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WEIGHTED L^q ESTIMATES FOR DERIVATIVES OF HARMONIC FUNCTIONS

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Dedicated to the memory of Filippo Chiarenza

The purpose of this article is to summarize some of the results obtained jointly by J. Michael Wilson and the author in [12] and [13]. The main problem in both [12] and [13] is the study of weighted L^q norms of derivatives of harmonic functions in the upper Euclidean half space. In [13], such norms are estimated from above by weighted Hardy H^p norms of the (undifferentiated) harmonic function, while in [12] the harmonic function is assumed to be the Poisson integral of a function f on the boundary, and the L^q norms are instead estimated in terms of weighted L^p norms of f.

More precisely, the H^p problem consists of characterizing nonnegative measures μ on $\mathbb{R}^{d+1}_+$ and weights v on $\mathbb{R}^d$ so that the following inequality holds for a given pair p, q of indices, $0 < p, q < \infty$, and all harmonic functions $u(x, y)$ on $\mathbb{R}^{d+1}_+$:

$$(1) \qquad \left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(x, y)|^q d\mu(x, y)\right)^{1/q} \leq C||u||_{H^p(v,\mathbb{R}^d)},$$

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with a constant C that is independent of u. Here we use the notation $\mathbb{R}^d$ for d-dimensional Euclidean space and $\mathbb{R}^{d+1}_+$ for the upper $(d+1)$-dimensional Euclidean half-space $\{(x, y) : x \in \mathbb{R}^d, 0 < y < \infty\}$. Also, ∇ denotes the ordinary gradient in x and y, i.e.,

$$\nabla u = \langle \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial y} \rangle.$$

Finally, $H^p(v, \mathbb{R}^d)$ (or just $H^p(v)$ for simplicity) is the Hardy space associated with v, that is, if $N(u)(x)$ is the nontangential maximal function of u defined by

$$N(u)(x) = \sup_{(t,y)\in\mathbb{R}^{d+1}_+ : |t-x|<y} |u(t, y)|,$$

then by definition,

$$||u||_{H^p(v)} = \left(\int_{\mathbb{R}^d} N(u)(x)^p v(x)\, dx \right)^{1/p} \quad \left(= ||N(u)||_{L^p(v,\mathbb{R}^d)}\right).$$

The corresponding L^p problem is to determine all μ and v so that

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(x, y)|^q d\mu(x, y) \right)^{1/q} \leq C||f||_{L^p(v,\mathbb{R}^d)}, \tag{2}$$

where now u is the Poisson integral of $f(x)$, $x \in \mathbb{R}^d$, and

$$||f||_{L^p(v)} = \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx \right)^{1/p}.$$

For the L^p problem, we only consider the case $1 < p \leq q < \infty$.

In [13], we have obtained complete characterizations of μ, v for the H^p problem (1) for all $0 < p, q < \infty$, assuming that v belongs to the A_∞ class of C. Fefferman and B. Muckenhoupt. On the other hand, for the L^p problem (2), even assuming that v is an A_∞ weight, there is a gap between the necessary conditions and the sufficient conditions derived in [12]; this discrepancy has been reduced in the recent work [14] of Wilson, but it has not been eliminated. No complete characterization of μ, v for the L^p problem is presently known. Since the H^p and L^p results (and their proofs) are very different, they will be discussed separately below, beginning with the H^p results, which are much more complete. For weights v such that the spaces $H^p(v)$ and $L^p(v)$ coincide, the problems are of course the same; this occurs for example when $1 < p < \infty$ and v belongs to the class A_p of Muckenhoupt.

1. H^p results.

Only results for the H^p problems (1) are discussed in this section. In case $v \equiv 1$, the problem was solved earlier in a series of papers [9], [10] and [5], [6] dealing with various special ranges of p, q and d. These results are extended in [13] to the case when $v \in A_\infty$, i.e., in case there exist constants $C, \epsilon > 0$ such that whenever B is a ball in $\mathbb{R}^d$ and E is a (Lebesgue) measurable set with $E \subset B$, then

$$\frac{\int_E v(x)dx}{\int_B v(x)dx} \leq C\left(\frac{|E|}{|B|}\right)^\epsilon .$$

Here $|E|$ denotes the Lebesgue measure of E. The notation $v(E) = \int_E v(x)\,dx$ is used consistently below. The methods in [13] are similar to ones in the papers listed above, and in particular, to those in [6].

