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SEQUENTIAL ORDERS OF ADJUNCTION SPACES

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Let X, Y be two disjoint spaces, M be a closed subset of X, and $f : M \longrightarrow Y$ be a continuous map. In the direct sum $X \oplus Y$ of X and Y, define an equivalence relation $\sim$ by $a \sim f(a)$ for each $a \in M$. The quotient space $X \oplus Y/\sim$, is denoted by $X \cup_f Y$, usually called the adjunction space determined by X, Y and f. In this paper we prove that for two sequential spaces X and Y, $so(X \cup_f Y) \leq so(X) + so(Y)$ and, if $so(X \cup_f Y) > \max\{so(X), so(Y)\}$ and $so(X) \leq \omega$, then there exists a special map $p : S_2 \hookrightarrow X \cup_f Y$, where $so(X)$ denotes the sequential order of X and S_2 is the Arens' space. We also give an answer for a question of Kannan [4].

1. Introduction.

In [1], Arhangel'skii and Franklin constructed sequential spaces of its sequential order α for any $0 \leq \alpha \leq \omega_1$. It was done by attaching a sequential space to a sequential space by a continuous map. In Section 2, we give the relations between sequential orders of attaching space and original spaces. In Section 4, we answer a question of Kannan in [4].

Definition. Let X, Y be two disjoint spaces, M be a closed subset of X, and $f : M \longrightarrow Y$ be a continuous map. In the direct sum $X \oplus Y$ of X and Y,

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define an equivalence relation $\sim$ as follows: if $f(a) = f(b)$ then $a, b, f(b)$ are equivalent. The quotient space $X \oplus Y/\sim$, is denoted by $X \cup_f Y$, usually called the adjunction space determined by X, Y and f. If $a \in X \setminus M$, we denote by a the equivalent class of a when confusion does not occur. It is well-known that if X and Y are paracompact(normal), then $X \cup_f Y$ is also paracompact(normal). Nevertheless, a simple example shows that the Hausdorffness of X and Y does not imply that $X \cup_f Y$ is Hausdorff.

This indicates that the topological property what both of X and Y have, may not be transformed in $X \cup_f Y$.

Throughout this paper, we use q to denote the naturally quotient map from $X \oplus Y$ to $X \cup_f Y$, and N to denote the set of natural numbers. As a topological space, N has the discrete topology.

For a subset A of a topological space X, we denote by $\overline{A}^X$ (resp. $[A]_X^{Seq}$) the closure (resp. sequential closure, i.e., the set of limits of convergent sequences consisting of points of A) of A in X. We shall write $\overline{A}$ (resp. $[A]^{Seq}$) for $\overline{A}^X$ (resp.$[A]_X^{Seq}$) when confusion does not occur. A space X is sequential if, whenever $A \subseteq X$ and A is not closed, there is a sequence from A converging to a point outside the set A, and X is *Fréchet* if, whenever $x \in \overline{A}$, there is a sequence from A converging to x.

Let A be a subset of a space X.

We define $[A]_\alpha^X$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]_0^X = A$, $[A]_{\alpha+1}^X = [[A]_\alpha^X]_X^{Seq}$ and $[A]_\alpha^X = \cup\{[A]_\beta^X : \beta < \alpha\}$ for a limit α. We shall write $[A]_\alpha$ for $[A]_\alpha^X$ when confusion does not occur. One can easily see that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for all subsets A of X. For a sequential space X we define $so(X)$, the sequential order, by $so(X) = \min\{\alpha \in \omega_1 + 1 : \overline{A} = [A]_\alpha$ for every $A \subseteq X\}$. Obviously, if X is a Fréchet space, then $so(X) \leq 1$.

It is straightforward that if X and Y are both sequential spaces, then so is $X \cup_f Y$. Nevertheless, for two Fréchet spaces X and Y, $X \cup_f Y$ need not be Fréchet, but, as is shown in the sequel, $so(X \cup_f Y) \leq 2$.

2. Main results.

We first recall a well-known fact about the space $X \cup_f Y$ (cf. Theorem 6.3 of [3]) which is frequently used in the sequel.

Theorem 2.0 ([2]). *Let X, Y be two disjoint spaces. Then:*

(1) Y is embedded as a closed set in $X \cup_f Y$, and the restriction of q to Y is a homeomorphism.

(2) $X \setminus A$ is embedded as open set in $X \cup_f Y$, and the restriction of q to $X \setminus A$ is a homeomorphism.

