SEQUENTIAL ORDERS OF ADJUNCTION SPACES

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Let $X$, $Y$ be two disjoint spaces, $M$ be a closed subset of $X$, and $f : M \to Y$ be a continuous map. In the direct sum $X \oplus Y$ of $X$ and $Y$, define an equivalence relation $\sim$ by $a \sim f(a)$ for each $a \in M$. The quotient space $X \oplus Y/ \sim$, is denoted by $X \sqcup_f Y$, usually called the adjunction space determined by $X$, $Y$ and $f$. In this paper we prove that for two sequential spaces $X$ and $Y$, $so(X \sqcup_f Y) \leq so(X) + so(Y)$ and, if $so(X \sqcup_f Y) > \max\{so(X), so(Y)\}$ and $so(X) \leq \omega$, then there exists a special map $p : S_2 \to X \sqcup_f Y$, where $so(X)$ denotes the sequential order of $X$ and $S_2$ is the Arens’ space. We also give an answer for a question of Kannan [4].

1. Introduction.

In [1], Arhangel’skii and Franklin constructed sequential spaces of its sequential order $\alpha$ for any $0 \leq \alpha \leq \omega_1$. It was done by attaching a sequential space to a sequential space by a continuous map. In Section 2, we give the relations between sequential orders of attaching space and original spaces. In Section 4, we answer a question of Kannan in [4].

**Definition.** Let $X$, $Y$ be two disjoint spaces, $M$ be a closed subset of $X$, and $f : M \to Y$ be a continuous map. In the direct sum $X \oplus Y$ of $X$ and $Y$, 

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define an equivalence relation $\sim$ as follows: if $f(a) = f(b)$ then $a, b, f(b)$ are equivalent. The quotient space $X \oplus Y/\sim$, is denoted by $X \cup_f Y$, usually called the adjunction space determined by $X$, $Y$ and $f$. If $a \in X \setminus M$, we denote by $a$ the equivalent class of $a$ when confusion does not occur. It is well-known that if $X$ and $Y$ are paracompact(normal), then $X \cup_f Y$ is also paracompact(normal). Nevertheless, a simple example shows that the Hausdorffness of $X$ and $Y$ does not imply that $X \cup_f Y$ is Hausdorff.

This indicates that the topological property what both of $X$ and $Y$ have, may not be transformed in $X \cup_f Y$.

Throughout this paper, we use $q$ to denote the naturally quotient map from $X \oplus Y$ to $X \cup_f Y$, and $N$ to denote the set of natural numbers. As a topological space, $N$ has the discrete topology.

For a subset $A$ of a topological space $X$, we denote by $\overline{A}^X$ (resp. $[A]_X^{seq}$) the closure (resp. sequential closure, i.e., the set of limits of convergent sequences consisting of points of $A$) of $A$ in $X$. We shall write $\overline{A}$ (resp. $[A]^{seq}$) for $\overline{A}^X$ (resp.$[A]_X^{seq}$) when confusion does not occur. A space $X$ is sequential if, whenever $A \subseteq X$ and $A$ is not closed, there is a sequence from $A$ converging to a point outside the set $A$, and $X$ is Fréchet if, whenever $x \in \overline{A}$, there is a sequence from $A$ converging to $x$.

Let $A$ be a subset of a space $X$.

We define $[A]_\alpha^X$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]^X_0 = A$, $[A]^{X+1}_\alpha = ([A]^{X}_{\alpha})^{seq}_X$ and $[A]^{X}_\alpha = \cup\{[A]^{X}_\beta : \beta < \alpha\}$ for a limit $\alpha$. We shall write $[A]_\alpha$ for $[A]^{X}_\alpha$ when confusion does not occur. One can easily see that $[A]^{X+1}_{\omega_1} = [A]^{X}_{\omega_1}$, and that a space is sequential if and only if $\overline{A} = [A]^{X}_{\omega_1}$ for all subsets $A$ of $X$. For a sequential space $X$ we define $so(X)$, the sequential order, by $so(X) = \min\{\alpha \in \omega_1 + 1 : \overline{A} = [A]_{\alpha}$ for every $A \subseteq X\}$. Obviously, if $X$ is a Fréchet space, then $so(X) \leq 1$.

It is straightforward that if $X$ and $Y$ are both sequential spaces, then so is $X \cup_f Y$. Nevertheless, for two Fréchet spaces $X$ and $Y$, $X \cup_f Y$ need not be Fréchet, but, as is shown in the sequel, $so(X \cup_f Y) \leq 2$.

2. Main results.

We first recall a well-known fact about the space $X \cup_f Y$ (cf. Theorem 6.3 of [3]) which is frequently used in the sequel.