The simplest case occurs when $0 < p < q < \infty$. In this case, the A_∞ assumption on v can be weakened to the doubling condition

$$v(2B) \leq cv(B)$$

for all balls $B \subset \mathbb{R}^d$, with c independent of B, where $2B$ denotes the ball concentric with B whose radius is twice that of B. We say that such a weight v is doubling. In order to state the characterization of μ and v for (1) in this case, it is convenient to make several more definitions.

If $z = (x, y)$ is a point in $\mathbb{R}^{d+1}_+$, let

$$B(z) = \{t \in \mathbb{R}^d : |t - x| < y\};$$

thus $B(z)$ is the ball in $\mathbb{R}^d$ with center x and radius y. If $z = (x, y) \in \mathbb{R}^d$, let $z^* = (x, -y)$ denote the "complex conjugate" of z, and for $0 < \epsilon < 1$, let

$$D_\epsilon(z) = \{w \in \mathbb{R}^{d+1}_+ : \frac{|w - z|}{|w - z^*|} < \epsilon\}.$$

Then $D_\epsilon(z)$ is a subset of $\mathbb{R}^{d+1}_+$, and it is useful to think of $D_\epsilon(z)$ as the ϵ-ball centered at z with respect to the hyperbolic metric ρ defined by

$$\rho(w, z) = \frac{|w - z|}{|w - z^*|} \quad \left(= \frac{|z - w|}{|z - w^*|}\right), \; w, z \in \mathbb{R}^{d+1}_+;$$

$$D_\epsilon(z) = \{w : \rho(w, z) < \epsilon\}.$$

The characterizations involve $D_\epsilon(z)$ but turn out not to depend on any particular choice of ϵ, so that ϵ can be suppressed and the notation $D_\epsilon(z)$ can be replaced simply by $D(z)$. The absence of ϵ's in the statements of Theorem 1 and Theorem 2 below means that, if a sufficiency condition holds for *any* value of ϵ $(0 < \epsilon < 1)$, then the corresponding inequality (4) or (5) holds; and, conversely, if (4) or (5) holds, then the corresponding necessity condition holds for *all* ϵ $(0 < \epsilon < 1)$.

Theorem 1. *Let μ be a positive measure on $\mathbb{R}^{d+1}_+$ and v be a doubling weight on $\mathbb{R}^d$. Let $0 < p < q < \infty$. If there exists a constant $c > 0$ such that*

$$\mu(D(z)) \leq c\, y^q v(B(z))^{q/p} \tag{3}$$

for all $z = (x, y) \in \mathbb{R}^{d+1}_+$, then

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u|^q \, d\mu \right)^{1/q} \leq C\, ||u||_{H^p(v)} \tag{4}$$

for all harmonic functions u on $\mathbb{R}^{d+1}_+$, with C independent of u. Conversely, if (4) holds for all harmonic u, then (3) holds. In fact, if the analogue of (4) with ∇u replaced by any first partial derivative (in x or y) is valid, then (3) holds.

Versions of Theorem 1 (and also of Theorem 2 below) for derivatives of u of order higher than 1 are also derived in [13].

For the remaining ranges of p and q, the characterization involves weighted versions of the tent spaces introduced in [3], and we now define these spaces. When $v = 1$, they will agree with the ones used in [6]. If $t \in \mathbb{R}^d$ and $0 < \alpha < \infty$, let $\Gamma_\alpha(t)$ denote the usual nontangential cone in $\mathbb{R}^{d+1}_+$ with vertex t:

$$\Gamma_\alpha(t) = \{(x, y) \in \mathbb{R}^{d+1}_+ : |x - t| < \alpha y\}.$$

In case $\alpha = 1$, we write simply $\Gamma(t) = \Gamma_1(t)$. Now, if $0 < r \leq \infty$ and $f : \mathbb{R}^{d+1}_+ \to \mathbb{C}$ is a Borel measurable function, define

$$R_r f(t) = \begin{cases} \left(\displaystyle\int_{\Gamma(t)} |f(z)|^r \frac{dx\,dy}{y^{d+1}} \right)^{1/r} & \text{if } r < \infty; \\ ||f||_{L^\infty(\Gamma(t))} & \text{if } r = \infty. \end{cases}$$

