ON SZASZ-MIRAKYAN OPERATORS OF FUNCTIONS OF TWO VARIABLES

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We consider Szasz-Mirakyan operators $s^{(i)}_{m,n}$ in polynomial and exponential weighted spaces of functions of two variables. We give Voronowskaya type theorem and theorem on convergence of sequence $\left\{ \frac{1}{\pi^2} s^{(i)}_{m,n}(f) \right\}$.

1. Preliminaries.

1.1. Similarly as in [1] and [2], for fixed $p \in N_0 := \{0, 1, 2, \ldots \}$ and $q \in R_+ := (0, +\infty)$ and for all $x \in R_0 := R_+ \cup \{0\}$, we define

\[(1) \quad w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1,\]

\[(2) \quad v_q(x) := e^{-q x}.\]

Next, for fixed $p_1, p_2 \in N_0$, we define the weighted function

\[(3) \quad w_{p_1, p_2}(x, y) := w_{p_1}(x) w_{p_2}(y), \quad (x, y) \in R^2_0 := R_0 \times R_0,\]

and the polynomial weighted space $C_{1: p_1, p_2}$ of real-valued functions $f$ continuous on $R^2_0$ for which $f w_{p_1, p_2}$ is uniformly continuous and bounded on $R^2_0$. The norm in $C_{1: p_1, p_2}$ is defined by

\[(4) \quad \|f\|_{1: p_1, p_2} := \sup_{(x,y) \in R^2_0} w_{p_1, p_2}(x, y) |f(x, y)|.\]
Analogously, for fixed $q_1, q_2 \in R_+$, we define

$$v_{q_1, q_2}(x, y) := v_{q_1}(x)v_{q_2}(y), \quad (x, y) \in R_0^2,$$

and the exponential weighted space $C^{2, q_1, q_2}$ of real-valued functions $f$ continuous on $R_0^2$ for which $f v_{q_1, q_2}$ is uniformly continuous and bounded on $R_0^2$. The norm in $C^{2, q_1, q_2}$ is given by

$$\|f\|_{2, q_1, q_2} := \sup_{(x, y) \in R_0^2} v_{q_1, q_2}(x, y)|f(x, y)|.$$

Moreover, for fixed $m \in N := \{1, 2, \ldots\}$ and $p_1, p_2 \in N_0$, let $C^m_{1, p_1, p_2}$ be the class of all functions $f \in C_{1, p_1, p_2}$ which partial derivatives of the order $\leq m$ belong to $C_{1, p_1, p_2}$ also. Analogously we define the class $C^m_{2, q_1, q_2}, m \in N$ and $q_1, q_2 \in R_+$.

1.2. In [3] were examined the Szasz-Mirakyan operators for functions $f$ continuous on $R_0^2$

$$S^{[1]}_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)f\left(\frac{j}{m}, \frac{k}{n}\right),$$

$$S^{[2]}_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)a_{n,k}(y)mn \int_{\frac{j}{m}}^{\frac{j+1}{m}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t, z) \, dt \, dz,$$

$(x, y) \in R_0^2, m, n \in N$, where

$$a_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}, \quad x \in R_0, \quad k \in N_0, \quad n \in N.$$

These operators are analogues of the Szasz-Mirakyan operators, considered in [1] – [3] for functions $f$ of one variable

$$S^{[1]}_{n}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)f\left(\frac{k}{n}\right),$$

$$S^{[2]}_{n}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) \, dt, \quad x \in R_0, \quad n \in N.$$
From the results given in [3] we can deduce that if \( f \in C_{1; p_1, p_2} \) or \( f \in C_{2; q_1, q_2} \), with some \( p_1, p_2 \in \mathbb{N}_0 \) and \( q_1, q_2 \in \mathbb{R}_+ \), then
\[
\lim_{m, n \to \infty} S_{m,n}^{(i)}(f; x, y) = f(x, y), \quad i = 1, 2,
\]
for every \((x, y) \in \mathbb{R}_0^2\).