Theorem 2.1. *Let X, Y be two disjoint sequential spaces. Then*

$$so(X \cup_f Y) \leq so(X) + so(Y).$$

Corollary 2.2. *Let X, Y be two disjoint Fréchet spaces. Then*

$$so(X \cup_f Y) \leq 2.$$

Theorem 2.3. *Let X, Y be two disjoint sequential spaces, M a closed subset of X, $f : M \longrightarrow Y$ a continuous mapping. If $f(\overline{A}^X \cap M)$ is closed in Y for every $A \subseteq X \setminus M$, then*

$$so(X \cup_f Y) \leq \max\{so(X), so(Y)\}.$$

Corollary 2.4. *Let X, Y be two disjoint sequential spaces and let M be a closed subset of X. If M is countably compact, then $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$.*

Proof. From the countable compactness of A and sequentiality of Y, it follows that f is closed. According to Theorem 2.3, $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$.

Corollary 2.5. *Let X, Y be two disjoint Fréchet spaces, If X is countably compact, then $X \cup_f Y$ is also Fréchet.*

Remark. Obviously, the converses of Theorem 2.3, Corollary 2.4 and 2.5 need not be true.

Theorem 2.6. *Let X be a Hausdorff Fréchet space, Y be a Fréchet T_1-space, M be a closed subset of X and let $f : M \longrightarrow Y$ be continuous. Then, $X \cup_f Y$ is Fréchet if and only if $f(\overline{A} \cap M)$ is closed in Y for every $A \subseteq X \setminus M$.*

As we have showed above the sequential order of $X \cup_f Y$ is suppressed by the sum of $so(X)$ and $so(Y)$. On the other hand, by Theorem 2.3, if f is closed, then $so(X \cup_f Y) \leq \max\{so(X), so(Y)\}$. Therefore it is natural to ask when is $X \cup_f Y$ really large than both of $so(X)$ and $so(Y)$. The following theorem give a necessary condition for the question when $so(X) \leq \omega$.

Theorem 2.7. *Let X, Y be two disjoint Hausdorff sequential spaces, M be a closed subset of X, $f : M \longrightarrow Y$ be a continuous mapping and let $so(X) \leq \omega$. If*

$$so(X \cup_f Y) > \max\{so(X), so(Y)\},$$

then there exists an embedding map $p : S_2 \hookrightarrow X \cup_f Y$ such that

$$\{p(t_n) : n \in \mathbb{N}\} \subseteq q(M)$$

and

$$\{p(t_{nm}) : n, m \in \mathbb{N}\} \subseteq X \setminus M.$$

Recall the definition of S_2 (see also example 1.6.19 of [3]).

Let $T = \{t_n : n \in \mathbb{N}\}$ be a sequence converging to $t_0 \notin T$. Then S_2 is the space obtained by attaching the space $N \times \{t_n : n \in \omega\}$ to the space $T \cup \{t_0\}$ by the continuous map $f : \{(n, t_0) : n \in \mathbb{N}\} \longrightarrow \{t_n : n \in \mathbb{N}\}$ defined by $f((n, t_0)) = t_n$ for all $n \in \mathbb{N}$. For convenience, we write t_{nm} for (n, t_m).

Corollary 2.8. *Let X, Y be two disjoint Hausdorff Fréchet spaces, $M \subseteq X$ a closed subset and $f : M \longrightarrow Y$ a continuous map. Then the following conditions are equivalent:*

(1) $so(X \cup_f Y) = 2$;

(2) there exists an embedding map $p : S_2 \hookrightarrow X \cup_f Y$ such that

$$\{p(t_n) : n \in \mathbb{N}\} \subseteq q(M)$$

and

$$\{p(t_{nm}) : n, m \in \mathbb{N}\} \subseteq X \setminus M.$$

Question. (a) Let X, Y be two disjoint sequential spaces. Then, does

$$so(X \cup_f Y) \leq so(Y) + so(X)$$

hold?

(b) In Theorem 2.7, whether the condition of $so(X) \leq \omega$ can be removed?

3. The proofs of theorems.

Lemma 3.1. *Let X, Y be two topological spaces. Let $f : X \longrightarrow Y$ be a continuous map. Then, for any $A \subseteq X$ and ordinal number α,*

$$f([A]_\alpha) \subseteq [f(A)]_\alpha.$$

Proof. We show Lemma 3.1 by induction.

Suppose that $f([A]_\beta) \subseteq [f(A)]_\beta$ for all $\beta < \alpha$.