If v is any nonnegative locally integrable function on $\mathbb{R}^d$ and if r, s satisfy $0 < r \leq \infty$ and $0 < s < \infty$, the tent space $T^s_r(v)$ is defined by saying that $f \in T^s_r(v)$ if $R_r f \in L^s(\mathbb{R}^d, v)$. Define

$$||f||_{T^s_r(v)} = ||R_r f||_{L^s(\mathbb{R}^d, v)} = \left(\int_{\mathbb{R}^d} |R_r f(t)|^s v(t)\, dt \right)^{1/s}.$$

It is also necessary to define the spaces $T^\infty_r(v)$ when $0 < r < \infty$. The pattern of the definition is different in this case. If Q is a cube in $\mathbb{R}^d$ with sidelength $\ell(Q)$, let $\hat{Q}$ be the Carleson cube in $\mathbb{R}^{d+1}_+$ defined by

$$\hat{Q} = \{(t, y) : t \in Q, 0 < y < \ell(Q)\}.$$

Let $v \in L^1_{loc}(\mathbb{R}^d)$, $v \geq 0$, and let $0 < r < \infty$. If $f : \mathbb{R}^{d+1}_+ \to \mathcal{C}$ is Borel measurable and $t \in \mathbb{R}^d$, set

$$C_{r,v}(f)(t) = \sup_{Q:t\in Q} \left(\frac{1}{v(Q)} \int_{\hat{Q}} |f(z)|^r \, v(B(z)) \frac{dx\,dy}{y^{d+1}} \right)^{1/r},$$

where the sup is taken over all cubes in $\mathbb{R}^d$ which contain t. We say that $f \in T^\infty_r(v)$ if $C_{r,v}(f) \in L^\infty(\mathbb{R}^d, v)$, and we set

$$||f||_{T^\infty_r(v)} = ||C_{r,v}(f)||_{L^\infty(\mathbb{R}^d,v)}.$$

The characterization of (1) for the remaining cases of p, q is stated in the next theorem. It depends on an auxiliary function $G_q(z)$ defined on $\mathbb{R}^{d+1}_+$ by

$$G_q(z) = y^{-q} \, \frac{\mu(D(z))}{v(B(z))}.$$

Theorem 2. *Let μ be a positive measure on $\mathbb{R}^{d+1}_+$ and v be an A_∞ weight on $\mathbb{R}^d$. Suppose that $0 < q \leq p < \infty$. In order that*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(z)|^q \, d\mu(z) \right)^{1/q} \leq C||u||_{H^p(v)}, \tag{5}$$

it is necessary and sufficient that:

1) $G_q \in T^\infty_{2/(2-q)}(v)$ *if* $0 < p = q < 2$;
2) $G_q \in T^{p/(p-q)}_{2/(2-q)}(v)$ *if* $0 < q < p$ *and* $q < 2$;
3) $G_q \in T^{p/(p-q)}_\infty(v)$ *if* $2 \leq q < p$;
4) G_q *is bounded on* $\mathbb{R}^{d+1}_+$ *if* $p = q \geq 2$.

Moreover, it is shown in [13] that if the analogue of (5) with ∇u replaced by any first partial derivative of u holds, then the appropriate statement 1), 2), 3) or 4) also holds.

The reason for the stronger assumption $v \in A_\infty$ in Theorem 2 (only the doubling of v is required in Theorem 1) is that the proof uses the fact that for such v, the $L^p(v)$ norm of the classical Lusin area integral $S(u)(t)$ defined by

$$S(u)(t) = \left(\int_{\Gamma(t)} |y \nabla u(z)|^2 \, \frac{dx\,dy}{y^{d+1}} \right)^{1/2}$$

is bounded by a multiple of $||u||_{H^p(v)}$, i.e.,

$$||S(u)||_{L^p(v)} \leq c\, ||u||_{H^p(v)} \quad \text{if} \quad v \in A_\infty,$$

for all harmonic u with c independent of u (see [4]). Note that in the terminology of tent spaces, this means that $y|\nabla u(z)| \in T_2^p(v)$ if $u \in H^p(v)$. Similarly, by using the definition of $H^p(v)$ in terms of the nontangential maximal function, it follows that $u \in T_\infty^p(v)$ if $u \in H^p(v)$.

