LE MATEMATICHE
Vol. LIII (1998) – Fasc. I, pp. 35–50

REMARKS ON Q-OSCILLATORS REPRESENTATIONS OF HOPF-TYPE BOSON ALGEBRAS

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We present a method of constructing known deformed or undeformed oscillators as quotients of certain models of Hopf-type oscillator algebras, using similar techniques to those of determining fix point sets of the adjoint action of a Hopf algebra. Moreover we give a characterization of these models in terms of these quotients coupled to Euclidean Clifford algebra. A theorem is proved which provides representations of the models, induced from those of a certain type of quotient algebra.

1. Introduction.

Recently there has been an extensive interest in deformations of the original oscillator algebras mainly because of their significance in quantum algebras and superalgebras and their wide range of application in mathematics and physics [1]–[5]. Such deformations first appeared in [6], [7], [8] and [9], [10], [11], to be followed by other generalizations [12]–[17], while their consistency interrelation and representations have also been analysed [18]–[28]. Moreover extensive investigations of simplest deformations of Heisenberg algebras [29]–[33] have revealed an important role played by the Cuntz algebra [34] and in obtaining and classifying representations of deformed oscillator algebras [35].

On the other hand a possible quasitriangular Hopf algebra structure, in accordance with appropriately defined deformed or undeformed boson algebras,

Entrato in Redazione il 10 novembre 1997.

has recently been addressed [36]–[43]. In particular in [43] certain models of deformed and undeformed boson algebras (denoted as $B_{\zeta}(\alpha,\beta)$, $B^{+}_{\zeta}(\alpha,\beta)$ and $B^{q}_{\zeta}(\alpha,\beta)$, $B^{q+}_{\zeta}(\alpha,\beta)$ respectively, where $\zeta = 1,-1$, $\alpha,\beta \in \mathbb{R}$) have been proposed and shown to admit the above algebraic structure together with their relation to already known oscillator models such as the Calogero-Vasiliev algebra and its deformation, very important in physical models such as the Calogero-Sutherland, anyonic systems [44]–[48].

In this article firstly we shall introduce the basic ingredients of the Hopf-type boson algebras $B^{+}_{-1}(\alpha,\beta)$ and $B^{q+}_{-1}(\alpha,\beta)$. In section three it is shown how one can obtain already known undeformed and deformed oscillators as quotients of the above algebras using appropriately the adjoint action of a Hopf algebra. In section four and five we characterize $B^{q+}_{-1}(\alpha,\beta)$ and $B^{+}_{-1}(\alpha,\beta)$ respectively, in relation with the above quotients coupled with $1-\dim$ Euclidean Clifford algebra. Using smash product techniques [51] we provide theorems which give representations of the above algebras induced from those of their quotients. In what follows we shall set $B_{-1}(\alpha,\beta) = B(\alpha,\beta)$, $B^{+}_{-1}(\alpha,\beta) = B^{+}(\alpha,\beta)$, $B^{q}_{-1}(\alpha,\beta) = B_q(\alpha,\beta)$, $B^{q+}_{-1}(\alpha,\beta) = B^{+}_{q}(\alpha,\beta)$.

2. The boson algebras $B(\alpha,\beta)$, $B^{+}(\alpha,\beta)$, $B_q(\alpha,\beta)$, $B^{+}_q(\alpha,\beta)$.

We first state certain generalities on quasitriangular Hopf algebras needed in what will follow. Consider a unital associative algebra, over a field F, with multiplication $m : A\otimes A \to A$ (i.e. $m(a\otimes b) = ab, \forall a$ and $\forall b \in A$) and unit $u : F \to A$ (i.e. $u(1) = I$, the identity on A) endowed with a Hopf algebra structure (c.f.[50]), that is, having a copruduct $\Delta : A \to A\otimes A$, a counit $\varepsilon : A \to F$ (which is a homomorphism) and an antipode $S : A \to A$ (which is an antihomomorphism i.e. $S(ab) = S(b)S(a)$, and we shall assume that it has an inverse S^{-1}) subject to the following consistency condition:

$$(id\otimes\Delta)\Delta(a) = (\Delta\otimes id)\Delta(a)$$
$$(id\otimes\varepsilon)\Delta(a) = (\varepsilon\otimes id)\Delta(a) = a \tag{1}$$
$$m(id\otimes S)\Delta(a) = m(S\otimes id)\Delta(a) = \varepsilon(a)I \quad \forall a\in A$$