In the present paper we shall prove some analogues of (12) for derivatives of the operators (7) and (8). In Section 2 we shall give some auxiliary results and in Section 3 we shall prove the main theorems.

By \( M_k(a, b), k = 1, 2, \ldots \), we shall denote suitable positive constants depending only on indicated parameters \( a, b \). The partial derivative of function \( f \) we shall denote as usual by \( f'_x \) or \( \frac{\partial f}{\partial x} \).

2. Auxiliary results.

2.1. First we shall give some properties of the operators \( S_n^{(i)} \) and \( S_{m,n}^{(i)} \) proved in [1]–[3].

From (7)–(11) it follows that
\[
S_n^{(i)}(1; x) = 1, \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \quad i = 1, 2,
\]
\[
S_{m,n}^{(i)}(1; x, y) = 1, \quad (x, y) \in \mathbb{R}_0^2, \quad m, n \in \mathbb{N}, \quad i = 1, 2.
\]

Moreover, if \( f \in C_{1; p_1, p_2} \) or \( f \in C_{2; q_1, q_2} \) (\( p_1, p_2 \in \mathbb{N}_0, \ q_1, q_2 \in \mathbb{R}_+ \)) and if \( f(x, y) = f_1(x) f_2(y) \) for all \((x, y) \in \mathbb{R}_0^2\) then
\[
S_{m,n}^{(i)}(f(t, z); x, y) = S_{m,n}^{(i)}(f_1(t); x) S_{m,n}^{(i)}(f_2(z); y)
\]
for all \((x, y) \in \mathbb{R}_0^2, \ m, n \in \mathbb{N} \) and \( i = 1, 2 \).

For every fixed \( x \in \mathbb{R}_0 \) and for all \( n \in \mathbb{N} \) we have ([1])
\[
S_n^{(i)}(t - x; x) = \begin{cases} 
0 & \text{if } i = 1, \\
\frac{1}{2n} & \text{if } i = 2,
\end{cases}
\]
\[
S_n^{(i)}((t - x)^2; x) = \begin{cases} 
\frac{x}{n} & \text{if } i = 1, \\
\frac{x}{n} + \frac{1}{3n^2} & \text{if } i = 2.
\end{cases}
\]

Lemma 1 ([1]). For every fixed \( x_0 \in \mathbb{R}_0 \) there exists a positive constant \( M_1(x_0) \) such that for all \( n \in \mathbb{N} \) and \( i = 1, 2 \)
\[
S_n^{(i)}((t - x_0)^2; x_0) \leq M_1(x_0)n^{-2}.
\]
Lemma 2 ([1]). For every fixed \( p \in N_0 \) there exists a positive constant \( M_2(p) \) such that for all \( x \in R_0, n \in N \) and \( i = 1, 2 \)

\[
   w_p(x) S^{[i]}_n(1/w_p(t); x) \leq M_2(p),
\]

\[
   w_p(x) S^{[i]}_n((t - x)^2/w_p(t); x) \leq M_2(p) \begin{cases} \frac{x}{n} & \text{if } i = 1, \\ \frac{x + 1}{n} & \text{if } i = 2. \end{cases}
\]

Lemma 3 ([2]). Let \( r > q > 0 \) are fixed numbers. Then there exist \( M_3(q, r) = \text{const} > 0 \) and natural number \( n_0 > q(\ln(r/q))^{-1} \) such that for all \( x \in R_0, n \geq n_0 \) and \( i = 1, 2 \)

\[
   v_q(x) S^{[i]}_n(1/v_q(t); x) \leq M_3(q, r)
\]

\[
   v_q(x) S^{[i]}_n((t - x)^2/v_q(t); x) \leq M_3(q, r) \begin{cases} \frac{x}{n} & \text{if } i = 1, \\ \frac{x + 1}{n} & \text{if } i = 2. \end{cases}
\]

Applying these lemmas, (1) – (6) and (15), we immediately derive from (7) – (9) the following two lemmas.

