upon the invertibility of

$$I - \mu \boldsymbol{S}.$$

It is obvious that $I - \mu \boldsymbol{S}$ is invertible in all Lebesgue spaces $L^r(\mathbb{R}^n;\mathbb{R}^n)$ for which $||\mu||_\infty ||\boldsymbol{S}||_r < 1$. Less obvious is the following (see [14])

**Theorem 3.2.** *If* (3.2) *holds, then for* $\alpha \in \{-1, 0, 1\}$ *there exists a bounded linear operator*

$$\Pi : L^2 \log^\alpha L(\mathbb{R}^n, d\omega) \to L^2 \log^\alpha L(\mathbb{R}^n, dx)$$

*such that*

$$\Pi \circ (I - \mu \boldsymbol{S}) = (I - \mu \boldsymbol{S}) \circ \Pi = I$$

*where* $d\omega = K^2(x)dx$.

*Proof*. The proof requires some steps.
*First Step*. The Hilbert transform

$$\boldsymbol{S} : L^2(\mathbb{R}^n, d\omega) \to L^2(\mathbb{R}^n, d\omega)$$

is bounded:

$$\begin{aligned} ||\boldsymbol{S}F||_{L^2(\mathbb{R}^n,d\omega)} &= ||K\boldsymbol{S}F||_2 \le ||\boldsymbol{S}(KF)||_2 + ||(K\boldsymbol{S} - \boldsymbol{S}K)F||_2 \le \\ &\le ||KF||_2 + c(n)||K||_{BMO}||F||_2 \le c(n)||F||_{L^2(\mathbb{R}^n,d\omega)}. \end{aligned}$$

Here we have used a well known estimate for the commutator $K\boldsymbol{S} - \boldsymbol{S}K$ due to Coifman, Rochberg and Weiss.

*Second Step*. We obtain the estimate:

$$||\Pi g||_{L^2(\mathbb{R}^n,dx)} \le 2||g||_{L^2(\mathbb{R}^n,d\omega)}. \tag{3.7}$$

The proof is based on the following pointwise inequality

$$\frac{|\boldsymbol{S}F|^2 + |F|^2}{1 + \varepsilon K} \le \frac{2K}{1 + \varepsilon K}(|\boldsymbol{S}F|^2 - |F|^2) + 4K^2|(I - \mu \boldsymbol{S})F|^2 \tag{3.8}$$

which can be proved rather easily (see [14]).
Note that $k = \frac{K}{1+\varepsilon K}$ is bounded and its BMO-norm does not depend on $\varepsilon$:

$$||k||_{BMO} \le 2||K||_{BMO} \le 2\lambda(n).$$

The existence of the operator $\Pi$ follows from the estimate

$$||F||_{L^2(\mathbb{R}^n,dx)} \le 2||(I - \mu \boldsymbol{S})F||_{L^2(\mathbb{R}^n,d\omega)} \tag{3.9}$$

for $F \in L^2(\mathbb{R}^n, dx)$, which derives by integrating (3.8). Namely if we introduce $\boldsymbol{S}F = \boldsymbol{S}(E - B) = E + B$ and note that $|\boldsymbol{S}F|^2 - |F|^2 = 4\langle B, E\rangle$, we have

$$\int_{\mathbb{R}^n} \frac{|E|^2 + |B|^2}{1 + \varepsilon K} \le 4 \int_{\mathbb{R}^n} \langle kB, E\rangle + 2||F - \mu \boldsymbol{S}F||^2_{L^2(\mathbb{R}^n,d\omega)} \tag{3.10}$$

we then apply Hodge decomposition of the vector field $kB \in L^2(\mathbb{R}^n, dx)$

$$kB = B' + E'$$

and so, by Hölder's inequality,

$$\left|\int_{\mathbb{R}^n} \langle kB, E\rangle\right| = \left|\int_{\mathbb{R}^n} \langle E', E\rangle\right| \leq ||E'||_2 ||E||_2.$$