where, following Sweedler [50], we shall adopt the notation $\Delta(a) = \sum_{(a)} a^{(1)}\otimes a^{(2)}$. Let T be the twist map on $A\otimes A$ defined by $T(a\otimes b) = b\otimes a$. Then there also exists an opposite Hopf algebra structure on A with coproduct $T\Delta = \Delta^{T}$, antipode S^{-1} and counit as before. According to Drinfeld [1] a Hopf algebra A is called quasitriangular if there exists an invertible element $R\in A\otimes A$ such that

$$\begin{aligned} \Delta^T(a)R &= R\Delta(a),\ \forall a \in A \\ R_{13}R_{23} = (\Delta \otimes I)R,\ &R_{13}R_{12} = (I \otimes \Delta)R \end{aligned} \tag{2}$$

with usual meaning of R_{12}, R_{13}, R_{23} as embeddings of R in $A \otimes A \otimes A$. The inverse R^{-1} is then given by $R^{-1} = (S \otimes I)R$ and it is easily shown that R satisfies the Yang-Baxter equation, $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. We say that A is a Hopf *-algebra if there exists an involutive antilinear mapping $^* : A \longrightarrow A$, $a \longrightarrow a^*$, such that the comultiplication and the counit are *-homomorphisms and $S\circ^*$ is involutive, i.e. $S \circ^* \circ S \circ^* = I$.

Finally for any Hopf algebra we can define the adjoint operation ad given by

$$ada(b) = \sum_{(a)} a^{(1)} b S(a^{(2)}). \tag{3}$$

Let us now present the following undeformed Hopf type boson algebras $B(\alpha, \beta)$, $B^+(\alpha, \beta)$ considered in [43] and generated by $a, a^\dagger$ and N subject to the following relations:

$$\begin{aligned} \{a, a^\dagger\} &= \alpha N + \beta I, \\ [N, a] &= -a, \\ [N, a^\dagger] &= a^\dagger \end{aligned} \tag{4}$$

where $\alpha, \beta \in \mathbb{R}$ and where here and in the rest of the paper $\{x, y\} = xy + yx$. $B(\alpha, \beta)$ has to be enlarged to become a Hopf algebra by adding an invertible element $(-1)^N$ which will be treated as a supplementary generator satisfying the following relations:

$$\left\{(-1)^N, a\right\} = 0 = \left\{(-1)^N, a^\dagger\right\}, \quad \left[(-1)^N, N\right] = 0. \tag{5}$$

Similar considerations were used in [42] and in that paper's context our enlarged algebra $B(\alpha, \beta)$ can be thought of as a spectrum generating algebra for the ordinary harmonic oscillator, while the element g of [42] will be $g = (-1)^{\tilde{N}}$ provided that we impose the condition $g^2 = (-1)^{2\tilde{N}} = I$ where $\widetilde{N} = N + \frac{\beta}{\alpha}$. We shall denote by $B^+(\alpha, \beta)$ and $U(B^+(\alpha, \beta))$ this enlarged algebra and its universal enveloping algebras respectively. The coproduct counit and antipode of $B(\alpha, \beta)$ satisfying (1) are given by:

$$
\begin{gathered}
\Delta(N) = N \otimes I + I \otimes N + \frac{\beta}{\alpha} I \otimes I, \\
\Delta(a) = a \otimes I + (-1)^{\tilde{N}} \otimes a, \\
\Delta(a^\dagger) = a^\dagger \otimes I + (-1)^{-\tilde{N}} \otimes a^\dagger, \\
\varepsilon(N) = -\frac{\beta}{\alpha}, \ \varepsilon(a) = \varepsilon(a^\dagger) = 0, \quad \varepsilon(I) = 1 \\
S(N) = -N - \frac{2\beta}{\alpha}, \ S(a) = -(-1)^{-\tilde{N}} a, \ S(a^\dagger) = -a^\dagger (-1)^{\tilde{N}+1},
\end{gathered} \tag{6}
$$

$$
\begin{gathered}
\Delta\Big((-1)^{\pm\tilde{N}}\Big) = (-1)^{\pm\tilde{N}} \otimes (-1)^{\pm\tilde{N}}, \\
\varepsilon\Big((-1)^{\pm\tilde{N}}\Big) = I \ , \ S\Big((-1)^{\pm\tilde{N}}\Big) = (-1)^{\mp\tilde{N}},
\end{gathered} \tag{7}
$$

provided that $\alpha \neq 0$. Moreover an opposite Hopf algebra structure also exists for $B(\alpha, \beta)$ with coproduct Δ^T and antipode the inverse S^{-1} of S which can be immediately deduced from S given in (6), (7).