The proofs of both theorems above use the fact that the tent spaces $T_r^s(v)$ are naturally related to the Benedek-Panzone mixed norm spaces $L^s L^r(\nu, v)$ with $d\nu(z) = dx\, dy/y^{d+1}$, these spaces being defined as follows. For $0 < r, s < \infty$, $L^s L^r(\nu, v)$ is the collection of all functions $g : (t, z) \to g(t, z)$ mapping $\mathbb{R}^d \times \mathbb{R}_+^{d+1} \to \mathbb{C}$ such that

$$||g||_{L^s L^r(\nu, v)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+^{d+1}} |g(t, z)|^r \, d\nu(z) \right)^{s/r} v(t)\, dt \right)^{1/s} < \infty,$$

with an obvious modification in case $r = \infty$. A large part of the proofs of Theorems 1 and 2 depends on determining the dual spaces of certain of the tent spaces $T_r^s(v)$, and the known duality structure (see [1]) of the Benedek-Panzone spaces for $1 \leq r, s < \infty$ is used in many cases.

Another ingredient in the proofs is showing that when $s = 1$ and $1 < r < \infty$, functions in the $T_r^1(v)$ spaces have an atomic decomposition. This is helpful in determining their duals spaces. Given a Borel measure ν on $\mathbb{R}_+^{d+1}$ and a doubling weight v on $\mathbb{R}^d$, a $(1, r)$-atom is a Borel measurable function $a : \mathbb{R}_+^{d+1} \to \mathbb{C}$ whose support is contained in $\hat{Q}$ for some cube $Q \subset \mathbb{R}^d$ and which satisfies

$$\int_{\hat{Q}} |a(z)|^r \, v(B(z))\, d\nu(z) \leq v(Q)^{1-r}.$$

Choosing $d\nu(z) = dxdy/y^{d+1}$, it is shown in [13], Theorem 8 that any $f \in T_r^1(v)$, $1 < r < \infty$, has an atomic decomposition in terms of such atoms. Some remarks about atomic decompositions of weighted tent spaces are also made in [11], but the definition of an atom there is different from the one above.

The definition of $H^p(v)$ is given in terms of the nontangential maximal function over cones $\Gamma_\alpha(t)$ of aperture $\alpha = 1$, but as is well-known, the definition is independent of any fixed aperture as long as v is a doubling weight. Similarly, the tent spaces T_r^s for $s < \infty$ are defined with respect to cones of aperture 1, but it turns out that the definition is independent of aperture for any doubling weight v. See [13] for details.

2. L^p results.

We now assume that $u(x, y)$ is the Poisson integral of a function $f(x)$, and we consider the norm inequality (2) (with $C = 1$ for simplicity), i.e.,

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(x, y)|^q \, d\mu(x, y)\right)^{1/q} \le \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx\right)^{1/p} \tag{6}$$

in case $1 < p \le q < \infty$, μ is a non-negative Borel measure on $\mathbb{R}^{d+1}_+$, and v is a non-negative, locally integrable function on $\mathbb{R}^d$. The problem is to determine conditions on μ and v which ensure that (6) holds for all f. The Poisson integral of f is defined by $u(x, y) = P_y * f(x)$, where

$$P_y(x) = c_d \frac{y}{(y^2 + |x|^2)^{(d+1)/2}}$$

is the Poisson kernel for $\mathbb{R}^{d+1}_+$.

