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NEW SOLUTIONS OF THE CONFLUENT HEUN EQUATION

HAROLD EXTON

New compact triple series solutions of the confluent Heun equation (CHE) are obtained by the appropriate applications of the Laplace transform and its inverse to a suitably constructed system of soluble differential equations. The computer-algebra package MAPLE V is used to tackle an auxiliary system of non-linear algebraic equations. This study is partly motivated by the relationship between the CHE and certain Schröndinger equations.

1. Introduction.

Heun's equation and its confluent forms have recently aroused considerable interest on account of their occurence in various branches of pure and applied mathematics. For an overview, the reader is referred to Ronveaux [6], in particular to Part B, Slavyanov [9], where the confluent Heun equation (CHE) is treated in detail.
The CHE is a linear differential of the second order with three singularities, two of which are regular and the thrd is irregular of the second type. The standard form used in the present context is that employed by Exton [3], that is

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x - \chi x^2]y' - [\alpha\beta + \chi\kappa x]y = 0. \tag{1.1}$$

It is taken that all of the parameters of this equation are non-zero.

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The purpose of this study consists of obtaining new compact solutions of the CHE and is in part motivated by the association of this equation with certain Scrödinger equations. (See Exton [4], Leaver [5] and Roy, Roychoudhury and Roy [7] for example). While a great deal is known concerning the general theory of solutions of the CHE, the explicit solutions hitherto mentioned in the literature are those complicated expressions of perturbation type presented by Exton [3]. The solutions of the CHE given here are obtained by the appropriate use of the Laplace transform and its inverse applied to suitably constructed soluble differential equations. This leads to a system of non-linear algebraic equations tackled by the use of the computer algebra package MAPLE V. The solutions obtained are in the form of triple series and are of there types. Firstly solutions valid in restricted neighbourhoods of the regular singularities of the CHE, secondly, global solutions and thirdly, formal representations of solutions relative to the irregular singularity at infinity which in some cases reduce to polynomials. General solutions of the CHE can be deduced from these results by using the symmetries indicated by Exton [3]. Any values of parameters leading to results which do not make sense are tacitly excluded and indices of summation are taken to run over all of the non-negative integers unless otherwise indicated. the Pochhammer symbol $(a, n) = \Gamma(a+n)/\Gamma(a)$ is used frequently in the subsequent analysis. Trivial multipliers which have no essential bearing of final results will also be left out.

2. An auxiliary differential system.

Consider the soluble differential equations

$$(at+b)u' + cu = v \tag{2.1}$$

and

$$dtv'' + (ft+g)v' + hv = 0, \tag{2.2}$$

which are equivalent to the third-order equation

$$\begin{aligned}[adt^2 + bdt]u''' + [aft^2 + (ag+bf+cd+2ad)t + bg]u'' \\ +[(ah+af+cf)t + ag+bg+bh]u' + chu = 0.\end{aligned} \tag{2.3}$$

If the inverse Laplace transform is

$$u(t) = \int \exp(-tx)y(x)\,dx, \tag{2.4}$$

where the contour of integration consists of a simple loop on the Riemann surface of the integrand such that this integrand remains unchanged on the completion of one circuit. It is found that the function $y(x)$ is given by the differential equation

$$(2.5)\qquad (x^2 - adx^3/f)y'' + [-bdx^3/(af) + (g/f + b/d + \\ + cd/(af) - 4d/f)x^2 + (3 - h/f - c/a)x]y' + [(bg/(af) - 3bd/(af))x^2 + \\ +(cg/(af) - 2d/f + g/f + 3b/a - bh/(af))x + ch/(af) - h/f + 1 - c/a]y = 0.$$

Put

$$(2.6)\qquad x = fX/d,$$

when we have

$$(2.7)\ (X^2 - X^3)y'' + [-bf/(ad)X^3 + (g/d + bf/(ad) + c/a - 4)X^2 + \\ +(3d/f - dh/f^2 - cd/(af))X]y' + [(bgf/(ad^2) - 3bf/(ad))X^2 + (cg/(ad) - \\ - 2 + g/d + 3bf/(ad) - bh/(ad))X + ch/(af) - h/f + 1 - c/a]y = 0.$$