**Theorem 2.0** ([2]). Let $X, Y$ be two disjoint spaces. Then:

(I) $Y$ is embedded as a closed set in $X \cup_f Y$, and the restriction of $q$ to $Y$ is a homeomorphism.
(2) $X \setminus A$ is embedded as open set in $X \cup_f Y$, and the restriction of $q$ to $X \setminus A$ is a homeomorphism.

**Theorem 2.1.** Let $X$, $Y$ be two disjoint sequential spaces. Then

$$so(X \cup_f Y) \leq so(X) + so(Y).$$

**Corollary 2.2.** Let $X$, $Y$ be two disjoint Fréchet spaces. Then

$$so(X \cup_f Y) \leq 2.$$

**Theorem 2.3.** Let $X$, $Y$ be two disjoint sequential spaces, $M$ a closed subset of $X$, $f : M \to Y$ a continuous mapping. If $f(\overline{\Lambda} \cap M)$ is closed in $Y$ for every $A \subseteq X \setminus M$, then

$$so(X \cup_f Y) \leq \max\{so(X), so(Y)\}.$$

**Corollary 2.4.** Let $X$, $Y$ be two disjoint sequential spaces and let $M$ be a closed subset of $X$. If $M$ is countably compact, then $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$.

**Proof.** From the countable compactness of $A$ and sequentiality of $Y$, it follows that $f$ is closed. According to Theorem 2.3, $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$.

**Corollary 2.5.** Let $X$, $Y$ be two disjoint Fréchet spaces, If $X$ is countably compact, then $X \cup_f Y$ is also Fréchet.

**Remark.** Obviously, the converses of Theorem 2.3, Corollary 2.4 and 2.5 need not be true.

**Theorem 2.6.** Let $X$ be a Hausdorff Fréchet space, $Y$ be a Fréchet $T_1$-space, $M$ be a closed subset of $X$ and let $f : M \to Y$ be continuous. Then, $X \cup_f Y$ is Fréchet if and only if $f(\overline{\Lambda} \cap M)$ is closed in $Y$ for every $A \subseteq X \setminus M$.

As we have showed above the sequential order of $X \cup_f Y$ is suppressed by the sum of $so(X)$ and $so(Y)$. On the other hand, by Theorem 2.3, if $f$ is closed, then $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$. Therefore it is natural to ask when is $X \cup_f Y$ really large than both of $so(X)$ and $so(Y)$. The following theorem give a necessary condition for the question when $so(X) \leq \omega$. 
Theorem 2.7. Let $X, Y$ be two disjoint Hausdorff sequential spaces, $M$ be a closed subset of $X$, $f : M \rightarrow Y$ be a continuous mapping and let $so(X) \leq \omega$. If
\[ so(X \cup_f Y) > \max\{so(X), so(Y)\}, \]
then there exists an embedding map $p : S_2 \hookrightarrow X \cup_f Y$ such that
\[ \{p(t_n) : n \in \mathbb{N}\} \subseteq q(M) \]
and
\[ \{p(t_{mn}) : n, m \in \mathbb{N}\} \subseteq X \setminus M. \]

Recall the definition of $S_2$ (see also example 1.6.19 of [3]).

Let $T = \{t_n : n \in \mathbb{N}\}$ be a sequence converging to $t_0 \notin T$. Then $S_2$ is the space obtained by attaching the space $N \times \{t_n : n \in \omega\}$ to the space $T \cup \{t_0\}$ by the continuous map $f : \{(n, t_0) : n \in \mathbb{N}\} \rightarrow \{t_n : n \in \mathbb{N}\}$ defined by $f((n, t_0)) = t_n$ for all $n \in \mathbb{N}$. For convenience, we write $t_{mn}$ for $(n, t_m)$.

Corollary 2.8. Let $X, Y$ be two disjoint Hausdorff Fréchet spaces, $M \subseteq X$ a closed subset and $f : M \rightarrow Y$ a continuous map. Then the following conditions are equivalent:

1. $so(X \cup_f Y) = 2$;
2. there exists an embedding map $p : S_2 \hookrightarrow X \cup_f Y$ such that
\[ \{p(t_n) : n \in \mathbb{N}\} \subseteq q(M) \]
and
\[ \{p(t_{mn}) : n, m \in \mathbb{N}\} \subseteq X \setminus M. \]

Question. (a) Let $X, Y$ be two disjoint sequential spaces. Then, does
\[ so(X \cup_f Y) \leq so(Y) + so(X) \]
hold?

(b) In Theorem 2.7, whether the condition of $so(X) \leq \omega$ can be removed?
3. The proofs of theorems.

