LE MATEMATICHE
Vol. LIII (1998) – Fasc. I, pp. 3–9

ON TAIL ASYMPTOTICS FOR L^1-NORM OF CENTERED BROWNIAN BRIDGE

OLGA A. PODKORYTOVA

Asymptotics of the tail probability for L^1-norm of the centered Brownian bridge is obtained.

1. Introduction.

Let us consider the Gaussian process defined as follows:

$$\xi(t) = B(t) - \int_0^1 B(s)\, ds, \quad t \in [0, 1], \tag{1}$$

where B is the standard Brownian bridge on [0, 1]. In this paper we obtain the rough asymptotics of the tail probability of the L^1-norm of the process ξ. Distributions of different norms of Gaussian processes were investigated by many authors. Such results are very useful, in particular, in creating new nonparametric tests and exploring their limiting distributions, see Shorack and Wellner [9]. The tail probabilities of these distributions play the key role in calculating the Bahadur asymptotic efficiency of statistical tests. Details may be found in Bahadur [1] and Nikitin [6]. To exemplify let us refer to the papers closely connected to our subject, namely by Shepp [8] and Cifarelli [2] in which the distribution of of the L^1-norm of B was described, by Darling

Entrato in Redazione il 5 maggio 1997.

[3] who investigated the distribution of the L^∞-norm of ξ and by Watson [10], the distribution of the L^2-norm of ξ being discussed in it. One can also recall the paper by Kallianpur and Oodaira [5] that contains calculations of the tail probability asymptotics of L^p-norm of standard Wiener process and Brownian bridge for each $p > 0$.

2. Asymptotics of the tail probability.

Consider the process ξ defined by (1). Obviously, $E\xi(t) = 0$ for each $t \in [0, 1]$. Let $K(t, s)$ be the covariance function of ξ:

$$K(t, s) = E(\xi(t)\xi(s)).$$

One can easily see that

$$K(t, s) = \frac{1}{2}\left((t-s)^2 - |t-s| + \frac{1}{6}\right). \tag{2}$$

Let us denote the norm of ξ in the Lebesgue space L^1 as

$$||\xi||_1 : ||\xi||_1 = \int_0^1 |\xi(s)|\, ds.$$

Proving our main result we will use the general theorem of Donsker and Varadhan (see, for example, Deuschel and Stroock [4], p. 86).

Theorem 1 (Donsker, Varadhan)**.** *Let P be a centered Gaussian measure on a separable Banach space X with norm $||\cdot||_X$; X^* is the dual space of X. Let us define the covariance operator $\mathcal{K}$ as*

$$(\mathcal{K}x^*, y^*) = \int_X (x, x^*)(x, y^*)\, dP(x), \tag{3}$$

where $x^*, y^* \in X^*$; $||\mathcal{K}||$ being its norm. Then

$$\lim_{r\to\infty} \frac{1}{r^2} \ln P(x : ||x||_X > r) = -\frac{1}{2||\mathcal{K}||}.$$

Theorem 2. *Let ξ be defined by* (1)*. Then*

$$\lim_{r\to\infty} \frac{1}{r^2} \ln P(||\xi||_1 > r) = -24.$$

Proof. It is well known that the dual space of L^1 is L^∞. It is easy to see that the operator $\mathcal{K}$ defined by (3), in our case proves to be an integral operator mapping L^∞ into L^1

$$(\mathcal{K}f)(t) = \int_0^1 K(t,s)f(s)\,ds, \quad f \in L^\infty,$$

where K is given by (2). In view of the Theorem 1, all that we have to do is to show that $||\mathcal{K}|| = \frac{1}{48}$. The kernel K depends on the difference $t - s$ only, i.e. $K(t,s) = H(t-s)$, where $H(u) = \frac{1}{2}(u^2 - |u| + \frac{1}{6})$, so it is natural to consider 1-periodic extension of the functions f and H. We will denote them f and H as we did before. Obviously $H \in \mathbb{C}(\mathbb{R})$ and $\int_0^1 H(u)\,du = 0$. Hence $\int_0^1 H(t-s)\,dt = 0$ for each $s \in \mathbb{R}$ and therefore

$$y(t) = (\mathcal{K}f)(t) = \int_0^1 K(t,s)f(s)\,ds = \int_0^1 H(t-s)f(s)\,ds$$

is a 1-periodic function having the zero mean value:

$$\int_0^1 y(t)\,dt = \int_0^1 \left(\int_0^1 H(t-s)\,dt\right) f(s)\,ds = 0.$$

By applying to the function y theorem about integral with parameter (see for example [7], Theorem 115) we obtain that y is a smooth function and for $t \in [0,1]$ its derivative is

$$\begin{aligned} y'(t) &= \frac{1}{2}\int_0^1 (2(t-s) - \operatorname{sign}(t-s))f(s)\,ds = \\ &= t\int_0^1 f(s)\,ds - \int_0^1 sf(s)\,ds + \frac{1}{2}\int_t^1 f(s)\,ds - \frac{1}{2}\int_0^t f(s)\,ds. \end{aligned}$$