For convenience, we write (2.7) in the form

$$(2.8)\qquad (X^2 - X^3)y'' + [-AX^3 + BX^2 + CX]y' + \\ + [DX^2 + EX + ch/(af) - h/f + 1 - c/a]y = 0$$

on the assumption (which will be justified below) that A, B, C, D and E can be expressed independently in terms of a, b, c, d, f, g and h.
Let

$$(2.9)\qquad y = X^P Y$$

where P is to determined, when we obtain

$$(2.10)\qquad (X - X^2)Y'' = [-AX^2 + (B - 2P)X + C + 2P]Y' + \\ + [(D + AP)X + BP + E]Y = 0$$

if we put

$$(2.11)\qquad P(P - 1) + CP + ch/(af) - h/f + 1 - c/a = 0.$$

If we also put $A = \chi$, $B = 2P + 1 + \alpha + \beta$, $C = \gamma - 2P$, $D = -\chi(P + \kappa)$ and

$$E = -P(2P + 1 + \alpha + \beta) - \alpha\beta, \tag{2.12}$$

(2.8) becomes the same as the standard form of the CHE, namely (1.1).
The solution of the equations

$$\begin{gathered} bf/(ad) = A,\ g/d + bf/(ad) + c/a - 4 = B, \\ 3d/f - dh/f^2 - cd/(af) = C,\ bgf/(ad^2) - 3bf/(ad) = D, \\ cg/(ad) - 2 + g/d + 3bf/(ad) - bh/(ad) = E \\ \text{and } P(P-1) + CP + ch/(af) - h/f + 1 - c/a = 0 \end{gathered} \tag{2.13}$$

is obtained by applying the computer algebra package MAPLE V. Taking into account the equations (2.11) and (2.12), after eliminating A, B, C, D and E, we obtain the results

$$a = -bf(d\chi),\ b = c\chi d^2/f/[g - \chi d - 4d - (2P+1+\alpha+\beta)d],\ d = g/(P-\kappa-3),$$
$$h = (g - d + \chi d - (2P + a + \alpha + \beta)d - (\gamma - 2P)f)f/d,$$

and

$$\begin{gathered} g = (\gamma - 2P)f\chi^2(3\chi + \chi(P-\kappa) + \\ + 2\chi^3 - 3(2P + a + \alpha + \beta)\chi^2 + \chi^2(P-\kappa)^2 + 2\chi^3(P-\kappa) \\ -\chi^2(P-\kappa)(2P + a + \alpha + \beta) = \chi^4 - \chi^3(2P + a + \alpha + \beta) - \\ - ((2P + 1 + \alpha + \beta) - \alpha\beta)\chi^2. \end{gathered} \tag{2.14}$$

The parameter P is determined by the cubic equation

$$\begin{gathered} 3P^3 + (10 + 5\alpha + 5\beta + 3\kappa - 5\chi)P^2 + \\ +(2+12\alpha+12\beta+2\alpha^2+2\beta^2+5\alpha\beta-\kappa^2+[7+\alpha+\beta]\kappa = [\gamma-11-4\alpha-4\beta]\chi)p+ \\ +(7+2\alpha+2\beta-\alpha^2-\beta^2-\alpha\beta)\kappa-\kappa^2-\kappa^3+(2\kappa^2+2[\alpha+\beta]\kappa-\alpha\beta-\alpha-\beta-8)\chi+ \\ + (1-\kappa)\chi^2 = 0, \end{gathered} \tag{2.15}$$

the solutions of which give rise to three families of the quantities given by (2.14). The algebraic assumption made in connection with (2.8) is seen to be justified. The parameters c and f are disposable.