Lemma 3.1. Let $X$, $Y$ be two topological spaces. Let $f : X \to Y$ be a continuous map. Then, for any $A \subseteq X$ and ordinal number $\alpha$,

$$f([A]_\alpha) \subseteq [f(A)]_\alpha.$$  

Proof. We show Lemma 3.1 by induction.

Suppose that $f([A]_\beta) \subseteq [f(A)]_\beta$ for all $\beta < \alpha$.

If $\alpha$ is a limit, then

$$f([A]_\alpha) = \bigcup_{\beta < \alpha} f([A]_\beta) \subseteq \bigcup_{\beta < \alpha} [f(A)]_\beta = [f(A)]_\alpha.$$  

If $\alpha$ is not a limit, then $\alpha = \beta + 1$ for some $\beta < \alpha$. Fix $y \in f([A]_{\beta + 1})$. Then $y = f(x)$ for some $x \in [A]_{\beta + 1}$. Thus there is a sequence $\{x_i : i < \omega\}$ in $[A]_\beta$ such that $x_i \to x$ as $i \to \infty$. Since $f$ is continuous, $f(x_i) \to f(x)$ as $i \to \infty$. By the supposition, we have $\{f(x_i) : i < \omega\} \subseteq [f(A)]_\beta$. Therefore $f(x) \in [f(A)]_{\beta + 1}$ which completes the proof.

Lemma 3.2. Let $X$, $Y$ be two disjoint topological spaces. Also let $A \subseteq X \setminus M$. If $z = q(y) \in \overline{A}^{X \cup Y}$ for some $y \in Y$, then, $y \in f(\overline{A}^{X \cap M})$.

Proof. Let $V$ be a neighbourhood open in $Y$ of $y$. To complete the proof, it suffices to show that $f^{-1}(V) \cap \overline{A}^X \neq \emptyset$. It is obvious that $V \cap f(M) \neq \emptyset$. Therefore, $f^{-1}(V) \neq \emptyset$. Suppose $f^{-1}(V) \cap \overline{A}^X = \emptyset$. Then there is an open subset $U'$ of $X$ containing $f^{-1}(V)$ such that $U' \cap A = \emptyset$.

On the other hand, since $f$ is continuous, there is an open subset $U''$ of $X$ such that $f^{-1}(V) = U'' \cap M$. Let $U = U' \cap U''$ and $W = q(U \cup V)$. It is easy to see that $W \cap A = \emptyset$. If we can show that $W$ is an open neighbourhood of $z$ in $X \cup Y$, then this completes the proof of Lemma 3.2, because it contradicts $z \in \overline{A}^{X \cup Y}$. Obviously, $z \in W$. Notice that if $x \in M \setminus U$ and $q(x) \in W$, then $f(x) \in V$. In fact, there is $w \in U \cup V$ such that $q(w) = q(x)$. If $w \in U$, then $w \in M$. Since $f^{-1}(V) = M \cap U$, it follows that $w \in f^{-1}(V)$. So $x \in f^{-1}(V)$. Hence, $q^{-1}(W) = U \cup V$.

The proof of Theorem 2.1. Let $\text{so}(X) = \alpha$ and $\text{so}(Y) = \beta$. We will show that $\text{so}(X \cup_f Y) \leq \alpha + \beta$. Let $k = q|_Y$ and $j = q|_X$ be the restrictions of $q$ to $X$ and $Y$ respectively. Now let us fix $A \subseteq X \cup_f Y$ and $z \in \overline{A}^{X \cup_f Y}$. It is easy to see that $z \in A \cap (X \setminus M)^{X \cup_f Y} \cup A \cap k(Y)^{X \cup_f Y}$. Now we prove that $z \in [A]_{\alpha + \beta}$.  

Case 1. \( z \in \overline{A \cap k(Y)^{X \cup_f Y}} \).

By Theorem 2.0, \( k(Y) \) is a closed subset of \( X \cup_f Y \). Therefore, \( z \in \overline{A \cap k(Y)} \). By the facts that \( k(Y) \) and \( Y \) are homeomorphic (Theorem 2.0) and that \( \text{so}(Y) = \beta \), one has

\[
z \in [A \cap k(Y)]^Y_{\beta} \subseteq [A]^Y_{\beta} \subseteq [A]^{X \cup_f Y}_{\alpha+\beta}.
\]

Case 2. \( z \in \overline{A \cap (X \setminus M)^{X \cup_f Y}} \).