If α is a limit, then

$$f([A]_\alpha) = \cup_{\beta<\alpha} f([A]_\beta) \subseteq \cup_{\beta<\alpha}[f(A)]_\beta = [f(A)]_\alpha.$$

If α is not a limit, then $\alpha = \beta + 1$ for some $\beta < \alpha$. Fix $y \in f([A]_{\beta+1})$. Then $y = f(x)$ for some $x \in [A]_{\beta+1}$. Thus there is a sequence $\{x_i : i < \omega\}$ in $[A]_\beta$ such that $x_i \longrightarrow x$ as $i \longrightarrow \infty$. Since f is continuous, $f(x_i) \longrightarrow f(x)$ as $i \longrightarrow \infty$. By the supposition, we have $\{f(x_i) : i < \omega\} \subseteq [f(A)]_\beta$. Therefore $f(x) \in [f(A)]_{\beta+1}$ which completes the proof.

Lemma 3.2. *Let X, Y be two disjoint topological spaces. Also let $A \subseteq X \setminus M$. If $z = q(y) \in \overline{A}^{X\cup_f Y}$ for some $y \in Y$, then, $y \in \overline{f(\overline{A}^X \cap M)}^Y$.*

Proof. Let V be a neighbourhood open in Y of y. To complete the proof, it suffices to show that $f^{-1}(V) \cap \overline{A}^X \neq \emptyset$. It is obvious that $V \cap f(M) \neq \emptyset$. Therefore, $f^{-1}(V) \neq \emptyset$. Suppose $f^{-1}(V) \cap \overline{A}^X = \emptyset$. Then there is an open subset U' of X containing $f^{-1}(V)$ such that $U' \cap A = \emptyset$.

On the other hand, since f is continuous, there is an open subset U'' of X such that $f^{-1}(V) = U'' \cap M$. Let $U = U' \cap U''$ and $W = q(U \cup V)$. It is easy to see that $W \cap A = \emptyset$. If we can show that W is an open neighbourhood of z in $X \cup_f Y$, then this completes the proof of Lemma 3.2, because it contradicts $z \in \overline{A}^{X\cup_f Y}$. Obviously, $z \in W$. Notice that if $x \in M \setminus U$ and $q(x) \in W$, then $f(x) \in V$. In fact, there is $w \in U \cup V$ such that $q(w) = q(x)$. If $w \in U$, then $w \in M$. Since $f^{-1}(V) = M \cap U$, it follows that $w \in f^{-1}(V)$. So $x \in f^{-1}(V)$. Hence, $q^{-1}(W) = U \cup V$.

The proof of Theorem 2.1. Let $so(X) = \alpha$ and $so(Y) = \beta$. We will show that $so(X \cup_f Y) \leq \alpha + \beta$. Let $k = q|_Y$ and $j = q|_X$ be the restrictions of q to X and Y respectively. Now let us fix $A \subseteq X \cup_f Y$ and $z \in \overline{A}^{X\cup_f Y}$. It is easy to see that $z \in \overline{A \cap (X \setminus M)}^{X\cup_f Y} \cup \overline{A \cap k(Y)}^{X\cup_f Y}$. Now we prove that $z \in [A]_{\alpha+\beta}$.

Case 1. $z \in \overline{A \cap k(Y)}^{X \cup_f Y}$.

By Theorem 2.0, $k(Y)$ is a closed subset of $X \cup_f Y$. Therefore, $z \in \overline{A \cap k(Y)}^{k(Y)}$. By the facts that $k(Y)$ and Y are homeomorphic (Theorem 2.0) and that $so(Y) = \beta$, one has

$$z \in [A \cap k(Y)]_\beta^{k(Y)} \subseteq [A]_\beta^{X \cup_f Y} \subseteq [A]_{\alpha+\beta}^{X \cup_f Y}.$$

Case 2. $z \in \overline{A \cap (X \setminus M)}^{X \cup_f Y}$.

If $z \in X \setminus M$, then $z \in \overline{A \cap (X \setminus M)}^{X \setminus M}$ because $X \setminus M$ is embedded in $X \cup_f Y$ as an open subspace, and so $z \in \overline{A \cap (X \setminus M)}^{X}$. Since $so(X) = \alpha$, we have $z \in [A \cap (X \setminus M)]_\alpha$, and so, by Lemma 3.1,

$$z \in [A \cap (X \setminus M)]_\alpha^{X \cup_f Y} \subseteq [A]_{\alpha+\beta}^{X \cup_f Y}.$$