3. The solution of (2.1) and (2.2).

We recall that (2.2) is

$$dtv'' + (ft + g)v' + hv = 0; \tag{3.1}$$

put

$$t = -dz/f, \tag{3.2}$$

when we have the confluent hypergeometric equation

$$zv'' + (g/d - z)v' - hv/f = 0. \tag{3.3}$$

Two solutions of this equation are suitable in the present context, namely,

$$v = \psi(h/f;\, g/d;\, ft/d) = \tag{3.4}$$
$$= t^{-h/f} \int_{-i\infty}^{i\infty} \Gamma(-w)\Gamma(h/f + w)\Gamma(h/f - g/d + 1 + w)(-dt/f)^{-w}\, dw,$$

a Barnes integral representation and where ψ is a confluent hypergeometric function of the second kind. Another form of v is and

$$\begin{aligned} v &= t^{1-g/d}\phi(h/f;\, g/d;\, ft/d) \\ &= t_1^{1-g/d} F_1[1 + h/f - g/d; 2 - g/d; -ft/d], \end{aligned} \tag{3.5}$$

where ϕ is a confluent hypergeometric function of the first kind.
The corresponding series representation of (3.5) is a ${}_2F_0$ series, which does not converge. See Erdélyi [2], Chapter 6.
By means of the usual method of solution of the linear differential equation of the first order, we have from (2.1)

$$u = (t + b/a)^{-c/a} \int^{t} (t + b/a)^{c/a-1} v\, dt. \tag{3.6}$$

4. A solution of the CHE near to the origin with zero exponent.

Let v be given by (3.4), and by appealing to de la Valleé Poussin's theorem (Bromwich [1], p. 504) bearing in mind the absolute convergence of the integrals concerned, using the usual technique of the solution of linear differential equations of the first order,

$$u = (t + b/a)^{-c/a} \cdot \tag{4.1}$$

$$\cdot \int_{-i\infty}^{i\infty} T(-w)\Gamma(h/f + w)\Gamma(h/f - g/d + 1 + w)(-f/d)^{w} \cdot$$

$$\cdot \left[\int^{t} (t + b/a)^{c/a-1} t^{-h/f-w} dt \right] dw.$$

Then path of integration in the w-plane is dformed so that the poles of $\Gamma(-w)$ lie outside of the contour while any poles of the other gamma functions of the integrand are outside of the contour.
The binomial factors are now expanded in descending powers of t, and noting that the resulting series are uniformly convergent, the indefinite integration with respect to t is carried out term-by-term, so that

$$u = \sum_{m,n} [(c/a, m)(1 - c/a, n)(-b/a)^{m+n} t^{-h/f-m-n}]/[m!n!] \cdot \tag{4.2}$$

$$\cdot \int_{-i\infty}^{i\infty} \Gamma(-w)\Gamma(h/f + w) \cdot$$

$$\cdot \frac{\Gamma(h/f - g/d + 1 + w)\Gamma(h/f - c/a + n + w)(-ft^{-1}/d)^{w}}{\Gamma(h/f - c/a + 1 + n + w)} dw.$$

Paths of integration in the w-plane are similarly deformed to that of the w-integral of (4.1) From (2.4), by inversion, it follows that

$$y(x) = \int \exp(xt) u(t)\, dt, \tag{4.3}$$

when

$$y(x) = \sum_{m,n} [(c/a, m)(1 - c/a, n)(-b/a)^{m+n}]/[m!n!] \cdot \tag{4.4}$$

$$\cdot \int_{-i\infty}^{i\infty} \frac{\Gamma(-w)\Gamma(h/f + w)\Gamma(h/f - c/a + n + w)}{\Gamma(h/f - c/a + 1 + n + w)} \cdot$$

$$\cdot \left[\int \exp(xt) t^{-h/f-m-n-w}\, dt \right] dw.$$

Let the contour of integration in the t-plane of (4.4) consist of a simple loop beginning and ending at $-\infty$ if $\mathrm{Re}(x) > 0$ or at ∞ if $\mathrm{Re}(x) < 0$, and encircling the origin once. Hence the inner integral in t is proportional to

$$x^{h/f-1+m+n+w}/\,\Gamma(-h/f+m+n+w) \tag{4.5}$$

and if this result is inserted into (4.4), it is evident that

$$\begin{aligned} y(x) = x^{h/f-1} \sum_{m,n} (c/a,m)(1-c/a,n)(-bx/a)^{m+n}/[m!n!]\cdot \\ \cdot \int_{-i\infty}^{i\infty} \Gamma(-w)\Gamma(h/f+w)\Gamma(h/f-g/d+1+w)\cdot \\ \cdot \frac{\Gamma(h/f-c/a+n+w)(-fx/d)^w}{\Gamma(h/f-c/a+1+n+w)\Gamma(-h/f+m+n+w)}\,dw. \end{aligned} \tag{4.6}$$