If \( z \in X \setminus M \), then \( z \in \overline{A \cap (X \setminus M)^{X \cup_f M}} \) because \( X \setminus M \) is embedded in \( X \cup_f Y \) as an open subspace, and so \( z \in A \cap (X \setminus M)^{X} \). Since \( \text{so}(X) = \alpha \), we have \( z \in [A \cap (X \setminus M)]^Y_{\alpha} \), and so, by Lemma 3.1,

\[
z \in [A \cap (X \setminus M)]^Y_{\alpha} \subseteq [A]^{X \cup_f Y}_{\alpha+\beta}.
\]

If \( z \notin X \setminus M \), then \( z = k(y) \) for some \( y \in Y \). Thus,

\[
z = q(y) \in q(\overline{f(A \cap (X \setminus M)^{X \cup_f Y} \cap M)}) \quad \text{(by Lemma 3.2)}
\]

\[
= q(\overline{f(A \cap (X \setminus M)^{X \cup_f Y} \cap M)})^Y_{\beta}
\]

\[
\subseteq [q(\overline{f(A \cap (X \setminus M)^{X \cup_f Y} \cap M)})^Y_{\beta}]^{X \cup_f Y} \quad \text{(by Lemma 3.1)}
\]

\[
= [q(A \cap (X \setminus M)^{X \cup_f Y})]^Y_{\beta}
\]

\[
= [q([A \cap (X \setminus M)]^Y_{\alpha} \cup [A \cap (X \setminus M)]^Y_{\beta})]^{X \cup_f Y}
\]

\[
\subseteq [[q(A \cap (X \setminus M)]_{\alpha}^{X \cup_f Y} \cup [q(A \cap (X \setminus M)]_{\beta}^{X \cup_f Y})]^{X \cup_f Y} \quad \text{(by Lemma 3.1)}
\]

\[
= [A \cap (X \setminus M)]^{X \cup_f Y}_{\alpha+\beta}
\]

\[
\subseteq [A]^{X \cup_f Y}_{\alpha+\beta}.
\]

This completes the proof of Theorem 2.1.

Lemma 3.3. Let \( X \) and \( Y \) be two topological spaces, \( M \) be a closed subset of \( X \) and let \( f : M \longrightarrow Y \) be continuous. Also, let \( A \subseteq X \setminus M \) be such that \( f(\overline{A \cap M}) \) is closed in \( Y \). Then, \( q(\overline{A}^X) = \overline{A}^{X \cup_f Y} \).
Proof. Let \( z \in \overline{A}^{X_{\cup Y}} \). If \( z \in X \setminus M \), then by Theorem 2.0, \( z \in \overline{A}^{X,M} \subseteq \overline{A}^X \). Thus \( z = q(z) \in q(\overline{A}^X) \). If \( z = q(y) \in \overline{A}^{X_{\cup Y}} \cap q(Y) \) where \( y \in Y \), by Lemma 3.2, \( y \in f(A \cap M) \). Since \( f(A \cap M) \) is closed in \( Y \), there exists \( x \in A \cap M \) such that \( f(x) = y \), hence \( z \in q(\overline{A}^X) \).

The proof of Theorem 2.3. Take \( B \subseteq X \cup_f Y \). Then

\[
[B]^{X_{\cup Y}}_{\text{tot}(X_{\cup Y})} = B^{X_{\cup Y}}_{\text{tot}(X_{\cup Y})} = \frac{B \cap (X \setminus M)^X_{\text{tot}(X_{\cup Y})} \cup B \cap q(Y)^X_{\text{tot}(Y)}}{q((B \cap (X \setminus M)^X_{\text{tot}(X)}) \cup [B \cap q(Y)]^{X_{\cup Y}}_{\text{tot}(Y)} \subseteq [B \cap (X \setminus M)]^{X_{\cup Y}}_{\text{tot}(X)} \cup [B \cap q(Y)]^{X_{\cup Y}}_{\text{tot}(Y)} \subseteq [B]^{X_{\cup Y}}_{\text{max}(\text{tot}(X), \text{tot}(Y))} = [B]^{X_{\cup Y}}_{\text{max}(\text{tot}(X), \text{tot}(Y))}.}
\]

Lemma 3.4. Let \( X \) be a Hausdorff space, \( Y \) be a \( T_1 \)-space, \( M \) be a closed subset of \( X \) and let \( f : M \rightarrow Y \) be continuous. Suppose that \( \{x_n : n \in \mathbb{N}\} \subseteq X \setminus M \) is a sequence which is convergent in \( X \cap Y \) to a point \( q(y) \) where \( y \in Y \). Then there is \( x \in \{x_n : n \in \mathbb{N}\}^X \cap M \) such that \( y = f(x) \).