If $z \notin X \setminus M$, then $z = k(y)$ for some $y \in Y$. Thus,

$$\begin{aligned}
z = q(y) \in \quad & q(\overline{f(\overline{A \cap (X \setminus M)}^{X} \cap M)}^{Y}) \text{ (by Lemma 3.2)} \\
= \quad & q([f(\overline{A \cap (X \setminus M)}^{X} \cap M)]_\beta^{Y}) \\
\subseteq \quad & [q(f(\overline{A \cap (X \setminus M)}^{X} \cap M))]_\beta^{X \cup_f Y} \text{ (by Lemma 3.1)} \\
= \quad & [q(\overline{A \cap (X \setminus M)}^{X} \cap M)]_\beta^{X \cup_f Y} \text{ (by the definition of } q) \\
\subseteq \quad & [q(\overline{A \cap (X \setminus M)}^{X})]_\beta^{X \cup_f Y} \\
= \quad & [q([A \cap (X \setminus M)]_\alpha^{X})]_\beta^{X \cup_f Y} \\
\subseteq \quad & [[q(A \cap (X \setminus M))]_\alpha^{X \cup_f Y}]_\beta^{X \cup_f Y} \text{ (by Lemma 3.1)} \\
= \quad & [[A \cap (X \setminus M)]_\alpha^{X \cup_f Y}]_\beta^{X \cup_f Y} \\
= \quad & [A \cap (X \setminus M)]_{\alpha+\beta}^{X \cup_f Y} \\
\subseteq \quad & [A]_{\alpha+\beta}^{X \cup_f Y}.
\end{aligned}$$

This completes the proof of Theorem 2.1.

Lemma 3.3. *Let X and Y be two topological spaces, M be a closed subset of X and let $f : M \longrightarrow Y$ be continuous. Also, let $A \subseteq X \setminus M$ be such that $f(\overline{A} \cap M)$ is closed in Y. Then, $q(\overline{A}^{X}) = \overline{A}^{X \cup_f Y}$.*

Proof. Let $z \in \overline{A}^{X\cup_f Y}$. If $z \in X \setminus M$, then by Theorem 2.0, $z \in \overline{A}^{X\setminus M} \subseteq \overline{A}^{X}$. Thus $z = q(z) \in q(\overline{A}^{X})$. If $z = q(y) \in \overline{A}^{X\cup_f Y} \cap q(Y)$ where $y \in Y$, by Lemma 3.2, $y \in \overline{f(\overline{A} \cap M)}^{Y}$. Since $f(\overline{A} \cap M)$ is closed in Y, there exists $x \in \overline{A} \cap M$ such that $f(x) = y$, hence $z \in q(\overline{A}^{X})$.

The proof of Theorem 2.3. Take $B \subseteq X \cup_f Y$. Then

$$\begin{aligned}
[B]^{X\cup_f Y}_{so(X\cup_f Y)} &= \overline{B}^{X\cup_f Y} \\
&= \overline{B \cap (X \setminus M)}^{X\cup_f Y} \cup \overline{B \cap q(Y)}^{X\cup_f Y} \\
&= q(\overline{B \cap (X \setminus M)}^{X}) \cup \overline{B \cap q(Y)}^{q(Y)} \text{ (by Lemma 3.3)} \\
&= q([B \cap (X \setminus M)]^{X}_{so(X)}) \cup [B \cap q(Y)]^{q(Y)}_{so(Y)} \\
&\subseteq [B \cap (X \setminus M)]^{X\cup_f Y}_{so(X)} \cup [B \cap q(Y)]^{X\cup_f Y}_{so(Y)} \text{ (by Lemma 3.1)} \\
&\subseteq [B]^{X\cup_f Y}_{so(X)} \cup [B]^{X\cup_f Y}_{so(Y)} \\
&= [B]^{X\cup_f Y}_{\max\{so(X), so(Y)\}}.
\end{aligned}$$

Lemma 3.4. *Let X be a Hausdorff space, Y be a T_1-space, M be a closed subset of X and let $f : M \longrightarrow Y$ is continuous. Suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X \setminus M$ is a sequence which is convergent in $X \cap_f Y$ to a point $q(y)$ where $y \in Y$. Then there is $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{X} \cap M$ such that $y = f(x)$.*