Lemma 4. For fixed \( p_1, p_2 \in N_0 \) there exists \( M_4(p_1, p_2) = \text{const} > 0 \) such that for every \( f \in C_{1; p_1, p_2} \) and for all \( m, n \in N, i = 1, 2 \)

\[
   \|S^{[i]}_{m,n}(f; \cdot, \cdot)\|_{1; p_1, p_2} \leq M_4(p_1, p_2) \|f\|_{1; p_1, p_2}.
\]

In particular

\[
   \|S^{[i]}_{m,n}(1/w_{p_1, p_2}(t, z); \cdot, \cdot)\|_{1; p_1, p_2} \leq M_4(p_1, p_2) \text{ for } m, n \in N, i = 1, 2.
\]

From (7) – (9) and (18) we deduce that \( S^{[i]}_{m,n}, m, n \in N, i = 1, 2, \) is a linear positive operator from the space \( C_{1; p_1, p_2} \) into \( C_{1; p_1, p_2} \).

Lemma 5. For fixed \( r_1 > q_1 > 0 \) and \( r_2 > q_2 > 0 \) there exist \( M_5(q_1, q_2, r_1, r_2) = \text{const} > 0 \) and natural numbers \( m_0 > q_1(\ln(r_1/q_1))^{-1}, n_0 > q_2(\ln(r_2/q_2))^{-1} \) such that for all \( m \geq m_0, n \geq n_0 \) and \( i = 1, 2 \)

\[
   \|S^{[i]}_{m,n}(1/v_{q_1, q_2}(t, z); \cdot, \cdot)\|_{2; r_1, r_2} \leq M_5(q_1, q_2, r_1, r_2).
\]
Moreover, for every \( f \in C_{2, q_1, q_2} \) and for all \( m \geq m_0, \ n \geq n_0 \) and \( i = 1, 2 \) we have

\[
\|S_m^n(f; \cdot \cdot \cdot)\|_{2,r_1, r_2} \leq M_5(q_1, q_2, r_1, r_2)\|f\|_{2, q_1, q_2}.
\]

The formulas (7) – (9) and the inequality (20) prove that \( S_m^n, i = 1, 2, \) is a positive linear operator from the space \( C_{2, q_1, q_2} \) into \( C_{2, r_1, r_2} \) provided that \( r_1 > q_1 > 0, \ r_2 > q_2 > 0 \) and \( m \geq m_0, \ n \geq n_0. \)

3. Main results.

3.1. First we shall prove the Voronovskaya type theorem.

**Theorem 1.** Suppose that \( f \in C_{1, p_1, p_2} \) or \( f \in C_{2, q_1, q_2} \), with some \( p_1, p_2 \in N_0, \ q_1, q_2 \in R_+ \). Then, for every \( (x, y) \in R^2_{+} := R_+ \times R_+ \) and \( i = 1, 2 \), we have

\[
\lim_{n \to \infty} n \{ \{ S_m^n(f; x, y) - f(x, y) \} = \begin{cases} \frac{x}{2} f''_{xx}(x, y) + \frac{y}{2} f''_{yy}(x, y) + \\
0 \quad \text{if} \quad i = 1, \\
\frac{1}{2} f'_x(x, y) + \frac{1}{2} f'_y(x, y) \quad \text{if} \quad i = 2 \end{cases}
\]

**Proof.** Let \( i = 1, \ f \in C_{1, p_1, p_2} \) and let \( (x_0, y_0) \in R^2_{+} \) be fixed point. Then, by the Taylor formula, we can write

\[
f(t, z) = f(x_0, y_0) + f'_x(x_0, y_0)(t - x_0) + f'_y(x_0, y_0)(z - y_0) + \frac{1}{2} \{ f''_{xx}(x_0, y_0)(t - x_0)^2 + 2 f''_{xy}(x_0, y_0)(t - x_0)(z - y_0) + f''_{yy}(x_0, y_0)(z - y_0)^2 \} + \varphi(t, z; x_0, y_0)\sqrt{(t - x_0)^4 + (z - y_0)^4}, \quad (t, z) \in R^2_0,
\]

where \( \varphi(t, z) \equiv \varphi(t, z; x_0, y_0) \) belongs to \( C_{1, p_1, p_2} \) and \( \lim_{(t, z) \to (x_0, y_0)} \varphi(t, z) = 0. \)