Proof. By Lemma 3.2, \( \{x_n : n \in \mathbb{N}\} \) is not a closed subset of \( X \). Therefore, in particular, \( \{x_n : n \in \mathbb{N}\}^X \setminus \{x_n : n \in \mathbb{N}\} \neq \emptyset \), because \( X \) is sequential. As \( X \) is \( T_1 \), there exists a point \( x \in \{x_n : n \in \mathbb{N}\}^X \setminus \{x_n : n \in \mathbb{N}\} \) and a subsequence \( \{x_{k_n} : n \in \mathbb{N}\} \) of \( \{x_n : n \in \mathbb{N}\} \) such that \( x_{k_n} \rightarrow x \) as \( n \rightarrow \infty \). Since \( X \) is Hausdorff, we have \( \{x_{k_n} : n \in \mathbb{N}\}^X = \{x_{k_n} : n \in \mathbb{N}\} \cup \{x\} \). Note now that \( q(y) \in \overline{\{x_{k_n} : n \in \mathbb{N}\}^{X_{\cup Y}}} \). Hence, by Lemma 3.2, \( y \in f(\overline{\{x_{k_n} : n \in \mathbb{N}\}} \cap M) = \overline{\{f(x)\}} \). Since \( Y \) is \( T_1 \), it follows that \( y = f(x) \).

The proof of Theorem 2.6. By Theorem 2.3, we only need to show the necessary.
Let \( A \subseteq X \setminus M \). Suppose \( \overline{A}^X \cap M \neq \emptyset \), and let \( y \in f(\overline{A}^X \cap M) \). Therefore,

\[
q(y) \in q\left(f(\overline{A}^X \cap M)\right) = q(\overline{A}^X)^{X \cup_f Y}
\]

\[
\subseteq q(\overline{A}^X)^{X \cup_f Y} \quad \text{(by the definition of } q)\n\]

\[
\subseteq q(A)^{X \cup_f Y}^{X \cup_f Y} \quad \text{(by the continuity of } q)\n\]

\[
= \overline{A}^{X \cup_f Y}.
\]

Since \( X \cup_f Y \) is Fréchet, there exists a sequence \( \{x_n : n \in \mathbb{N}\} \) from \( A \) such that \( \{x_n : n \in \mathbb{N}\} \) is convergent in \( X \cup_f Y \) to the point \( q(y) \). By Lemma 3.4, this implies that \( y \in f(\overline{A} \cap M) \).

**Lemma 3.5.** Let \( X \) be a Hausdorff space, \( Y \) be a \( T_1 \)-sequential space, \( M \) be a closed subset of \( X \) and let \( f : M \rightarrow Y \) is continuous. Also let \( A \subseteq X \setminus M \). If

\[
\alpha = \min\{\beta : [A]^Y_{X \cup_f Y} \cap q(Y) \text{ is not closed in } q(Y)\},
\]

then \( [A]^Y_{X \cup_f Y} \cap q(Y) \subseteq q(M) \).

**Proof.** Take a point \( q(y) \in [A]^Y_{X \cup_f Y} \cap q(Y) \) where \( y \in Y \). Since \( y \notin A \), the following ordinal is well-defined:

\[
\beta(y) = \min\{\beta : q(y) \in [A]^Y_{X \cup_f Y} \setminus [A]^Y_{X \cup_f Y} \}.
\]

Now, \( q(y) \in [A]^Y_{X \cup_f Y} \setminus [A]^Y_{X \cup_f Y} \) implies the existence of a sequence \( \{x_n : n \in \mathbb{N}\} \) from \( [A]^Y_{X \cup_f Y} \) which is convergent in \( X \cup_f Y \) to the point \( q(y) \). In the case, the set \( \{n : x_n \notin X \setminus M\} \) is finite. Indeed, otherwise we can find a subsequence \( \{x_{k_n} : n \in \mathbb{N}\} \) of \( \{x_n : n \in \mathbb{N}\} \cap q(Y) \) such that \( x_{k_n} \rightarrow q(y) \) as \( n \rightarrow \infty \). However, this will finally imply that \( q(y) \in [A]^Y_{X \cup_f Y} \beta(y) \) because, by construction, \( \beta(y) < \alpha \) and \( [A]^Y_{X \cup_f Y} \cap q(Y) \) is closed. So, there is \( n_0 \in \mathbb{N} \) such that \( \{x_n : n \geq n_0\} \subseteq X \setminus M \). By Lemma 3.4, it follows that \( y \in f(\overline{A} \cap M) \).