A Fock-type representation $B(\alpha, \beta)$, with $a|0>= 0$, $N|n>= n|n>$, $n \in \mathbb{Z}_+$ and $<0|0>= 1$, exists such that, when $\alpha > 0, \beta > 0$ it is unitary and it is provided by:

$$
\begin{gathered}
\big|n>= \frac{1}{([n]!)^{\frac{1}{2}}} (a^\dagger)^n \big|0>, \\
a\big|n>= [n]^{\frac{1}{2}} \big|n-1>, \ a^\dagger\big|n>= [n+1]^{\frac{1}{2}} \big|n+1>,
\end{gathered}
$$

where

$$
\begin{gathered}
[n] = \Big(\frac{\alpha n}{2} + \frac{2\beta - \alpha}{4}(1 + (-1)^{n+1})\Big), \\
[n]! = \prod_{l=1}^{n} [l] \quad \text{and} \quad <n\big|n'>= \delta_{nn'}.
\end{gathered} \tag{8}
$$

With the definition $(-1)^{\pm N}\big|n>= (-1)^{\pm n}\big|n>$ this Fock space provides also a representation of $B^+(\alpha, \beta)$.

We shall now turn to the presentation of the q-deformation of the above algebras. $B_q(\alpha, \beta)$ is generated by $a_q, a_q^\dagger$ and N subject to the following relations:

$$
\begin{aligned}
a_q a_q^\dagger + a_q^\dagger a_q &= [\alpha N + \beta]_q, \\
[N, a_q] &= -a_q, \\
[N, a_q^\dagger] &= a_q^\dagger
\end{aligned}
\tag{9}
$$

where $\alpha, \beta \in \mathbb{R}$ and $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. Similarly to the undeformed case, in order to obtain a Hopf algebra structure for $B_q(\alpha, \beta)$, we have to enlarge it by adding an invertible element $(-1)^{\tilde{N}}$ which will be treated as a supplementary generator satisfying relations (5) (with a_q and $a_q^\dagger$ in the place of a and $a^\dagger$ respectively) and (7). We shall denote this extended algebra (its universal enveloping algebra) as $B_q^+(\alpha, \beta)$ $(U(B_q^+(\alpha, \beta)))$. The coproduct, counit and antipode satisfy

$$
\begin{gathered}
\Delta(N) = N \otimes I + I \otimes N + \frac{\beta}{\alpha} I \otimes I, \\
\Delta(a) = a_q \otimes q^{\frac{\alpha\tilde{N}}{2}} + (-1)^{\tilde{N}} q^{-\frac{\alpha\tilde{N}}{2}} \otimes a_q\ , \\
\Delta(a_q^\dagger) = a_q^\dagger \otimes q^{\frac{\alpha\tilde{N}}{2}} + (-1)^{-\tilde{N}} q^{-\frac{\alpha\tilde{N}}{2}} \otimes a_q^\dagger\ , \\
\varepsilon(N) = -\frac{\beta}{\alpha}, \varepsilon(a_q) = \varepsilon(a_q^\dagger) = 0, \varepsilon(I) = 1, \\
S(N) = -N - \frac{2\beta}{\alpha}\ , \\
S(a_q) = -(-1)^{-\tilde{N}} q^{-\frac{\alpha}{2}} a\ ,\ S(a_q^\dagger) = -a_q^\dagger (-1)^{\tilde{N}+1} q^{\frac{\alpha}{2}}
\end{gathered}
\tag{10}
$$

provided that $\alpha \neq 0$. An opposite Hopf algebra structure also exists with coproduct Δ^T and antipode the inverse S^{-1} of S, which can be immediately deduced from S given in (10).