Proof. By Lemma 3.2, $\{x_n : n \in \mathbb{N}\}$ is not a closed subset of X. Therefore, in particular, $[\{x_n : n \in \mathbb{N}\}]^X_1 \setminus \{x_n : n \in \mathbb{N}\} \neq \emptyset$, because X is sequential. As X is T_1, there exists a point $x \in [\{x_n : n \in \mathbb{N}\}]^X_1 \setminus \{x_n : n \in \mathbb{N}\}$ and a subsequence $\{x_{k_n} : n \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ such that $x_{k_n} \longrightarrow x$ as $n \longrightarrow \infty$. Since X is Hausdorff, we have $\overline{\{x_{k_n} : n \in \mathbb{N}\}}^{X} = \{x_{k_n} : n \in \mathbb{N}\} \cup \{x\}$. Note now that $q(y) \in \overline{\{x_{k_n} : n \in \mathbb{N}\}}^{X\cup_f Y}$. Hence, by Lemma 3.2, $y \in \overline{f(\overline{\{x_{k_n} : n \in \mathbb{N}\}}^{X} \cap M)}^{Y} = \overline{\{f(x)\}}^{Y}$. Since Y is T_1, it follows that $y = f(x)$.

The proof of Theorem 2.6. By Theorem 2.3, we only need to show the necessary.

Let $A \subseteq X \setminus M$. Suppose $\overline{A}^X \cap M \neq \emptyset$, and let $y \in \overline{f(\overline{A}^X \cap M)}^Y$. Therefore,

$$\begin{aligned} q(y) \in & \ \overline{q(f(\overline{A}^X \cap M))}^{X\cup_f Y} \\ \subseteq & \ \overline{q(\overline{A}^X)}^{X\cup_f Y} \quad \text{(by the definition of } q\text{)} \\ \subseteq & \ \overline{\overline{q(A)}^{X\cup_f Y}}^{X\cup_f Y} \quad \text{(by the continuity of } q\text{)} \\ = & \ \overline{A}^{X\cup_f Y}. \end{aligned}$$

Since $X \cup_f Y$ is Fréchet, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ from A such that $\{x_n : n \in \mathbb{N}\}$ is convergent in $X \cup_f Y$ to the point $q(y)$. By Lemma 3.4, this implies that $y \in f(\overline{A} \cap M)$.

Lemma 3.5. *Let X be a Hausdorff space, Y be a T_1-sequential space, M be a closed subset of X and let $f : M \longrightarrow Y$ is continuous. Also let $A \subseteq X \setminus M$. If*

$$\alpha = \min\{\beta : [A]_\beta^{X\cup_f Y} \cap q(Y) \textit{ is not closed in } q(Y)\},$$

then $[A]_\alpha^{X\cup_f Y} \cap q(Y) \subseteq q(M)$.

Proof. Take a point $q(y) \in [A]_\alpha^{X\cup_f Y} \cap q(Y)$ where $y \in Y$. Since $y \notin A$, the following ordinal is well-defined:

$$\beta(y) = \min\{\beta : q(y) \in [A]_{\beta+1}^{X\cup_f Y} \setminus [A]_\beta^{X\cup_f Y}\}.$$

Now, $q(y) \in [A]_{\beta(y)+1}^{X\cup_f Y} \setminus [A]_{\beta(y)}^{X\cup_f Y}$ implies the existence of a sequence $\{x_n : n \in \mathbb{N}\}$ from $[A]_{\beta(y)}^{X\cup_f Y}$ which is convergent in $X \cup_f Y$ to the point $q(y)$. In the case, the set $\{n : x_n \notin X \setminus M\}$ is finite. Indeed, otherwise we can find a subsequence $\{x_{k_n} : n \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\} \cap q(Y)$ such that $x_{k_n} \longrightarrow q(y)$ as $n \longrightarrow \infty$. However, this will finally imply that $q(y) \in [A]_{\beta(y)}^{X\cup_f Y}$ because, by construction, $\beta(y) < \alpha$ and $[A]_{\beta(y)}^{X\cup_f Y} \cap q(Y)$ is closed. So, there is $n_0 \in \mathbb{N}$ such that $\{x_n : n \geq n_0\} \subseteq X \setminus M$. By Lemma 3.4, it follows that $y \in f(\overline{A} \cap M)$.

The proof of Theorem 2.7. Let $\max\{so(X), so(Y)\} = \alpha$. Since $so(X \cup_f Y) > \alpha$, there exists $A \subseteq X \cup_f Y$ such that

$$\overline{A}^{X\cup_f Y} \setminus [A]_\alpha^{X\cup_f Y} \neq \emptyset.$$

We pick

$$z \in \overline{A}^{X\cup_f Y} \setminus [A]_\alpha^{X\cup_f Y}.$$

Since

$$\overline{A}^{X\cup_f Y} = \overline{A \cap q(Y)}^{X\cup_f Y} \cup \overline{A \cap (X \setminus M)}^{X\cup_f Y},$$

by Theorem 2.0 and the hypothesis of $\max\{so(X), so(Y)\} = \alpha$ we have

$$(*) \qquad z = q(y) \in \overline{A \cap (X \setminus M)}^{X\cup_f Y} \setminus [A \cap (X \setminus M)]_\alpha^{X\cup_f Y}$$

where $y \in Y$. For the covenience, without loss of generality, we may write $A \cap (X \setminus M) = A$.