From this, applying (13) – (15), we get

\[
S_{n, n}^{(1)}(f(t, z); x_0, y_0) = f(x_0, y_0) + \frac{1}{2} \{ f''_{xx}(x_0, y_0)S_{n}^{(1)}(t - x_0)^2 + 2 f''_{xy}(x_0, y_0)S_{n}^{(1)}(t - x_0)(z - y_0) + f''_{yy}(x_0, y_0)(z - y_0)^2 \} + S_{n, n}^{(1)}(\varphi(t, z)\sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0),
\]

\( n \in N. \)
But from (16) and (17) it follows that
\[
\lim_{n \to \infty} n S_n^{(i)}(t - x_0; x_0) = 0 = \lim_{n \to \infty} n S_n^{(i)}(z - y_0; y_0),
\]
(23)
\[
\lim_{n \to \infty} n S_n^{(1)}((t - x_0)^2; x_0) = x_0, \quad \lim_{n \to \infty} n S_n^{(1)}((z - y_0)^2; y_0) = y_0.
\]
(24)

By the Hölder inequality and by the linearity of $S_{n,n}^{(1)}$ and (13) – (15) we get
\[
\left| S_{n,n}^{(1)}(\varphi(t, z) \sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0) \right| \leq \left[ S_{n,n}^{(1)}(\varphi^2(t, z); x_0, y_0) \right]^{1/2} \left[ S_{n,n}^{(1)}((t - x_0)^4; x_0) + S_{n,n}^{(1)}((z - y_0)^4; y_0) \right]^{1/2}, \ n \in \mathbb{N}.
\]

But by properties of $\varphi$ and (12), we have
\[
\lim_{n \to \infty} S_{n,n}^{(1)}(\varphi^2(t, z); x_0, y_0) = \varphi^2(x_0, y_0) = 0.
\]

From the foregoing facts and Lemma 1 we obtain
\[
\lim_{n \to \infty} n S_{n,n}^{(1)}(\varphi(t, z) \sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0) = 0.
\]
(25)

Next, using (23) – (25), we derive from (22)
\[
\lim_{n \to \infty} n \left\{ S_{n,n}^{(1)}(f(t, z); x_0, y_0) - f(x_0, y_0) \right\} = \frac{x_0}{2} f''_{xx}(x_0, y_0) + \frac{y_0}{2} f''_{yy}(x_0, y_0).
\]

Thus the proof of (21) for $i = 1$ and $f \in C^2_{1;p_1,p_2}$ is completed. The proof of (21) in the other cases is analogous. \[\square\]

3.2. Now we shall give analogues of (12) for partial derivatives of $S_{n,n}^{(i)}(f; \cdot, \cdot)$.

Theorem 2. Suppose that $f \in C^1_{1;p_1,p_2}$ or $f \in C^1_{2;q_1,q_2}$ with some $p_1, p_2 \in \mathbb{N}_0$, $q_1, q_2 \in \mathbb{R}_+$. Then for every $(x, y) \in \mathbb{R}^2_+$ and $i = 1, 2$
\[
\lim_{n \to \infty} \frac{\partial}{\partial x} S_{n,n}^{(i)}(f; x, y) = \frac{\partial f}{\partial x}(x, y),
\]
(26)
\[
\lim_{n \to \infty} \frac{\partial}{\partial y} S_{n,n}^{(i)}(f; x, y) = \frac{\partial f}{\partial y}(x, y).
\]
(27)
\textbf{Proof.} We shall prove only (26), because the proof of (27) is identical. Let \( i = 1, f \in C^1_{1; p_1, p_2} \) and let \((x, y)\) be a fixed point in \( R^2_0 \). From (7) and (9) it follows that
\[
\frac{\partial}{\partial x} S_{n,n}^{(1)}(f(t, z); x, y) = -n S_{n,n}^{(1)}(f(t, z); x, y) + \frac{n}{x} S_{n,n}^{(1)}(tf(t, z); x, y)
\]
for every \( n \in N \). Applying the Taylor formula for \( f \in C^1_{1; p_1, p_2} \), we can write
\[
f(t, z) = f(x, y) + f_x'(x, y)(t - x) + f_y'(x, y)(z - y) + \psi(t, z; x, y)\sqrt{(t - x)^2 + (z - y)^2}, \quad (t, z) \in R^2_0,
\]
where \( \psi(t, z) \equiv \psi(t, z; x, y) \) is function of the class \( C_{1; p_1, p_2} \) and
\[
\lim_{(t, z) \to (x, y)} \psi(t, z) = 0.
\]