A Fock-type representation of $B_q(\alpha, \beta)$, and $B_q^+(\alpha, \beta)$ with $a_q|0>_q = 0$, $N|n>_q = n|n>_q$, $n \in \mathbb{Z}_+$ and ${}_q<0|0>_q = 1$, exists such that with $\alpha \neq 0, \beta \neq 0$

$$
\begin{gathered}
|n>_q = \frac{1}{\sqrt{(n)^q!}} (a_q^\dagger)^n |0>_q, \\
a_q|n>_q = \sqrt{(n)^q}\,|n-1>_q,\ a_q^\dagger|n>_q = \sqrt{(n+1)^q}\,|n+1>_q,
\end{gathered}
$$

where

$$(n)^q = \frac{1}{q-q^{-1}}\Big(q^{\frac{\alpha(n-1)}{2}+\beta} \frac{(q^{\frac{\alpha n}{2}} - (-1)^n q^{-\frac{\alpha n}{2}})}{(q^{\frac{\alpha}{2}} + q^{-\frac{\alpha}{2}})} -$$

$$- q^{-\frac{\alpha(n-1)}{2}-\beta} \frac{(q^{-\frac{\alpha n}{2}} - (-1)^n q^{\frac{\alpha n}{2}})}{(q^{\frac{-\alpha}{2}} + q^{\frac{\alpha}{2}})}\Big) \tag{11}$$

$$(n)^q! = \prod_{m=1}^{n} (m)^q, \;\; {}_q < n|n' >_q = \delta_{nn'}$$

and $(-1)^{\pm N}|n >= (-1)^{\pm n}|n >$. In the limit $q \to 1$ we get the Fock space of the undeformed algebra $B(\alpha, \beta)$ (and $B^+(\alpha, \beta)$).

Finally an R-matrix for $B_q^+(\alpha, \beta)$ exists and is given by

$$R = R_0 q^{\alpha \tilde{N} \otimes \tilde{N}} \sum_{l=0}^{\infty} (q-q^{-1})^l q^{-\frac{\alpha}{4}l(l+1)} \frac{(-1)^{\frac{l}{4}(l-1)}}{[l]_x!} q^{\frac{\alpha}{2} l \tilde{N}} \cdot \tag{12}$$

$$\cdot (-1)^{l\tilde{N}} (a_q^\dagger)^l \otimes q^{-\frac{\alpha}{2} l \tilde{N}} a_q^l$$

where $\widetilde{N} = N + \beta/\alpha$, $x = (-q^{-\alpha})^{1/2}$ and

$$R_0 = \frac{1}{2}(I \otimes I + I \otimes (-1)^{\tilde{N}} + (-1)^{\tilde{N}} \otimes I - (-1)^{\tilde{N}} \otimes (-1)^{\tilde{N}}) \tag{13}$$

provided that we demand that $(-1)^{2\tilde{N}} = I$. In [43] it was also shown that similarly to [42], $B_q^+(\alpha, \beta)$ is the spectrum generating quantum group for the ordinary q-deformed harmonic oscillator defined by the relations $a_q a_q^\dagger - q^{\pm 1} a_q^\dagger a_q = q^{\mp N}$ and the last two of (9). The R-matrices for the undeformed Hopf algebra can now be read off from (12), (13) at the limit $q \to 1$, where $R \to R_0$.

3. Quotients of $B^+(\alpha, \beta)$ and $B_q^+(\alpha, \beta)$.

Consider an element $C \in U(B^+(\alpha, \beta))$ and the two-sided, not necessarily Hopf ideal, I generated by $I_x = ad_x C - \varepsilon(x) C$ for x being each one of the generators of $B^+(\alpha, \beta)$. The quotient algebra $B' = B^+(\alpha, \beta)/I$, which will not be a Hopf algebra in general, will have C as central. In the special case where C is such that I_x is identically zero (for all x), C belongs to the fixed points of the adjoint action. The aim here is to show that appropriately chosen $C's$ can lead together with the use of the *equation* $ad_x C - \varepsilon(x)C = 0$ to known oscillator models as quotients B'. It is very important to state what the word equation means: we do not regard $ad_x C - \varepsilon(x)C = 0$ in the usual way in the context of fixed points and centralizers, but rather as an algebraic equation which helps deducing the defining relations that will hold in B'. We shall demonstrate the above considerations with examples. Assume that C has the form

$$C = a^\dagger a f(N) + l(N) \tag{14}$$

for some functions $f(N)$ and $l(N)$ of N. Then it can be easily seen that I_N and $I_{(-1)^N}$ are identically zero, while

$$\begin{aligned} I_a &= -a^\dagger a^2(f(N) + f(N-1)) + a[\alpha(N-1) + \beta]f(N) + \\ &\qquad + a(l(N) - l(N-1)) \\ I_{a^\dagger} &= (a^\dagger)^2 a(f(N) + f(N+1)) - a^\dagger[\alpha N + \beta]f(N+1) + \\ &\qquad + a^\dagger(-l(N+1) + l(N)). \end{aligned} \tag{15}$$