Claim 1. $[A]_{so(X)}^{X\cup_f Y} \cap q(Y)$ is not closed in $q(Y)$.

Since $X \cup_f Y$ is sequential, by $(*)$, there exists a sequence $\{z_n : n \in \mathbb{N}\}$ from $[A]_{so(X)}^{X\cup_f Y}$ converging to a point $z' = q(y)$ outside $[A]_{so(X)}^{X\cup_f Y}$ where $y \in Y$. If $\{z_n : n \in \mathbb{N}\} \cap (X \setminus M)$ is infinite, then, by Lemma 3.4, there is a subsequence $\{z_{k_n} : n \in \mathbb{N}\}$ of $\{z_n : n \in \mathbb{N}\}$ and $x \in X$ such that $x \in \overline{\{z_{k_n} : n \in \mathbb{N}\}}^X$ and $y = f(x)$.

On the other hand, for each $n \in \mathbb{N}$, $z_{k_n} \in [A]_{so(X)}^{X\cup_f Y} \cap (X \setminus M) = [A]_{so(X)}^{X\setminus M} \subseteq [A]_{so(X)}^X = \overline{A}^X$. Therefore, $x \in [A]_{so(X)}^X$. Thus, by Lemma 3.1, $q(y) = q(x) \in [A]_{so(X)}^{X\cup_f Y}$ which is a contradiction. So, $\{z_n : n \in \mathbb{N}\} \cap q(Y)$ is infinite, this completes the proof of Claim 1.

Now let us define

$$\beta = \min\{\alpha : [A]_\alpha^{X\cup_f Y} \cap q(Y) \text{ is not closed in } q(Y)\}.$$

By Lemma 3.5, $[A]_\alpha^{X\cup_f Y} \cap q(Y) \subseteq q(M)$.

On the other hand, since $X \cup_f Y$ is sequential, we can choose

$$z_0 \in [[A]_\beta^{X\cup_f Y} \cap q(Y)]_1^{X\cup_f Y} \setminus [A]_\beta^{X\cup_f Y} \cap q(Y).$$

There is a sequence $\{z_n : n \in \mathbb{N}\}$ from $[A]_\beta^{X\cup_f Y} \cap q(Y) (\subseteq q(M))$ converging to z_0. Since X and Y are T_1, so is $X \cup_f Y$. Hence $\{z_n : n \in \mathbb{N}\}$ is a infinite subset. As Y is Hausdorff, there exists a subsequence $\{z_{k_n} : n \in \mathbb{N}\}$ of $\{z_n : n \in \mathbb{N}\}$ and a family $\{V_n : n \in \mathbb{N}\}$ of pairwise disjoint open subsets of $X \cup_f Y$ such that $z_{k_n} \in V_n$. Let

$$\beta_n = \min\{\gamma : z_{k_n} \in [A]_\gamma^{X\cup_f Y} \cap q(Y)\}.$$

Obviously, for every $n \in \mathbb{N}$, there is an ordinal number α_n such that $\beta_n = \alpha_n + 1$ and

$$z_{k_n} \in [A]^{X\cup_f Y}_{\alpha_n+1} \setminus [A]^{X\cup_f Y}_{\alpha_n}.$$

Therefore, for each $n \in \mathbb{N}$, there is a sequence $\{z_{nm} : j \in \mathbb{N}\}$ from $[A]^{X\cup_f Y}_{\alpha_n}$ converging to z_{k_n}. Since $X \cup_f Y$ is T_1, $\{z_{nm} : j \in \mathbb{N}\}$ is infinite for each $n \in \mathbb{N}$. By the definition of β and the fact that $\alpha_n < \alpha_n + 1 = \beta_n \leq \beta$, it follows that $|\{z_{nm} : m \in \mathbb{N}\} \cap q(Y)| < \aleph_0$ for each $n \in \mathbb{N}$. For convenience, we still denote by $\{z_{nm} : m \in \mathbb{N}\}$, the intersection of $\{z_{nm} : m \in \mathbb{N}\}$ and $X \setminus M$.