The proof of Theorem 2.7. Let \( \max\{\text{so}(X), \text{so}(Y)\} = \alpha \). Since \( \text{so}(X \cup_f Y) > \alpha \), there exists \( A \subseteq X \cup_f Y \) such that

\[
\overline{A}^{X \cup_f Y} \setminus [A]^Y_{X \cup_f Y} \neq \emptyset.
\]
We pick
\[ z \in A^{Y \cup Y} \setminus [A]^Y[X \cup Y]. \]

Since
\[ A^{Y \cup Y} = A \cap q(Y)^{Y \cup Y} \cup A \cap (X \setminus M)^{Y \cup Y}, \]
by Theorem 2.0 and the hypothesis of \( \max\{so(X), so(Y)\} = \alpha \) we have
\[ z = q(y) \in A \cap (X \setminus M)^{Y \cup Y} \setminus [A \cap (X \setminus M)]^Y[X \cup Y] \]
where \( y \in Y \). For the convenience, without loss of generality, we may write \( A \cap (X \setminus M) = A \).

**Claim 1.** \([A]^Y \cup Y \cap q(Y)\) is not closed in \( q(Y) \).

Since \( X \cup Y \) is sequential, by (\#), there exists a sequence \( \{z_n : n \in \mathbb{N}\} \) from \([A]^Y \cup Y \cap (X \setminus M)\) converging to a point \( z' = q(y) \) outside \([A]^Y \cup Y \cap (X \setminus M)\), where \( y \in Y \). If \( \{z_n : n \in \mathbb{N}\} \cap (X \setminus M) \) is infinite, then, by Lemma 3.4, there is a subsequence \( \{z_{k_n} : n \in \mathbb{N}\} \) of \( \{z_n : n \in \mathbb{N}\} \) and \( x \in X \) such that \( x \in \{z_{k_n} : n \in \mathbb{N}\} \) and \( y = f(x) \).

On the other hand, for each \( n \in \mathbb{N} \), \( z_{k_n} \in [A]^Y \cup Y \cap (X \setminus M) = [A]^{Y \setminus M} \subseteq [A]^Y \cup Y \cap (X \setminus M) \). Therefore, \( x \in [A]^Y \cup Y \cap (X \setminus M) \). Thus, by Lemma 3.1, \( q(y) = q(x) \in [A]^Y \cup Y \cap (X \setminus M) \) which is a contradiction. So, \( \{z_n : n \in \mathbb{N}\} \cap q(Y) \) is infinite, this completes the proof of Claim 1.

Now let us define
\[ \beta = \min\{\alpha : [A]^{Y \cup Y} \cap q(Y) \text{ is not closed in } q(Y)\}. \]

By Lemma 3.5, \([A]^{Y \cup Y} \cap q(Y) \subseteq q(M)\).

On the other hand, since \( X \cup Y \) is sequential, we can choose
\[ z_0 \in [[A]^{Y \cup Y} \cap q(Y)]^{Y \cup Y} \setminus [A]^{Y \cup Y} \cap q(Y). \]

There is a sequence \( \{z_n : n \in \mathbb{N}\} \) from \([A]^{Y \cup Y} \cap q(Y) \subseteq q(M)\) converging to \( z_0 \). Since \( X \) and \( Y \) are \( T_1 \), so is \( X \cup Y \). Hence \( \{z_n : n \in \mathbb{N}\} \) is an infinite subset. As \( Y \) is Hausdorff, there exists a subsequence \( \{z_{k_n} : n \in \mathbb{N}\} \) of \( \{z_n : n \in \mathbb{N}\} \) and a family \( \{V_n : n \in \mathbb{N}\} \) of pairwise disjoint open subsets of \( X \cup Y \) such that \( z_{k_n} \in V_n \). Let
\[ \beta_n = \min\{\gamma : z_{k_n} \in [A]^Y \cup Y \cap q(Y)\}. \]
Obviously, for every \( n \in \mathbb{N} \), there is an ordinal number \( \alpha_n \) such that \( \beta_n = \alpha_n + 1 \) and
\[
z_k \in [A]^{X \cup_f Y}_{\alpha_n + 1} \setminus [A]^{X \cup_f Y}_{\alpha_n}.
\]
Therefore, for each \( n \in \mathbb{N} \), there is a sequence \( \{z_{nm} : j \in \mathbb{N}\} \) from \( [A]^{X \cup_f Y}_{\alpha_n} \) converging to \( z_k \). Since \( X \cup_f Y \) is \( T_1 \), \( \{z_{nm} : j \in \mathbb{N}\} \) is infinite for each \( n \in \mathbb{N} \). By the definition of \( \beta \) and the fact that \( \alpha_n < \alpha_n + 1 = \beta_n \leq \beta \), it follows that \( \|z_{nm} : m \in \mathbb{N}\| \cap q(Y) \leq \aleph_0 \) for each \( n \in \mathbb{N} \). For convenience, we still denote by \( \{z_{nm} : m \in \mathbb{N}\} \), the intersection of \( \{z_{nm} : m \in \mathbb{N}\} \) and \( X \setminus M \).