Therefore y' is absolutely continuous so y'' exists almost everywhere and

$$y''(t) = \int_0^1 f(s)\,ds - f(t).$$

As $||\mathcal{K}|| = \sup\left\{\int_0^1 |y(t)|\,dt \;:\; |f(s)| \le 1 \text{ a.e.}\right\}$, we need to find the upper bound of the integral $\int_0^1 |y(t)|\,dt$. Its value will not change if we consider $f(s-\tau)$ instead of $f(s)$. This leads to replacement of function $y(t)$ by $y(t-\tau)$, and for every $\tau \in \mathbb{R}$

$$\int_0^1 |y(t)|\,dt = \int_0^1 |y(t-\tau)|\,dt.$$

For further considerations it is convenient to let τ be equal to a root of y (there exists at least one root because of continuity and the zero mean value of y). Thus, without loss of generality we can assume that $y(0) = 0$ (and hence $y(1) = 0$). Let us define $E_+ = \{t \in [0,1] : y(t) > 0\}$ and $E_- = \{t \in [0,1] : y(t) < 0\}$. These sets are open for y is continuous and $y(1) = y(0) = 0$ (the last condition implies $E_\pm \subset (0,1)$). The statement $\int_0^1 y(t)\,dt = 0$ is equivalent to

$$\int_{E_+} y(t)\,dt = \int_{E_-} (-y(t))\,dt,$$

thus

$$\int_0^1 |y(t)|\,dt = 2\int_{E_+} y(t)\,dt = 2\int_{E_-} (-y(t))\,dt.$$

The set E_+ may be decomposed as $E_+ = \cup_k (a_k, b_k)$, where $y(a_k) = y(b_k) = 0$. For the sake of concision let us denote $\gamma_k = b_k - a_k$ and $y_k(t) = y(t + a_k)$. Using the equalities $y_k(0) = y_k(\gamma_k) = 0$ we get

$$\int_{a_k}^{b_k} y(t)\,dt = \int_0^{\gamma_k} y_k(t)\,d\left(t - \frac{\gamma_k}{2}\right) = -\int_0^{\gamma_k} \left(t - \frac{\gamma_k}{2}\right) y_k'(t)\,dt =$$

$$= -\frac{1}{2}\int_0^{\gamma_k} y_k'(t)\,d(t^2 - \gamma_k t) = \frac{1}{2}\int_0^{\gamma_k} y_k''(t)(t^2 - \gamma_k t)\,dt.$$

To estimate the integral appeared, let us denote $c_0 = \int_0^1 f(t)\,dt$ and remark that $|f(t)| \le 1$ implies $|c_0| \le 1$ and $|y''(t) - c_0| = |f(t)| \le 1$ for almost every t.

Thus

$$-y_k''(t) = -y''(t + a_k) \le 1 - c_0$$

and

$$\int_{a_k}^{b_k} y(t)\,dt = \frac{1}{2}\int_0^{\gamma_k} (-y_k''(t))t(\gamma_k - t)\,dt \le$$

$$\le \frac{1 - c_0}{2}\int_0^{\gamma_k} t(\gamma_k - t)\,dt = \frac{1 - c_0}{12}{\gamma_k}^3.$$

Therefore,

$$\int_0^1 |y(t)|\,dt = 2\int_{E_+} y(t)\,dt \le \frac{1 - c_0}{6}\sum_k {\gamma_k}^3 \le$$

$$\le \frac{1 - c_0}{6}(\sum_k \gamma_k)^3 = \frac{1 - c_0}{6}(\operatorname{mes} E_+)^3.$$

The proof of inequality

$$\int_0^1 |y(t)|\,dt = 2\int_{E_-}(-y(t))\,dt \le \frac{1+c_0}{6}(\mathrm{mes}\,E_-)^3$$

is similar to the previous one.