At this point we can decide what form the functions f and l must have, then set $I_\alpha = 0 = I_{a^\dagger}$ and solve to obtain the relations that will hold in B'. So assuming that the functions f and l have already been given, then from (15) we obtain that in B'

$$\begin{aligned} a^\dagger a(f(N+1) + f(N)) &= \\ &= [\alpha N + \beta]f(N+1) + l(N+1) - l(N). \end{aligned} \tag{16}$$

If we consider for example the choices $\alpha = 2$, $\beta = 1$, $f = I/2$, $l = -N/2$, we obtain from (16) that in the quotient algebra B' the following relations should hold: $a^\dagger a = N$ and $aa^\dagger = N + 1$, together with the last two of (4) which are easily recognized as the defining relations of the well known boson algebra extended by $(-1)^{a^\dagger a}$. In a similar way we can proceed to investigate other choices of C, f, l in relation to B'. In fact it is possible to start with a desired B' and follow the inverse way which will determine if this B' can indeed be a quotient of $B(\alpha, \beta)$.

Finally if in the above method we first demand that I_a and $I_{a^\dagger}$ are zero and try to find f, l, we shall obtain the necessary and sufficient conditions for C to be in the fixed point set of the adjoint action. $I_a = 0$ will then imply (after considering I_a and $I_{a^\dagger}$ as monomials in the generators) that:

$$f(N) + f(N-1) = 0 \tag{17}$$

and we obtain a similar result from $I_{a^\dagger} = 0$, with N replaced by $N + 1$ above. At this point we cannot state anything about the functions $l(N)$ since we cannot formulate the rest of (15) as appropriate order monomials in the generators. Relations (17) suggest admissible functions f which in turn will give admissible l. Here admissibility means choices of functions that do not contradict with the Hopf algebra structure (or else it would not be possible to implement the adjoint action). The simpletest solution to (17) is $f(N) = (-1)^N$. Then the choice $l(N) = \gamma N(-1)^N$ leads to I_a, $I_{a^\dagger}$ being identically zero *provided* that $\alpha + 2\gamma = 0 = \beta - \alpha - \gamma$.

Passing now to the deformed case $B_q^+(\alpha, \beta)$ the analysis goes through in exactly the same way as above. Let for example

$$C = a_q^\dagger a_q f(N) + l(N). \tag{18}$$

Then it can easily be checked by using the definition of the adjoint action and relations (9) that $I^q_{\tilde{N}} = 0 = I^q_{(-1)^{\tilde{N}}}$, while

$$\begin{aligned} I^q_{a^\dagger} &= ((a^\dagger)^2 a(f(N) + f(N+1)) - \\ &- a^\dagger[\alpha N + \beta]_q f(N+1) + a^\dagger(l(N) - l(N+1)))q^{-\frac{\alpha}{2}\tilde{N}} \\ I^q_a &= (-a^\dagger a^2(f(N) + f(N-1)) + \\ &+ [\alpha N + \beta]_q f(N+1)a + a(l(N) - l(N-1))q^{-\frac{\alpha}{2}\tilde{N}}. \end{aligned} \tag{19}$$

If we now take for simplicity $\alpha = 2, \beta = 1$ then the choice $f(N) = -(q + q^{-1}), l(N) = [2N]_q$ leads to a quotient B' where the following familiar q-boson relations hold:

$$a_q a_q^\dagger = [N+1]_{q^2} \quad a_q^\dagger a_q = [N+1]_{q^2}, \tag{20}$$

and thus

$$a_q a_q^\dagger - q^2 a_q^\dagger a_q = q^{-2N}$$

together with the last two of (9) and an invertible element g that will anticommute with $a_q, a_q^\dagger$ and squares to I. Similarly with the undeformed case, if we want the above C to be a fixed point of the adjoint action, say for $\alpha = 2, \beta = 1$, then we obtain a solution where $f(N) = -(-1)^N$, $l(N) = (-1)^N[N]_{q^2}$.

Generalization of the above considerations to obtain other quotients is straightforward in both the deformed and undeformed cases, and they can also involve more that one C. In particular the quotients obtained above of the undeformed and q-deformed bosons, will be used in what follows in characterizing the proposed deformed and undeformed algebras and also in inducing appropriate representations.