Using again Coifman-Rochberg-Weiss theorem we obtain

$$||E'||_2 \leq c(n)||k||_{BMO}||B||_2 \leq \lambda(n)c(n)||B||_2$$

and therefore (3.10) implies

$$\int_{\mathbb{R}^n} \frac{|E|^2 + |B|^2}{1+\varepsilon K} \leq 4\lambda(n)c(n)\int_{\mathbb{R}^n}(|E|^2 + |B|^2) + 2||F - \mu \boldsymbol{S} F||^2_{L^2(\mathbb{R}^n, d\omega)}.$$

By monotone convergence theorem and choosing $\varepsilon$ small enough we deduce

$$\frac{1}{2}\int_{\mathbb{R}^n} |F|^2 \leq 2||F - \mu \boldsymbol{S} F||^2_{L^2(\mathbb{R}^n, d\omega)}$$

and the second step is achieved, after an approximation argument of $\mu$ based on the sequence of bounded matrices

$$\mu_h(x) = \begin{cases} \mu(x) & \text{if } |\mu(x)| \leq 1 - \frac{1}{h} \\ \dfrac{(h-1)\mu(x)}{h|\mu(x)|} & \text{otherwise} \end{cases}$$

which satisfies

$$|\mu_h(x)| \leq \frac{K(x) - 1}{K(x) + 1}$$

and the operator $I - \mu_h \boldsymbol{S}$ is invertible in $L^2(\mathbb{R}^n, dx)$.

*Third Step*. The operator $\Pi$, originally defined in $L^2(\mathbb{R}^n, d\omega)$ with values in $L^2(\mathbb{R}^n, dx)$ extends to a continuous operator

$$\Pi : L^2 \log L(\mathbb{R}^n, d\omega) \to L^2 \log L(\mathbb{R}^n, dx).$$

A crucial role is played by inequality

$$\left|\int_{\mathbb{R}^n} k\langle B, E\rangle\right| \leq c(n)||k||_{BMO}||B||_{L^2 \log L}||E||_{L^2 \log^{-1} L} \tag{3.11}$$

for $\operatorname{div} B = 0$ and $\operatorname{curl} E = 0$, which can be proved much the same way as the corresponding inequality in the second step, plus a suitable tool for establishing boundedness of singular integrals, maximal operators and some commutators in $L^p \log^\alpha L$ spaces, see [11]–[14].
Another useful fact is that the norms

$$||g||_{L^2 \log L(\mathbb{R}^n, d\omega)} \,, \; ||Kg||_{L^2 \log L(\mathbb{R}^n, dx)}$$

are comparable.
The above mentioned extension of the operator $\Pi$ is established by mean of the inequality

$$||F||_{L^2 \log L(\mathbb{R}^n, d\omega)} \leq c(n)||K(I - \mu S)F||_{L^2 \log L(\mathbb{R}^n, dx)} + c(n)[K]||F||_2$$

where

$$[K] = ||K_0||_\infty + \varepsilon(n) \int_{\mathbb{R}^n} \left[ e^{\frac{K-K_0}{\varepsilon(n)}} - 1 \right]$$

and $K_0 \in L^\infty(\mathbb{R}^n)$ is such that $1 \leq K_0(x) \leq K(x)$ and $e^{\frac{K-K_0}{\varepsilon(n)}} - 1 \in L^1(\mathbb{R}^n)$. But this is a rather technical step that we don't pursue here (see [14]).

*Fourth Step*. The operator $\Pi$ satisfies also

$$||\Pi g||_{L^2 \log^{-1} L(\mathbb{R}^n, dx)} \leq c(n)[K]||g||_{L^2 \log^{-1} L(\mathbb{R}^n, d\omega)}. \tag{3.12}$$

The verification of (3.12) is another technical piece of work that the interested reader will find in paper [14]. □

By the previous Theorem 3.2 it is possible to obtain local estimates from which Theorem 3.1 follows.

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