Consequently we have

$$\int_0^1 |y(t)|\,dt \le \frac{1}{6}\min\left((1-c_0)(\mathrm{mes}\,E_+)^3,\,(1+c_0)(\mathrm{mes}\,E_-)^3\right) \le$$

$$\le \frac{1}{6}\sqrt{(1-c_0)(\mathrm{mes}\,E_+)^3(1+c_0)(\mathrm{mes}\,E_-)^3} \le$$

$$\le \frac{\sqrt{1-c_0^2}}{6}(\mathrm{mes}\,E_+(1-\mathrm{mes}\,E_+))^{\frac{3}{2}}.$$

Obviously, $0 \le \mathrm{mes}\,E_+ \le 1$, hence $\mathrm{mes}\,E_+(1-\mathrm{mes}\,E_+) \le \frac{1}{4}$. Consequently,

$$\int_0^1 |y(t)|\,dt \le \frac{\sqrt{1-c_0^2}}{48} \le \frac{1}{48}.$$

To show that this bound is accurate, note that each inequality becomes an equality if and only if

$$\begin{cases} \mathrm{mes}\,E_+ = \mathrm{mes}\,E_- = \frac{1}{2} \\ c_0 = 0 \\ |y''(t)| = 1, \quad \text{a.e.} \\ \sum_k {\gamma_k}^3 = (\sum_k \gamma_k)^3. \end{cases}$$

The last condition is equivalent to the following: $\gamma_k = 0$ for each k except possibly one, i.e. the set E_+ consists of the only interval of the length $\frac{1}{2}$. Similarly, E_- should be an interval of the length $\frac{1}{2}$. It is easy to see that there are only two smooth functions y_+ and y_-, such that $y_\pm(0) = 0$, satisfying this requirement:

$$y_\pm(t) = \pm\begin{cases} \frac{t}{2}(\frac{1}{2}-t), & t \in \left[0, \frac{1}{2}\right] \\ \frac{1-t}{2}(\frac{1}{2}-t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

These functions correspond to $f(t) = \pm\,\mathrm{sign}\,(\frac{1}{2}-t)$, $t \in [0,1]$, because

$$f_\pm(t) = \int_0^1 f(s)\,ds - y''_\pm(t) = c_0 - y''_\pm(t) = -y''_\pm(t) = \pm\,\mathrm{sign}\left(t-\frac{1}{2}\right).$$

$\square$

Remark. We have also proved that the extremal functions f_0 ($|f_0(s)| \leq 1$ a.e.), i.e. $||\mathcal{K} f_0||_1 = \frac{1}{48}$, will remain extremal under the shift transformation $f_0(s) \longrightarrow f_0(s-\tau)$. Provided additionally $0 = y(0) = (\mathcal{K} f)(0)$, we necessarily have $f_0(s) = \pm \operatorname{sign}(\frac{1}{2} - s), s \in [0,1]$. So we may conclude that all the extremal functions can be obtained by a shift of $\pm \operatorname{sign}(\frac{1}{2} - \{s\})$ ($\{x\}$=fractional part of x). It is obvious that each of these two functions is the shift of the other, when $\tau = \frac{1}{2}$. Therefore the conditions $||\mathcal{K} f||_1 = \frac{1}{48}, |f(s)| \leq 1$ for a.e. t are equivalent to existence of $\tau \in \mathbb{R}$ such that $f_0(s) = \operatorname{sign}(\frac{1}{2} - \{s-\tau\})$ a.e.

This remark can be usefull in searching for test statistics with the property of local asymptotic optimality in the Bahadur sense, see Bahadur [1] and Nikitin [6].

Acknowledgement. This work was partially supported by the grant No.95-01-1260 of the Russian Fund of Fundamental research. The author thanks M.A. Lifshits and Ya.Yu. Nikitin for helpful discussion during the preparation of this paper.

REFERENCES

[1] R. Bahadur, *Some limit theorems in statistics,* SIAM, Philadelphia, 1971.

[2] D.M. Cifarelli, *Contributi intorno ad un test per l'omogenita tra due campioni,* G. Econom. Ann. Econ. (N.S.), 34 (1975), pp. 233–249.

[3] D.A. Darling, *On the supremum of a certain Gaussian Process,* Ann. Prob., 11 (1983), pp. 807–808.

[4] J.D. Deuschel - D.W. Stroock, *Large deviations,* Academic Press, 1989.

[5] G. Kallianpur - H. Oodaira, *Freidlin-Wentzell estimates for abstract Wiener spaces,* Sankhyā, The Indian J. of Stat., 40 (1978), pp. 116–137.

[6] Ya.Yu. Nikitin, *Asymptotic efficiency of nonparametric tests,* Cambridge University Press, 1995.

[7] L. Schwartz, *Analyse mathematique, vol.1,* Hermann, 1967.

[8] L.A. Shepp, *On the integral of absolute value of the pinned Wiener process,* Ann. Prob., 10 (1982), pp. 234–239.

[9] G. Shorack - J. Wellner, *Empirical processes with application to Statistics,* J. Wiley and Sons, 1986.

[10] G.S. Watson, *Goodness of fit test on a circle,I,* Biometrika, 48 (1961), pp. 109–114.

Faculty of Math. and Mech.,
Dept. of Theory of Prob. and Math. Stat.,
St.Petersburg University,
S.Petergof, Bibliotechnaya sq., 2,
198904 St. Petersburg (RUSSIA),
e-mail: pan@pan.usr.pu.ru