4. Characterization of $B_q(2, 1)$ (and $B_q^+(2, 1)$).

Let $C(f)$ be associated to the bilinear forms $\mathbf{1}\langle f, f\rangle = (f \otimes f)$ so that the relations between the generators for $C(f)$ are $\{f, f\} = 2$. Similar considerations hold for a complex Clifford algebra $C(f, f^\dagger)$ generated by $f, f^\dagger$ (i.e. $\{f^\dagger, f^\dagger\} = 2I$ and $\{f^\dagger, f\} = 2I$). We shall construct oscillator representations of B_q through the action of the Clifford algebras $C(f)$. The need of the Clifford algebra comes from the existence of the invertible element $(-1)^N$ in B_q^+. B_q had been enlarged by adding $(-1)^N$ to obtain an Hopf algebra. A similar process was called bosonization in [42]. In fact set $g = (-1)^N$, and demand that $g^2 = I$, then $\{g, g\} = 2I$ so g is a Clifford element. Similarly we can considered together with g also $g^\dagger = g^{-1}$. Then $\{g, g^\dagger\} = 2I$.

Let $E_q(\mathbb{C})$ be the algebra defined by (20) whose annihilation and creation operators we shall denote by $b, b^\dagger$ respectively, to avoid complications in what will follow. Denote by $A_f = C(f) \otimes E_q(\mathbb{C})$ the tensor of the oscillator algebra by the real Clifford algebras. A_f is considered as enveloping algebra, generated by typical elements of the form $\left(\tilde{f}, \tilde{b}\right)$ with $\tilde{f} \in C(f)$ and $\tilde{b} \in E_q(\mathbb{C})$, where the multiplication is given as usual by

$$\left(\tilde{f}, \tilde{b}\right)\left(\tilde{f}', \tilde{b}'\right) = \left(\tilde{f}\tilde{f}', \tilde{b}\tilde{b}'\right).$$

The adjoint is $\left(\tilde{f}, \tilde{b}\right)^\dagger = \left(\tilde{f}^\dagger, \tilde{b}^\dagger\right)$. If we restrict ourselves to the tensor algebra $\mathcal{T}(E)$ generated only by $f, f^\dagger$ (and where E is the vector space associated to $C(f, f^\dagger)$), it is a known fact that the creation and annihilation operators are seen to generate a Clifford algebra which is isomorphic to an endomorphism subalgebra of $\mathcal{T}(E)$. It is not an algebra isomorphism. The creation operator is

$$e_f(a) = f \otimes a, \forall a \in \mathcal{T}(E)$$

and the annihilation operator is given by the unique linear map $i_f^* : \mathcal{T}(E) \to \mathcal{T}(E)$, $f^* \in E^*$

$$i_f^*(1) = 0, i_f^* \circ e_f = \langle x, f^*\rangle,\ x \in \mathcal{T}(E),\ f^* \in E^*.$$

We may think of $\langle x, f^*\rangle$ as the inner product in the vector space E with $\dim E = 2$. There exists a natural extension to an antiderivation on the exterior algebra so that

$$i_f^*(x_1 \times x_2 \times \ldots \times x_h) = \sum_{j=1}^{h} (-1)^{j+1} \langle x_j, f^*\rangle (x_1 \times \ldots \times \hat{x}_j \times \ldots x_h).$$

Thus there is an action $\alpha_f : C(f) \to End(A_f)$ s.t.

$$\alpha_f(x) = f \otimes x, \quad x \in A_f.$$

Consider now the algebra A_f and the quotient map $\phi : B_q \to E_q(\mathbb{C})$ such that

$$\begin{aligned} a_q &\longrightarrow b_q \\ a_q^\dagger &\longrightarrow b_q^\dagger \\ N &\longrightarrow \log_q q^N, \end{aligned}$$

where $b_q b_q^\dagger = [N+1]_{q^2}$ and $b_q^\dagger b_q = [N]_{q^2}$. This quotient map is nothing else but the restriction to the subalgebra B_q of a quotient map of B_q^+ to its quotient (isomorphic to A_f), to which we referred in the previous section. Thus $a_q^\dagger a_q \longrightarrow [N]_{q^2}$, and $a_q a_q^\dagger \longrightarrow [N+1]_{q^2}$ as it should be. It is a well defined algebra homomorphism. Consider now $\pi \in Hom(A_f, E_q(\mathbb{C}))$, defined by