We first summarize some of the results of [12] related to this problem. Some variants of (6) are also studied in [12], including similar inequalities with $\nabla u(x, y)$ replaced by $y^{-1}(f * \phi_y)(x)$, where ϕ is a smooth function which decays at infinity and satisfies $\int_{\mathbb{R}^d} \phi(x)\, dx = 0$, and where ϕ_y is the standard dilation of ϕ defined by $\phi_y(x) = y^{-d}\phi(x/y)$, $x \in \mathbb{R}^d$, $y > 0$. The derivatives $\partial u/\partial x_i$ and $\partial u/\partial y$ are included in this situation if ϕ is chosen appropriately. It turns out that the rate of decay of ϕ plays an important role in the results. For example, it is easy to see by direct computation that the function $\phi = \phi_0$ which corresponds to $\partial u/\partial y$ is

$$\phi_0(x) = -dP(x) - \sum_{i=1}^{d} x_i \frac{\partial P}{\partial x_i}(x) = c_d \frac{|x|^2 - d}{(1 + |x|^2)^{(d+3)/2}},$$

while the function $\phi = \phi_i$, $i = 1, \ldots, d$, that corresponds to $\partial u/\partial x_i$ is

$$\phi_i(x) = \frac{\partial P}{\partial x_i} = -c_d(d+1)\frac{x_i}{(1 + |x|^2)^{(d+3)/2}}, \qquad i = 1, \ldots, d.$$

It follows that ϕ_i decays like $|x|^{-d-2}$, while ϕ_0 decays at the slower rate $|x|^{-d-1}$. This difference is reflected in the results below. The simpler case when ϕ has compact support as well as analogues of (6) for Haar-function and wavelet expansions are also studied in [12].

For the full gradient of the Poisson integral, we have the following result. In stating it, if Q is a cube in $\mathbb{R}^d$ with center x_Q and edgelength $\ell(Q)$, then $T(Q)$ denotes the set in $\mathbb{R}^{d+1}_+$ defined by

$$T(Q) = \{(x, y)\colon x \in Q,\ \ell(Q)/2 < y \le \ell(Q)\}.$$

Thus, $T(Q)$ is only the top half of the usual Carleson cube $\hat{Q}$ that was defined earlier.

Theorem 3. *Let v be a non-negative weight in $L^1_{loc}(\mathbb{R}^d)$ and μ be a positive measure on $\mathbb{R}^{d+1}_+$. Let $1 < p < \infty$, $1/p + 1/p' = 1$, and $\sigma = v^{1-p'}$.*

(i) *If $0 < q < \infty$ and the norm inequality*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(x,y)|^q \, d\mu(x,y)\right)^{1/q} \le \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, dx\right)^{1/p} \tag{6}$$

holds for all f, where $u(x,y)$ is the Poisson integral of f, then there is a constant C depending only on p, q, and d such that

$$\mu(T(Q))^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q) + |x - x_Q|)^{(d+1)p'}}\, dx\right)^{1/p'} \le C \tag{7}$$

for all cubes Q in $\mathbb{R}^d$.

(ii) *Conversely, suppose that $1 < p \le q < \infty$, that $q \ge 2$, and that σ is an A_∞ weight. If*

$$\mu(T(Q))^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q) + |x - x_Q|)^{(d+1)p'/q'}}\, dx\right)^{1/p'} \le C\ell(Q)^{(d+1)/q} \tag{8}$$

for all cubes Q, for a suitable constant C that depends only on p, d, and σ, then (6) holds for all f.

The sufficiency result (i.e., part (ii) of Theorem 3) has an analogue in case σ is not an A_∞ weight, and the same is true for Theorems 4 and 5 below. In fact, in part (ii) of Theorem 3, instead of assuming that $\sigma \in A_\infty$, it is enough to assume that there is a weight $w(x)$ satisfying $\sigma(x) \le w(x)$ such that the analogue of (8) with σ replaced by w holds, and such that

$$\int_Q \sigma(x) \log^\tau(e + \sigma(x)/\sigma_Q)\, dx \le \int_Q w(x)\, dx$$

for all cubes Q and some $\tau > p'/2$, where $\sigma_Q = \sigma(Q)/|Q|$. If $\sigma \in A_\infty$, this condition is automatically fulfilled for any $\tau > 0$ with $w = \sigma$. (For details, see [15] or the proof of Theorem 4 in [12]).