Claim 2. No there is a sequence from $\{z_{nm} : n, m \in \mathbb{N}\}$ converging to z_0.

In fact, if $\beta = so(X)$ and if there is a sequence from $\{z_{nm} : n, m \in \mathbb{N}\}$ converging to z_0, then, by Lemmas 3.1 and 3.4, we have $z_0 \in [A]^{X\cup_f Y}_{\beta} \cap q(Y)$ which is a contradiction.

If $\beta < so(X)$, since $so(X) \leq \omega$, so β is not limit. Rest of the proof of Claim 2 is evident.

Since $X \setminus M$ is embedding in $X \cup_f Y$ as an open subset, therefore if we define map

$$p : S_2 \longrightarrow X \cup_f Y$$

by $p(t_n) = z_{k_n}$, $p(t_{nm}) = z_{nm}$ and $p(t_0) = z_0$, then it is not difficult to verify that the map p satisfies all of conditions required in the statement.

4. Incidental observation.

Definition 4.0. Let X, Y be two topological spaces. Let $f : X \longrightarrow Y$ be a mapping. Let α be an ordinal number and C a subset of Y. We define C^α_f as follows:

$$C^\alpha_f = C \qquad \text{if} \quad \alpha = 0,$$

$$C^\alpha_f = f(\overline{f^{-1}(C^\beta_f)}) \qquad \text{if} \quad \alpha = \beta + 1,$$

$$C^\alpha_f = \cup_{\beta<\alpha} C^\beta_f \qquad \text{if} \quad \text{if } \alpha \text{ is a limit ordinal number.}$$

In [4], Kannan asked that if $f : X \longrightarrow Y$ is a quotient mapping, A and B are both open subsets of Y, and $A \cup B = Y$, then for any $C \subseteq Y$, does

$$C^\alpha_f = (A \cap C)^\alpha_{f_A} \cup (B \cap C)^\alpha_{f_B}$$

holds? Where f_A and f_B are the restrictions of f to $f^{-1}(A)$ and $f^{-1}(B)$ respectively. The following theorem completely answers the question above in positive.

Theorem 4.1. *If $f : X \longrightarrow Y$ is a quotient mapping, A and B are both open subsets of Y, and $A \cup B = Y$, then for any $C \subseteq Y$, $C_f^\alpha = (A \cap C)_{f_A}^\alpha \cup (B \cap C)_{f_B}^\alpha$ where f_A and f_B are the restrictions of f to $f^{-1}(A)$ and $f^{-1}(B)$ respectively.*

To prepare for the proof of Theorem 4.1, we first introduce a lemma.

Lemma 4.2. *If $f : X \longrightarrow Y$ is a quotient mapping, B is an open subsets of Y, then, for any $C \subseteq Y$ and any ordinal number α, $(B \cap C)_{f_B}^\alpha = B \cap C_f^\alpha$ where f_B means the restriction of f to $f^{-1}(B)$.*

Proof. We first show that $(B \cap C)_{f_B}^\alpha \subseteq B \cap C_f^\alpha$.

Suppose that $(B \cap C)_{f_B}^\beta \subseteq C_f^\beta$ for all $\beta < \alpha$. If $\alpha = \beta + 1$, then

$$\begin{aligned}
(B \cap C)_{f_B}^\alpha = \quad & (B \cap C)_{f_B}^{\beta+1} \\
= \quad & f_B(\overline{f_B^{-1}((B \cap C)_{f_B}^\beta)}^{f^{-1}(B)}) \\
= \quad & f(f^{-1}(B) \cap \overline{f^{-}((B \cap C)_{f_B}^\beta)}) \\
\subseteq \quad & f(\overline{f^{-1}(C_f^\beta)}) \\
= \quad & C_f^{\beta+1}.
\end{aligned}$$

If α is a limit, then $(B \cap C)_{f_B}^\alpha = \cup_{\beta<\alpha}(B \cap C)_{f_B}^\beta \subseteq \cup_{\beta<\alpha} C_f^\beta = C_f^\alpha$.

Next, we show that $B \cap C_f^\alpha \supseteq (B \cap C)_{f_B}^\alpha$.

Suppose that $B \cap C_f^\beta \subseteq (B \cap C)_f^\beta$ for all $\beta < \alpha$. If $\alpha = \beta + 1$, we will show that $B \cap C_f^{\beta+1} \subseteq (B \cap C)_{f_B}^{\beta+1}$.