$$\pi(a_f) = \sum_{f'} \langle f, a_{f'}\rangle \hat{a}_{f'}, \quad a_f \in A_f, \quad a_{f'} \in E_q(\mathbb{C})$$

where $\hat{a}_{f'}$ is the word a_f in which we delete the part containing $f' \in C(f)$. Observe that there exists at most one $f' \in C(f)$ in any word a_f because $f \otimes f + f \otimes f = 2$, so that $\pi(a_f) = \langle f, a_f\rangle \hat{a}_f$. Thus $\hat{a}_f$ is a word containing no $f's$. Hence $\pi(a_f) \in E_q(\mathbb{C})$, for every $a_f \in A_f$. Note that the map π is uniquely defined by the following condition between the creation and annihilation operators in $T(E)$

$$\pi \circ \alpha_{\tilde{f}} = \langle \tilde{f}, f^*\rangle, \;\; \tilde{f} \in T(E), \;\; \forall f^* \in E^*,$$

thus

$$\pi \circ \alpha_{\tilde{f}}(a_f) = \langle \tilde{f}, [a_f]_{f'}\rangle \hat{a}_f$$

where $[a_f]_{f'}$ denotes the part of a_f containing $f' \in C(f)$ since the maps extend to the algebra A_f. Then the following theorem holds.

Theorem. *Let B_q^+ be the deformed boson algebra defined in* (9) *and let $E_q(\mathbb{C})$ be the q-oscillator algebra defined in* (20). *There exists a representation ϕ of B_q into $E_q(\mathbb{C})$ and a vector space V_ϕ such that the map*

$$\pi : B_q \times V_\phi \longrightarrow V_\phi$$

gives a representation of B_q induced from that of $E_q(\mathbb{C})$.

Proof. The existence of the representation $\phi : B_q \longrightarrow E_q(\mathbb{C})$ follows from the existence of the element C where the adjoint map acts trivially. Thus there exists at least one such ϕ and it is an algebra homomorphism.

Consider $\alpha : C(f) \times E_q(\mathbb{C}) \longrightarrow E_q(\mathbb{C})$ given by

$$\alpha_f(e_q) = -e_q, \;\; \alpha_1(e_q) = e_q.$$

The action α is such that f changes the sign of the element $e_q \in E_q(\mathbb{C})$ and 1 acts as the identity on $E_q(\mathbb{C})$. This is an action of $C(f)$ on $End(E_q(\mathbb{C}))$. Define the algebra $\tilde{A}_f = C(f) \otimes_\alpha E_q(\mathbb{C})$ to be the smash product of $C(f)$ by $E_q(\mathbb{C})$ under the action α. The elements of $\tilde{A}_f$ can be seen as pairs of the form (f, e_q). The product of any two such pairs is given by the following rule

$$(f, e_q)(f', e'_q) = (ff', \alpha_f(e'_q)e_q).$$

In fact, for example when $e_q = b_q$ or $e_q = b_q^\dagger$ then

$$(f, 1)(1, e_q) = (f1, \alpha_f(e_q)1) = (f1, -e_q)$$

and

$$(1, e_q)(f, 1) = (1f, \alpha_1(1)e_q) = (f, e_q)$$

thus $(f, 1)$ and $(1, e_q)$ anticommute in agreement with $\left\{(-1)^N, a_q\right\} = 0$ in B_q^+. Moreover if for example $e_q = b_q b_q^\dagger$ or N, then with a similar reasoning $(1, e_q)$ commutes with $(f, 1)$ as it should.

The Clifford algebra $C(f)$ can be realized as endomorphism subalgebra of the tensor algebra $T(E)$, E being the vector space associated to $C(f)$. There exists a unique linear map

$$i_\Phi : T(E) \longrightarrow T(E), \;\; \Phi \in E^*$$

such that

$$i_\Phi(1) = 0, i_\Phi \circ e_f = \langle x, \Phi\rangle \tag{21}$$

where e_f is the creation operator on $T(E)$. Define $\hat{i} : C(f) \otimes E_q(\mathbb{C}) \longrightarrow C(f) \otimes E_q(\mathbb{C})$ to be the extension of i,

$$\left(\hat{i} \circ e_f\right)|_{C(f)} = \langle x, \Phi\rangle, \text{ for every } \Phi \in E^*, x \in C(f) \subset T(E). \tag{22}$$