In order to prove that (7) is necessary for (6), we need the presence of the full gradient in (6). On the other hand, for part (ii), the analogue of (6) for just the x-derivatives $\partial u/\partial x_i$, excluding $\partial u/\partial y$, is true if we replace (8) by a weaker condition. This result is stated in the next theorem.

Theorem 4. *Let $1 < p \leq q < \infty$ and $q \geq 2$. Let v and μ be as above and suppose that $\sigma = v^{1-p'} \in A_\infty$. Let $i = 1, \dots, d$ and $u(x, y)$ be the Poisson integral of $f(x)$. In order that*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\frac{\partial}{\partial x_i} u(x, y)|^q d\mu(x, y)\right)^{1/q} \leq \left(\int_{\mathbb{R}^d} |f|^p v\, dx\right)^{1/p} \tag{9}$$

for all f, it is sufficient that

$$\mu(T(Q))^{1/q} \left(\int_{\mathbb{R}^d} \frac{\log^{p'/q'}(e + |x - x_Q|/\ell(Q))\sigma(x)}{(\ell(Q) + |x - x_Q|)^{(d+2)p'/q'}}\, dx\right)^{1/p'} \leq \tag{10}$$
$$\leq C\ell(Q)^{d+1-(d+2)/q'}$$

for all cubes Q. The constant C in (10) depends on p, q and d.

No necessity results for just the x-derivatives of $u(x, y)$ are proved in [12]. The sufficiency statement in Theorem 3 as well as the conclusion of Theorem 4 have been improved in [14] for some ranges of q. In fact, the following result is proved there; the first part concerns the y-derivative and the second part concerns the x-derivatives.

Theorem 5. *Let $1 < p \leq q < \infty$ and $q \geq 2$. Let v and μ be as above and suppose that $\sigma = v^{1-p'} \in A_\infty$. Let $u(x, y)$ be the Poisson integral of $f(x)$.*

(i) *In order that*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\frac{\partial}{\partial y} u(x, y)|^q d\mu(x, y)\right)^{1/q} \leq \left(\int_{\mathbb{R}^d} |f|^p v\, dx\right)^{1/p}$$

for all f, it is sufficient that there exist $\epsilon > 0$ such that

$$\mu(T(Q))^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(1 + |x - x_Q|/\ell(Q))^{(p'd/2)+p'-\epsilon}}\, dx\right)^{1/p'} \leq C\ell(Q)^{d+1}$$

for all cubes Q and a suitable constant C that depends on p, q and d.

(ii) *If $i = 1, \dots, d$, then in order that*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\frac{\partial}{\partial x_i} u(x, y)|^q d\mu(x, y)\right)^{1/q} \leq \left(\int_{\mathbb{R}^d} |f|^p v\, dx\right)^{1/p}$$

for all f, it is sufficient that there exist $\epsilon > 0$ such that

$$\mu(T(Q))^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(1 + |x - x_Q|/\ell(Q))^{(p'd/2)+2p'-\epsilon}}\, dx\right)^{1/p'} \leq C\ell(Q)^{d+1}$$

for all cubes Q and a suitable constant C that depends on p, q and d.

The only difference between the conditions in parts (i) and (ii) of Theorem 5 is the order of the exponent that appears in the two denominators. Theorem 5 improves the earlier results for $\partial u/\partial y$ in case $q < 2 + (2/d)$, and it improves the results for $\partial u/\partial x_i$ when $q < 2 + (4/d)$. Similar estimates but with the derivatives of $u(x, y)$ replaced by $y^{-1}\phi_y * f(x)$ for appropriate ϕ with $\int \phi = 0$ are also derived in [14].

The cancellation properties of the derivatives of the Poisson integral play an important role in the proofs of all the results above. There is, however, an alternate approach to the sufficiency results which ignores this cancellation. In this approach, one simply majorizes the derivatives in absolute value by integral operators with positive kernels, and then applies known results for operators with nonnegative kernels, such as the Carleson-type results in [8] and [7]. This process yields conclusions like (6) under conditions on μ and σ which are generally different from those that have been imposed so far. The new conditions involve an integral of μ extended over a set which is larger than $T(Q)$, such as $\hat{Q}$ or all of $\mathbb{R}^{d+1}_+$, but in the integral for σ, they involve a smaller integrand, after normalization. Also, the new conditions do not require the restriction that $q \geq 2$. Sometimes the new conditions are weaker than those already considered, but there are also cases, most notably when $p = q$, when the opposite is true. Also, we have never succeeded in proving any necessity results involving these sorts of conditions.