From the foregoing formulas and by (13) – (15) we get
\[
\begin{align*}
\frac{\partial}{\partial x} S_{n,n}^{(1)}(f(t, z); x, y) = & -n \left\{ f(x, y) + f_x'(x, y) S_n^{(1)}(t - x); x \right\} + \\
& + f_y'(x, y) S_n^{(1)}(z - y; y) + S_{n,n}^{(1)}(\psi(t, z)\sqrt{(t - x)^2 + (z - y)^2}; x, y) + \\
& + \frac{n}{x} \left\{ f(x, y) S_n^{(1)}(t; x) + f_x'(x, y) S_n^{(1)}(t(t - x); x) + \\
& + f_y'(x, y) S_n^{(1)}(t; x) S_n^{(1)}(z - y; y) + S_{n,n}^{(1)}(t \psi(t, z)\sqrt{(t - x)^2 + (z - y)^2}; x, y) \right\},
\end{align*}
\]
where by
\[
S_n^{(1)}(t(t - x); x) = S_n^{(1)}((t - x)^2; x) + x S_n^{(1)}(t - x; x)
\]
and by (16) and (17) implies
\[
\begin{align*}
\frac{\partial}{\partial x} S_{n,n}^{(1)}(f(t, z); x, y) = & f_x'(x, y) + \\
& + \frac{n}{x} S_{n,n}^{(1)}(t - x) \psi(t, z)\sqrt{(t - x)^2 + (z - y)^2}; x, y)
\end{align*}
\]
for all \( n \in \mathbb{N} \). Next, applying the Hölder inequality and (13) – (15), we have

\[
|S_{n,n}^{[1]}((t-x)\psi(t,z)\sqrt{(t-x)^2 + (z-y)^2}; x, y)| \leq \\
\leq \left[ S_{n,n}^{[1]}(\psi^2(t,z); x, y) \right]^{1/2} \left[ S_{n,n}^{[1]}((t-x)^4 + (t-x)^2(z-y)^2; x, y) \right]^{1/2} = \\
= \left[ S_{n,n}^{[1]}(\psi^2(t,z); x, y) \right]^{1/2} \left\{ S_{n,n}^{[1]}((t-x)^4; x) + S_{n,n}^{[1]}((t-x^2); x) S_{n,n}^{[1]}((z-y)^2; y) \right\}^{1/2},
\]

\( n \in \mathbb{N} \).

From the foregoing inequality and by (17), Lemma 1 and (12) we deduce that

\[
\lim_{n \to \infty} n S_{n,n}^{[1]}((t-x)\psi(t,z)\sqrt{(t-x)^2 + (z-y)^2}; x, y) = 0.
\]

Hence, from (29) we obtain

\[
\lim_{n \to \infty} \frac{\partial}{\partial x} S_{n,n}^{[1]}(f(t,z); x, y) = \frac{\partial f}{\partial x}(x, y) \quad \text{for} \, (x, y) \in \mathbb{R}_+^2,
\]

which completes the proof of (26) for \( i = 1 \).