The inner Barnes integral is now evaluated by summing the residues of its integrand, and after taking note of (2.7) and (2.10), we have, apart from a trivial constant factor,

$$\begin{aligned} Y(X) = X^{-P+h/f-1} \sum_{m,n,q} (c/a,m)(1-c/a,n)(h/f-c/a,n+q) \\ \cdot \frac{(h/f,q)(h/f-g/d+1,q)(-bd/(af)X)^{m+n}X^q}{(h/f-c/a+1,n+q)(-h/f,m+n+q)m!n!q!}. \end{aligned} \tag{4.7}$$

If the relations (2.13) are taken into account, this convergent triple series is a solution of the CHE, and recalling that the parameter f is disposable, this may be so selected that the exponent of (4.7) vanishes, and the solution of the CHE relative to the origin with zero exponent is obtained.

5. A global solution of the CHE.

If the form of v as given by (3.4) is now used on expanding the functions of the integrand of (3.5) in ascending powers of t, term-by-term integration gives the expression

$$\begin{aligned} u(t) = (t+a/b)^{-c/a} \sum_{m,n} (1-c/a,m)(1+h/f-g/d,n)\cdot \\ \cdot \frac{(2-g/d,m+n)(-a/b)^m(-f/d)^n t^{m+n}}{(2-g/d,n)(3-g/d,m+n)m!n!}. \end{aligned} \tag{5.1}$$

The exponential function of the Laplace integral (4.3) is also expanded in ascending powers of t, so that

$$y(x) = \sum_{m,n,q} (1 - c/a, m)(1 + h/f - g/d, n) \cdot \tag{5.2}$$

$$\cdot \frac{(2 - g/d, m + n)(-a/b)^m (-f/d)^n x^q}{(2 - g/d, n)(3 - g/d, m = n) m! n! q!} \int t^{2-g/d+m+n+q} (t + b/a)^{c/d} \, dt.$$

The contour of integration is now taken to be a Pochahmmer double-loop slung around the point b/a and the origin in the t-plane, when the inner integral of (5.2) is found to be proportional to

$$\frac{(-b/a)^{m+n+q}}{\Gamma(g/d - 2 - m - n - q)\Gamma(4 - g/d - c/a + m + n + q)}, \tag{5.3}$$

so that the corresponding solution of the CHE is

$$Y(X) = X^{-P} \sum_{m,n,q} (1 - c/a, m)(1 + h/f - g/d, n) \cdot \tag{5.4}$$

$$\cdot \frac{(2 - g/d, m + n)(3 - g/d, m + n + q)(-b/a)^n (bd/(af)X)^q}{(2 - g/d, n)(3 - g/d, m + n)(4 - g/d - c/a, m + n + q) m! n! q!}.$$

The summation relative to the index of summation m involves a hypergeometric series of unit argument and unit radius of convergence, namely

$$\begin{aligned} &{}_3F_2[1 - c/a, 2 - g/d + n, 3 - g/d + n + q; \\ &\qquad 3 - g/d + n, 4 - g/d - c/a + n + q; 1], \end{aligned} \tag{5.5}$$

which must be examined for convergence.
With reference to Slater [8], page 45, the function

$$ {}_3F_2[a_1, a_2, a_3; b_1, b_2; 1] \tag{5.6}$$

converges absolutely if

$$\mathrm{Re}(b_1 + b_2 - a_1 - a_2 - a_3) > 0 \tag{5.7}$$

and diverges otherwise. In the case of (5.5), the convergence condition (5.7) is clearly satisfied. It the follows that the representation (5.4) converges throughout the whole of the X-plane and is the global solution of the CHE sought.