For example, in the case when $p < q$, we obtain the following result in this way.

Theorem 6. *Let $1 < p < q < \infty$ and let ϕ satisfy $|\phi(x)| \leq (1 + |x|)^{-M}$ for some $M > 0$ and all $x \in \mathbb{R}^d$. Then the inequality*

$$\left(\int_{\mathbb{R}^{d+1}_+} |y^{-1}\phi_y * f(x)|^q d\mu(x, y)\right)^{1/q} \leq \left(\int_{\mathbb{R}^d} |f|^p v\, dx\right)^{1/p}$$

holds for all f if

$$\left(\int_{\mathbb{R}^{d+1}_+} \frac{y^{(M-d-1)q}}{(\ell(Q) + y + |x - x_Q|)^{Mq}} d\mu(x, y)\right)^{1/q} \cdot \\ \cdot \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q) + |x - x_Q|)^{Mp'}}\, dx\right)^{1/p'} \leq C\ell(Q)^{-M} \tag{11}$$

for all cubes Q. The constant C depends on p, q, d, and M.

The x-derivatives $\partial u/\partial x_i$ of the Poisson integral correspond to choosing $M = d+2$ above, and the y-derivative $\partial u/\partial y$ corresponds to $M = d+1$. It is not assumed that $\sigma \in A_\infty$ in Theorem 6.

In order to see how (11) is related to the earlier conditions, note that arguments in [7] and [8] show that (11) is equivalent to the following two weaker conditions together when $q > p$:

$$(11'a)\quad \left(\int_{\hat{Q}} y^{(M-d-1)q} d\mu(x,y)\right)^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q)+|x-x_Q|)^{Mp'}}\,dx\right)^{1/p'} \leq C$$

$$(11'b)\quad \left(\int_{\mathbb{R}^{d+1}} \frac{y^{(M-d-1)q}}{(\ell(Q)+y+|x-x_Q|)^{Mq}} d\mu(x,y)\right)^{1/q} \sigma(Q)^{1/p'} \leq C.$$

If σ also satisfies the reverse doubling condition

$$(RD)\qquad \sigma(2Q) \geq \alpha\sigma(Q)$$

for some $\alpha > 1$ for all cubes Q (e.g., if σ is an A_∞ weight), then as in ([7], p. 658), the condition (11′b) is the same as

$$\left(\int_{\hat{Q}} y^{(M-d-1)q} d\mu(x,y)\right)^{1/q} \sigma(Q)^{1/p'} \leq C\ell(Q)^M,$$

which clearly follows from (11′a). Thus, for $q > p$ and σ satisfying (RD), (11) is equivalent to

$$(11'')\quad \left(\int_{\hat{Q}} y^{(M-d-1)q} d\mu(x,y)\right)^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q)+|x-x_Q|)^{Mp'}} dx\right)^{1/p'} \leq C.$$

If we assume that (11″) holds with $M = d+1$, then it is easy to see that it also holds with $M = d+2$, and consequently we obtain the following corollary of Theorem 6.

Theorem 7. *Let $1 < p < q < \infty$ and suppose that $\sigma = v^{1-p'}$ satisfies (RD). Let $u(x,y)$ be the Poisson integral of f. Then the inequality*

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u(x,y)|^q d\mu(x,y)\right)^{1/q} \leq \left(\int_{\mathbb{R}^d} |f|^p v\,dx\right)^{1/p}$$

holds for all f if

$$(12)\qquad \mu(\hat{Q})^{1/q} \left(\int_{\mathbb{R}^d} \frac{\sigma(x)}{(\ell(Q)+|x-x_Q|)^{(d+1)p'}}\,dx\right)^{1/p'} \leq C$$

for all cubes Q. The constant C depends on p, q, d and σ.

Recall from Theorem 3 that the analogue of (12) with $\hat{Q}$ replaced by $T(Q)$ is necessary for the norm inequality (for the full gradient of u).