**Claim 2.** No there is a sequence from \( \{z_{nm} : n, m \in \mathbb{N}\} \) converging to \( z_0 \).

In fact, if \( \beta = so(X) \) and if there is a sequence from \( \{z_{nm} : n, m \in \mathbb{N}\} \) converging to \( z_0 \), then, by Lemmas 3.1 and 3.4, we have \( z_0 \in [A]^{X \cup_f Y}_{\beta + 1} \cap q(Y) \) which is a contradiction.

If \( \beta < so(X) \), since \( so(X) \leq \omega \), so \( \beta \) is not limit. Rest of the proof of Claim 2 is evident.

Since \( X \setminus M \) is embedding in \( X \cup_f Y \) as an open subset, therefore if we define map
\[
p : S_2 \longrightarrow X \cup_f Y
\]
by \( p(t_a) = z_k \), \( p(t_{anm}) = z_{nm} \) and \( p(t_0) = z_0 \), then it is not difficult to verify that the map \( p \) satisfies all of conditions required in the statement.

4. **Incidental observation.**

**Definition 4.0.** Let \( X, Y \) be two topological spaces. Let \( f : X \rightarrow Y \) be a mapping. Let \( \alpha \) be an ordinal number and \( C \) a subset of \( Y \). We define \( C_f^\alpha \) as follows:
\[
C_f^\alpha = C \quad \text{if} \quad \alpha = 0,
\]
\[
C_f^\alpha = f(f^{-1}(C^\beta)) \quad \text{if} \quad \alpha = \beta + 1,
\]
\[
C_f^\alpha = \cup_{\beta < \alpha} C_f^\beta \quad \text{if} \quad \alpha \text{ is a limit ordinal number}.
\]

In [4], Kannan asked if \( f : X \rightarrow Y \) is a quotient mapping, \( A \) and \( B \) are both open subsets of \( Y \), and \( A \cup B = Y \), then for any \( C \subseteq Y \), does
\[
C_f^\alpha = (A \cap C)^\alpha_{f_A} \cup (B \cap C)^\alpha_{f_B}
\]
holds? Where \( f_A \) and \( f_B \) are the restrictions of \( f \) to \( f^{-1}(A) \) and \( f^{-1}(B) \) respectively. The following theorem completely answers the question above in positive.
Theorem 4.1. If \( f : X \rightarrow Y \) is a quotient mapping, \( A \) and \( B \) are both open subsets of \( Y \), and \( A \cup B = Y \), then for any \( C \subseteq Y \), \( C^\alpha = (A \cap C)^\alpha_f \cup (B \cap C)^\alpha_f \) where \( f_A \) and \( f_B \) are the restrictions of \( f \) to \( f^{-1}(A) \) and \( f^{-1}(B) \) respectively.

To prepare for the proof of Theorem 4.1, we first introduce a lemma.

Lemma 4.2. If \( f : X \rightarrow Y \) is a quotient mapping, \( B \) is an open subsets of \( Y \), then, for any \( C \subseteq Y \) and any ordinal number \( \alpha \), \( (B \cap C)^\alpha_f = B \cap C^\alpha_f \) where \( f_B \) means the restriction of \( f \) to \( f^{-1}(B) \).

Proof. We first show that \( (B \cap C)^\alpha_f \subseteq B \cap C^\alpha_f \).

Suppose that \( (B \cap C)^\beta_f \subseteq C^\beta_f \) for all \( \beta < \alpha \). If \( \alpha = \beta + 1 \), then
\[
(B \cap C)^\alpha_f = (B \cap C)^{\beta+1}_f = f_B(f_B^{-1}((B \cap C)^\beta_f))^{f^{-1}(B)} = f^{-1}(B) \cap f^{-1}((B \cap C)^\beta_f) = B \cap C^{\beta+1}_f.
\]

If \( \alpha \) is a limit, then \( (B \cap C)^\alpha_f = \bigcup_{\beta < \alpha} (B \cap C)^\beta_f \subseteq C^\alpha_f = C_f^\alpha \).

Next, we show that \( B \cap C^\alpha_f = (B \cap C)^\alpha_f \).

Suppose that \( B \cap C^\beta_f \subseteq (B \cap C)^\beta_f \) for all \( \beta < \alpha \). If \( \alpha = \beta + 1 \), we will show that \( B \cap C^{\beta+1}_f \subseteq (B \cap C)^{\beta+1}_f \).