6. A formal solution of the CHE relative to its irregular singularity at infinity.

If the binomial factor of (5.1) is expanded in ascending powers of t, we have *formally*

$$(6.1)\qquad y(x) \sim \sum_{m,n,q} (1-c/a,m)(1+h/f-g/d,n)\cdot$$
$$\cdot\frac{(2-g/d,m+n)(-a/b)^{m+q}(-f/d)^n}{(2-g/d,n)(3-g/d,m+n)m!n!q!}\int \exp\,(xt)t^{2-g/d+m+n+q}\,dt.$$

On taking the contour of integration to be the same as that used in connection with (4.4), the inner integral is seen to be proportional to

$$(6.2)\qquad x^{g/d-3-m-n-q}/\,\Gamma(g/d-2-m-n-q).$$

Hence,

$$(6.3)\qquad Y(X) \sim X^{g/d-3-P} \sum_{m,n,q} (1-c/a,m)(1+h/f-g/d,n)\cdot$$
$$\cdot\frac{(2-g/d,m+n)(3-g/d,m+n+q)}{(2-g/d,n)(3-g/d,m+n)m!n!q!}(ad/(bfX))^m X^{-n}(-d/(fX))^q.$$

This results is a formal representation of a solution of the CHE near its irregular singularity at infinity, and, as expected, does not converge as it stands. If, however

$$(6.4)\qquad g/d = N+3, \quad N = 0,1,2,\ldots,$$

a polynomial solution results. The expression (6.4) is an eigenfunction equation and its solutions are related to the equations (2.13), including the cubic equation in P.

7. Applications.

The following Schrödinger equations are examples of cases which can be reduced to forms of the CHE:

$$(7.1)\qquad y'' + [2E + 2Ze^2/(r+b) - q(q+1)/r^2]y = 0,$$

$$(7.2)\qquad y'' + [E + r^2 + \lambda r^2/(1+gr^2)]y = 0,$$

and

$$(7.3) \qquad (r^2-1)y' + [-\lambda r^2 + 2Dr - m^2/(r^2-1) + \mu]y = 0.$$

The equation (7.1) arises in connection with a model of the potential due to a smeared charge (Exton [4]), (7.2) quantum optics (Roy, Roychoudary and Roy [7]) and (7.3) the hydrogen molecule-ion (Leaver [5]). Teukolsky's equations which occur in the theory of black holes can also be shown to be reducible to the CHE (Leaver [5]).

REFERENCES

[1] T. Bromwich, *Infinite series,* Macmillan, London, 1931.

[2] A. Erdélyi, *Higher transcendental functions,* Vol. I, McGraw Hill, New York, 1953.

[3] H. Exton, *On the confluent Heun equation* $[0, 2, 1_2]$, Ann. Soc. Sci. Bruxelles, 105 (1991), pp. 3–15.

[4] H. Exton, *The interaction* $V(r) = -Ze^2/(r+b)$ *and the confluent Heun equation,* J. Phys., A 24 (1991), pp. L329–L330.

[5] E.W. Leaver, *Solution to a generalized spheroidal wave equation: Teukolsky's equation in general relativity and the two-center problem in molecular quantum mechanics,* J. Math. Phys., 27 (1986), pp. 1238–1265.

[6] A. Ronveaux, *Heun's differential equations,* Oxford University Press, 1995.

[7] B. Roy - R. Roychoudhury - P. Roy, *Shifted* $1/N$ *approach to the interaction* $V(r) = r^2 + \lambda r^2/(1+gr^2)$, J. Phys., A 21 (1988), pp. 1579–1588.

[8] L.J. Slater, *Generalized hypergeometric functions,* Cambridge University Press, 1966.

[9] S. Yu Slavyanov, *Confluent Heun equation,* Part B of Ronveaux [6].

"Nyuggel",
Lunabister, Dunrossness,
Shetland ZE2 9JH (UNITED KINGDOM)