Even though the first factor of $(11'')$ involves all of $\hat{Q}$, a simple addition argument shows that $(11'')$ is weaker than the conditions required in the earlier results. Of course, it is important to remember that Theorems 3–5 have analogues where it is not assumed that σ satisfies either A_∞ or (RD).

It is also possible to simplify (11) if $y^{(M-d-1)q}d\mu(x,y)$ satisfies a sort of (RD) condition in the half-space, but we have to chosen to emphasize the case when σ satisfies (RD) for purposes of comparison with our earlier results.

The technique indicated above for obtaining results by ignoring cancellation does not work as well when $p = q$. For example, for the full gradient ∇u, if the results of [7] are used in the simple case when $v = 1$ and $1 < p < \infty$, this requires assuming that μ satisfies

$$\int_{\hat{Q}} y^{-p} d\mu(x,y) \le C|Q|, \tag{13}$$

whereas Theorem 3 (ii) shows that the weaker condition

$$\ell(Q)^{-p} \int_{T(Q)} d\mu(x,y) \le C|Q| \tag{14}$$

is enough when $p \ge 2$. Clearly, $d\mu = y^{p-1}\,dxdy$ satisfies (14) but not (13).

For any σ in A_∞ and $1 < p = q < \infty$, the condition required for (6) by using this approach is

$$\left(\int_{\hat{Q}} y^{-p} d\mu(x,y)\right)^{1/p} \sigma(Q)^{1/p'} \le C|Q| \tag{15}$$

(see [7], p. 645). For general σ, it is also possible to formulate an appropriate analogue based on Theorem 2 in [7].

REFERENCES

[1] A. Benedek - R. Panzone, *The spaces L^p with mixed norm,* Duke Math. Journal, 28 (1961), pp. 301–324.

[2] L. Carleson, *Interpolation by bounded analytic functions and the corona problem,* Annals of Math., 76 (1962), pp. 547–559.

[3] R.R. Coifman - Y. Meyer - E.M. Stein, *Some new function spaces and their applications to harmonic analysis,* J. Funct. Anal., 62 (1985), pp. 304–335.

[4] R.F. Gundy - R.L. Wheeden, *Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh-Paley series,* Studia Math., 49 (1974), pp. 107–124.

[5] D. Luecking, *Forward and reverse Carleson inequalities for functions in the Bergman spaces and their derivatives,* American Journal of Math., 107 (1985), pp. 85–111.

[6] D. Luecking, *Embedding derivatives of Hardy spaces into Lebesgue spaces,* Proc. London Math. Soc., 3 (1991), pp. 595–619.

[7] E.T. Sawyer - R.L. Wheeden, *Carleson conditions for the Poisson integral,* Indiana U. Math. J., 40 (1991), pp. 639–676.

[8] E.T. Sawyer - R.L. Wheeden - S. Zhao, *Weighted norm inequalities for operators of potential type and fractional maximal functions,* Potential Analy., 5 (1996), pp. 523–580.

[9] N.A. Shirokov, *Some generalizations of the Littlewood-Paley theorem,* Journal of Soviet Math., 8 (1977), pp. 119–129.

[10] N.A. Shirokov, *Some embedding theorems for spaces of harmonic functions,* Jour of Soviet Math., 14 (1980), pp. 1173–1176.

[11] G. Welland - S. Zhao, *ϵ-families of operators in Triebel-Lizorkin and tent spaces,* Canadian Math. Journal, 47 (1995), pp. 1095–1120.

[12] R.L. Wheeden - J.M. Wilson, *Weighted norm estimates for gradients of half-space extensions,* Indiana Univ. Math. Journal, 44 (1995), pp. 917–969.

[13] R.L. Wheeden - J.M. Wilson, *Weighted L^q estimates for derivatives of weighted H^p functions,* J. Fourier Analysis and Applications, 4 (1998), pp. 595–628.

[14] J.M. Wilson, *A semi-discrete Littlewood-Paley inequality,* Preprint, 1998.

[15] J.M. Wilson, *Weighted norm inequalities for the continuous square function,* Trans. Amer. Math. Soc., 314 (1989), pp. 661–692.

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