Then $\hat{i}$ is uniquely defined by the map i via the condition (21). Since $C(f)$ is a one-dimensional Clifford algebra then $\hat{i} : C(f) \otimes E_q(\mathbb{C}) \longrightarrow E_q(\mathbb{C})$. In particular $\hat{i}$ gives a map from $\tilde{A}_f \longrightarrow E_q(\mathbb{C})$. $\tilde{A}_f$ is a universal algebra with respect to the tensor product such that condition (22) is satisfied. From the above discussion there exists $\Psi : B_q \longrightarrow \tilde{A}_f$ linear mapping and also an algebra homomorphism. Let V_ϕ be the vector space defined as the set of linear mappings

$$h : \tilde{A}_f \longrightarrow E_q(\mathbb{C}) \text{ such that } h \circ \Psi = \phi. \tag{23}$$

Observe that $E_q(\mathbb{C})$ is naturally imbedded in $\tilde{A}_f$ as the pairs $(1, e_q)$ for every $e_q \in E_q(\mathbb{C})$, $e_q \longrightarrow (1, e_q)$. Thus $E_q(\mathbb{C})$ is a subalgebra of $\tilde{A}_f$. The map $\hat{i} : C(f) \otimes E_q(\mathbb{C}) \longrightarrow E_q(\mathbb{C})$ is a projection onto $E_q(\mathbb{C})$ of A_f, $\hat{i}^2((f, e_q)) = \hat{i}(\langle f, \Phi\rangle e_q) = \langle f, \Phi\rangle e_q = i((f, e_q))$. It also satisfies

$$\hat{i}(a_f e_q) = \hat{i}(a_q)e_q.$$

In fact, $\hat{i}(a_f e_q) = \langle f, \Phi\rangle e'_q e_q$, since e_q does not contain any $f's$, so that it is left invariant by $\hat{i}$. Thus

$$\hat{i}(a_f e_q) = \hat{i}(a_f)e_q.$$

We want to show that V_ϕ is a B_q-module. Consider $B_q \times V_\phi \longrightarrow V_\phi$; the action is given by

$$(x_q h)(a_f) = h(\Psi(x_q)a_f) \in E_q(\mathbb{C}).$$

Now let us prove that $(x_q h)$ satisfies (24). In fact

$$\begin{aligned}((x_q \circ h) \circ \Psi)(x'_q) &= (x_q \circ h)\Psi(x'_q) = h(\Psi(x_q)\Psi(x'_q)) \\ &= h(\Psi(x_q x'_q))\end{aligned}$$

in view of the fact that $x_q x'_q \in B_q^+$, and

$$h(\Psi(x_q x'_q)) = \phi(x_q x'_q)$$

so that (23) is satisfied. Thus V_ϕ is a B_q-module and by its construction it is easily seen to be an induced module from the representation of $E_q(\mathbb{C})$. □

The discussion of the undeformed case follows along the same line of argument and the element C is the natural one to give the normal boson algebra as a quotient.

5. Characterization of the undeformed boson algebra $B(2,1)$.

Let $B(2,1)$ be the boson algebra generated by $a, a^\dagger, N, (-1)^N$ satisfying relations (4).

We have the following two classes of *-homomorphisms of B into A_f and $A_{f,f^\dagger}$ respectively. Let $\pi_f : \mathcal{B} \longrightarrow End(A_f)$ as $a(b,f) = b \otimes f, a^\dagger(b,f) = b^\dagger \otimes f, (-1)^N(b,f) = f\sqrt{2}, N(b,f) = \frac{1}{2}(b^\dagger b - 1 \otimes 1)$. Instead of $b \otimes f$ we write bf and so on.

Thus $(aa^\dagger)(b,f) = bb^\dagger$ and $(a^\dagger a)(b,f) = b^\dagger b$ so from $aa^\dagger + a^\dagger a = 2N - 1$ it follows that

$$(aa^\dagger)(b,f) = bb^\dagger = 1 - b^\dagger b = 1 - a^\dagger a$$

implies $[a, a^\dagger](b,f) = 1$. The image of π_f is just the oscillator algebra $E_1(\mathbb{C})$ enlarged with the element $f\sqrt{2} = (-1)^N(b,f)$. It is a bosonization of the normal boson algebra.

Acknowledgments. The authors would like to thank P.E. Jorgensen, P.D. Jarvis, D. McAnally and R. Zhang for their sincere interest, support and fruitful comments. One of us (I.T.) would also like to thank the Mathematical Physics group of the Mathematics Department, University of Queensland for warm hospitality, where this work was completed.

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