If $y \in B \cap C_f^{\beta+1} \setminus (B \cap C)_{f_B}^{\beta+1}$, then $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta)} \neq \emptyset$ and $f^{-1}(y) \cap \overline{f_B^{-1}((B \cap C)_{f_B}^\beta)}^{f^{-1}(B)} = \emptyset$. By the soppostion of induction, it follows that $f^{-1}(y) \cap f^{-1}(B) \cap \overline{f^{-1}(C_f^\alpha)} = \emptyset$. Since $f^{-1}(B)$ is open and $f^{-1}(y) \subseteq f^{-1}(B)$, one has $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta)} = \emptyset$, which is a contradiction. Therefore $B \cap C_f^{\beta+1} \subseteq (B \cap C)_{f_B}^{\beta+1}$

If α is a limit, then $B \cap C_f^\alpha = B \cap (\cup_{\beta<\alpha}) C_f^\beta = \cap_{\beta<\alpha}(B \cap C_f^\beta) \subseteq \cup_{\beta<\alpha}(B \cap C)_{f_B}^\beta = (B \cap C)_{f_B}^\alpha$, which completes the proof of Lemma 4.2.

The proof of Theorem 4.1. Suppose that $C_f^\beta = (A \cap C)_{f_A}^\beta \cup (B \cap C)_{f_B}^\beta$ for all

$\beta < \alpha$. If $\alpha = \beta + 1$, then $C_f^\alpha = C_f^{\beta+1} = f(\overline{f^{-1}(C_f^\beta)})$ and

$$\begin{aligned}(C \cap A)_{f_A}^\alpha = & \quad (C \cap A)_{f_A}^{\beta+1} \\ = & \quad f_A(\overline{f^{-1}((C \cap A)_{f_A}^\beta)}^{f^{-1}(A)}) \\ = & \quad f(f^{-1}(A) \cap \overline{f^{-1}((C \cap A)_{f_A}^\beta)}).\end{aligned}$$

Similarly, $(C \cap B)_{f_B}^\alpha = f(f^{-1}(B) \cap \overline{f^{-1}((C \cap B)_{f_B}^\beta)})$. Therefore

$$\begin{aligned}&(C \cap A)_{f_A}^\alpha \cup (C \cap B)_{f_B}^\alpha = (C \cap A)_{f_A}^{\beta+1} \cup (C \cap B)_{f_B}^{\beta+1} \\ &= f((f^{-1}(A) \cup \overline{f^{-1}(C \cap B)_{f_B}^\beta}) \cap (f^{-1}(B) \cup \overline{f^{-1}((C \cap A)_{f_A}^\beta)}) \cap \overline{f^{-1}(C_f^\beta)}) \\ &\subseteq f(\overline{f^{-1}(C_f^\beta)}) = C_f^{\beta+1} = C_f^\alpha.\end{aligned}$$

On the other hand, if $y \in C_f^{\beta+1}$, then $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta)} \neq \emptyset$. Next we show that

$$(**) \qquad y \in (C \cap A)_{f_A}^{\beta+1} \cup (C \cap B)_{f_B}^{\beta+1}.$$

If $y \in A \cap B$, then $f^{-1}(y) \subseteq f^{-1}(A) \cap f^{-1}(B)$ and so, by the supposition of induction, $(**)$ follows. If $y \in B \setminus A$, then $f^{-1}(y) \cap \overline{f^{-1}((C \cap B)_{f_B}^\beta)} \neq \emptyset$. Indeed, since $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta)} \neq \emptyset$ and $f^{-1}(y) \subseteq f^{-1}(B)$, one has $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta)} \cap f^{-1}(B) \neq \emptyset$. Since $f^{-1}(B)$ is open, $f^{-1}(y) \cap \overline{f^{-1}(C_f^\beta \cap B)} \neq \emptyset$. By Lemma 4.2 and the supposition of induction, $f^{-1}(y) \cap \overline{f^{-1}((B \cap C)_{f_B}^\beta)} \neq \emptyset$, and so $y \in (C \cap B)_{f_B}^{\beta+1}$. Similarly, we can show $y \in (C \cap A)_{f_A}^{\beta+1}$ in the case of $y \in A \setminus B$.

If α is a limite, then by the supposition of induction, $C_f^\alpha = \cup_{\beta<\alpha} C_f^\beta = \cup_{\beta<\alpha}((A \cap C)_{f_A}^\beta \cup (B \cap C)_{f_B}^\beta) = (A \cap C)_{f_A}^\alpha \cup (B \cap C)_{f_B}^\alpha$, which completes the proof of Theorem 4.1.

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