If \( y \in B \cap C^{\beta+1}_f \setminus (B \cap C)^{\beta+1}_f \), then \( f^{-1}(y) \cap f^{-1}(C^\beta_f) \neq \emptyset \) and \( f^{-1}(y) \cap f_B^{-1}((B \cap C)^\beta_f) \neq \emptyset \). By the supposition of induction, it follows that \( f^{-1}(y) \cap f^{-1}(B) \cap f^{-1}(C^\beta_f) = \emptyset \). Since \( f^{-1}(B) \) is open and \( f^{-1}(y) \subseteq f^{-1}(B) \), one has \( f^{-1}(y) \cap f^{-1}(C^\beta_f) = \emptyset \), which is a contradiction.

Therefore \( B \cap C^{\beta+1}_f \subseteq (B \cap C)^{\beta+1}_f \).

If \( \alpha \) is a limit, then \( B \cap C^\alpha_f = B \cap (\bigcup_{\beta < \alpha} C^\beta_f) = \bigcup_{\beta < \alpha} (B \cap C^\beta_f) \subseteq \bigcup_{\beta < \alpha} (B \cap C)^\beta_f = (B \cap C)^\alpha_f \), which completes the proof of Lemma 4.2.

The proof of Theorem 4.1. Suppose that \( C^\beta_f = (A \cap C)^\beta_f \cup (B \cap C)^\beta_f \) for all
\(\beta < \alpha\). If \(\alpha = \beta + 1\), then \(C_j^\alpha = C_j^{\beta+1} = f(\overline{f^{-1}(C_j^\beta)})\) and

\[
(C \cap A)^f_{\beta} = (C \cap A)^{\beta+1}_{\beta} = fA(f^{-1}((C \cap A)^\beta_{\beta})f^{-1}(A)) = f(f^{-1}(A) \cap f^{-1}((C \cap A)^\beta_{\beta})).
\]

Similarly, \((C \cap B)^f_{\beta} = f(\overline{f^{-1}(B) \cap f^{-1}((C \cap B)^\beta_{\beta})}).\) Therefore

\[
(C \cap A)^f_{\beta} \cup (C \cap B)^f_{\beta} = (C \cap A)^{\beta+1}_{\beta} \cup (C \cap B)^{\beta+1}_{\beta}
\]

\[
= f((f^{-1}(A) \cup f^{-1}(C \cap B)^\beta) \cap f^{-1}(B) \cup f^{-1}((C \cap A)^\beta_{\beta})) \cap f^{-1}(C_j^\beta)) \subseteq f(f^{-1}(C_j^\beta)) = C_j^{\beta+1} = C_j^\alpha.
\]

On the other hand, if \(y \in C_j^{\beta+1}\), then \(f^{-1}(y) \cap f^{-1}(C_j^\beta) \neq \emptyset\). Next we show that

\[
(*) y \in (C \cap A)^{\beta+1}_{\beta} \cup (C \cap B)^{\beta+1}_{\beta}.
\]

If \(y \in A \cap B\), then \(f^{-1}(y) \subseteq f^{-1}(A) \cap f^{-1}(B)\) and so, by the supposition of induction, \((*)\) follows. If \(y \in B \setminus A\), then \(f^{-1}(y) \cap f^{-1}((C \cap B)^\beta_{\beta}) \neq \emptyset\).

Indeed, since \(f^{-1}(y) \cap f^{-1}(C_j^\beta) \neq \emptyset\) and \(f^{-1}(y) \subseteq f^{-1}(B)\), one has \(f^{-1}(y) \cap f^{-1}(C_j^\beta) \cap f^{-1}(B) \neq \emptyset\). Since \(f^{-1}(B)\) is open, \(f^{-1}(y) \cap f^{-1}(C_j^\beta) \cap B \neq \emptyset\).

By Lemma 4.2 and the supposition of induction, \(f^{-1}(y) \cap f^{-1}((B \cap C)^\beta_{\beta}) \neq \emptyset\), and so \(y \in (C \cap B)^{\beta+1}_{\beta}\). Similarly, we can show \(y \in (C \cap A)^{\beta+1}_{\beta}\) in the case of \(y \in A \setminus B\).

If \(\alpha\) is a limite, then by the supposition of induction, \(C_j^\alpha = \bigcup_{\beta < \alpha} C_j^\beta = C_j^{\beta+1} = \bigcup_{\beta < \alpha} ((A \cap C)^\beta_{\beta} \cup (B \cap C)^\beta_{\beta}) = (A \cap C)^\alpha_{\beta} \cup (B \cap C)^\alpha_{\beta}\), which completes the proof of Theorem 4.1.

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