Let \( f \in C^1_{p_1, p_2, i = 2} \) and \((x, y)\) be a fixed point in \( \mathbb{R}_+^2 \). From (7) – (11) it follows that

\[
\frac{\partial}{\partial x} S_{n,n}^{[2]}(f; x, y) = -n S_{n,n}^{[2]}(f; x, y) + \\
+ \frac{n}{x} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,j}(x) a_{n,k}(y) jn \int_{\frac{1}{x}}^{\frac{j+1}{x}} \int_{\frac{k}{z}}^{\frac{k+1}{z}} f(t, z) \, dt \, dz.
\]

Similarly as in the case \( i = 1 \), by (28) and by (13) – (17), we get

\[
\frac{\partial}{\partial x} S_{n,n}^{[2]}(f(t,z); x, y) = -n \left\{ f(x, y) + f_x^*(x, y) S_{n,n}^{[2]}(t-x; x) + \\
+ f_y^*(x, y) S_{n,n}^{[2]}(z-y; y) + S_{n,n}^{[2]}(\psi(t,z)\sqrt{(t-x)^2 + (z-y)^2}; x, y) \right\} + \\
+ \frac{n}{x} \left\{ f(x, y) S_{n,n}^{[1]}(t; x) + f_x^*(x, y) \sum_{j=0}^{\infty} a_{n,j}(x) j \int_{\frac{1}{x}}^{\frac{j+1}{x}} (t-x) \, dt + \\
+ \int_{\frac{1}{x}}^{\frac{j+1}{x}} \right\}
\]
\[ f_n^i(x, y)S_n^{[1]}(t; x)S_n^{[2]}(z-y; y) + \]
\[ + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y) \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(t, z) \sqrt{(t-x)^2 + (z-y)^2} \, dt \, dz \]
\[ = f_n^i(x, y) + \frac{n}{x} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y) \left( \frac{j}{n} - x \right) n^2. \]
\[ \cdot \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(t, z) \sqrt{(t-x)^2 + (z-y)^2} \, dt \, dz := f_n^i(x, y) + \frac{n}{x} A_n(x, y) \]

for \( n \in N \). Applying Hölder inequalities, we get for \( n \in N \)
\[ |A_n(x, y)| \leq \]
\[ \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,j}(x)a_{n,k}(y) \left| \frac{j}{n} - x \right| n \left\{ \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi^2(t, z) [(t-x)^2 + (z-y)^2] \, dt \, dz \right\}^{1/2} \]
\[ \leq \left\{ S_n^{[1]}((t-x)^2; x, y) \right\}^{1/2} \left\{ S_n^{[2]}(\psi^2(t, z)(t-x)^2; x, y) + S_n^{[2]}(\psi^2(t, z)(z-y)^2; x, y) \right\}^{1/2} \]
and, by (13) – (15),
\[ S_n^{[2]}(\psi^2(t, z)(t-x)^2; x, y) \leq \left\{ S_n^{[2]}(\psi^4(t, z); x, y) \right\}^{1/2} \left\{ S_n^{[2]}((t-x)^4; x) \right\}^{1/2}, \]
\[ S_n^{[2]}(\psi^2(t, z)(z-y)^2; x, y) \leq \left\{ S_n^{[2]}(\psi^4(t, z); x, y) \right\}^{1/2} \left\{ S_n^{[2]}((z-y)^4; y) \right\}^{1/2}, \]

which by Lemma 1, (12) and \( \psi(x, y) = 0 \) yield
\[ \lim_{n \to \infty} n S_n^{[2]}(\psi^2(t, z)(t-x)^2; x, y) = 0, \]
\[ \lim_{n \to \infty} n S_n^{[2]}(\psi^2(t, z)(z-y)^2; x, y) = 0. \]

From the above facts and by (17) we deduce that
\[ \lim_{n \to \infty} n A_n(x, y) = 0, \quad \text{for every fixed} \quad (x, y) \in R_+. \]

Using (31) to (30), we obtain (26) for \( i = 2 \).

The proof of (26) for \( f \in C_{1, q_1, q_2}^1 \) is identical. \( \square \)

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