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CERTAIN BANACH SPACES IN CONNECTION WITH BEST APPROXIMATIONS

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Given an increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_k \subset \ldots$ of subspaces of a Banach space X, for $x \in X$ we consider the series

$$|x| = \sum_{k=0}^{\infty} \operatorname{dist}(x, X_k).$$

Certain subspaces Z of X are Banach spaces with the norm $|\cdot|$. In order to prove this, a notion of equiconvergence of a family of numerical series is introduced.

1. Introduction.

Let $(X, \|\cdot\|)$ be a (real or complex) Banach space. Let $P \subset X$ a subspace of X. Given $x \in X$, if there exists $p \in P$ such that

$$\|x - p\| = \operatorname{dist}(x, P) = \inf\{\|x - u\| : u \in P\},$$

then p is called *a best* $\|\cdot\|$*-approximation to* x *from* P. If for any $x \in X$ there exists a unique best $\|\cdot\|$-approximation to x from P, then P is called a *Chebyshev subspace*.

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Let us consider an increasing sequence of Chebyshev subspaces of X,

$$X_0 \subset X_1 \subset \ldots \subset X_k \subset \ldots$$

For each $x \in X$ and each $k = 0, 1, 2, \ldots$, let us consider

$$\|x\|_k = \operatorname{dist}(x, X_k) = \inf\{\|x - u\| : u \in X_k\}.$$

It is immediate to prove that $\|\cdot\|_k$ is a seminorm on X. Consider the series

$$|x| = \sum_{k=0}^{\infty} \|x\|_k.$$

It is clear that the set

$$Y = \{x \in X : |x| < \infty\}$$

is a linear subspace of X and any X_k is contained in Y. Moreover $|\cdot|$ is a seminorm on Y and the kernel of the seminorm is X_0. The associated notions to $\|\cdot\|$ and $|\cdot|$ are distinguished by means of those symbols: $\|\cdot\|$-limit, $|\cdot|$-Cauchy, etc.

In Section 2 we relate the best $\|\cdot\|$-approximation with the best $|\cdot|$-approximations, and we prove that the difference belongs to X_0.

In Section 3, given a $\|\cdot\|$-closed subspace $M \subset X$, if $Z = M \cap Y$ satisfies certain conditions, then $|\cdot|$ is a norm on Z and Z is a Banach space. For this purpose we consider a notion of equiconvergence of a family of numerical series.

Finally, in Section 4, we give some examples.

2. Best approximations.

In this section we prove that the difference between the best $\|\cdot\|$-approximation and a best $|\cdot|$-approximation and a best $|\cdot|$-approximation to elements of Y from X_k belongs to X_0. For this purpose several simple results are necessary, the proofs of which are straightforward and so omitted.

Lemma 1.
(1) *If* $x \in X$, $k \geq n \geq 0$ *and* $u \in X_n$, *then* $\|x - u\|_k = \|x\|_k$.
(2) *If* $x \in X$, $n \geq k \geq 0$ *and* $u \in X_n$, *then* $\|x - u\|_k \geq \|x\|_n$.
(3) *If* $x \in X$, $n \geq k \geq 0$ *and if* p_n *is the best* $\|\cdot\|$*-approximation to* x *from* X_n, *then* $\|x - p_n\|_k = \|x\|_n$.

(4) *If* $y \in Y$, $n \geq 1$ *and* $u \in X_n$, *then*

$$|y - u| = \sum_{k=0}^{n-1} \|y - u\|_k + \sum_{k=n}^{\infty} \|y\|_k.$$

(5) *If* $y \in Y$ *and* $n \geq 1$, *and if* p_n *is the best* $\| \cdot \|$*-approximation to* y *from* X_n, *then*

$$|y - p_n| = |y| + n\|y\|_n - \sum_{k=0}^{n-1} \|y\|_k = n\|y\|_n + \sum_{k=n}^{\infty} \|y\|_k.$$

Theorem 2. *If* p_n *is the unique best* $\| \cdot \|$*-approximation to* $y \in Y$ *from* X_n, *then*

$$p_n + X_0 = \{p_n + w : w \in X_0\}$$

is the set of all the best $| \cdot |$*-approximations to* y *from* X_n.

Proof. Let $w \in X_0$. We prove that $p_n + w$ is an best $| \cdot |$-approximation to y from X_n. Applying the Lemma 1 (4) and (2), for any $u \in X_n$, result

$$|y - u| = \sum_{k=0}^{n-1} \|y - u\|_k + \sum_{k=n}^{\infty} \|y\|_k \geq n\|y\|_n + \sum_{k=n}^{\infty} \|y\|_k.$$

On the other hand, by Lemma 1 (1), we obtain

$$|y - p_n - w| = \sum_{k=0}^{\infty} \|y - p_n - w\|_k =$$

$$= \sum_{k=0}^{n-1} \|y - p_n - w\|_k + \sum_{k=n}^{\infty} \|y - p_n - w\|_k = n\|y\|_n + \sum_{k=n}^{\infty} \|y\|_k.$$

Then $|y - u| \geq |y - p_n - w|$, for every $u \in X_n$; that is, $p_n + w$ is a best $| \cdot |$-approximation to y from X_n.

Conversely, let q_n be a best $| \cdot |$-approximation to y from X_n. Taking into account that p_n is also a best $| \cdot |$-approximation as really we have proved in the first part of this proof, and by Lemma 1 (4) and 1 (5), we have that

$$|y - q_n| = \sum_{k=0}^{n-1} \|y - q_n\|_k + \sum_{k=n}^{\infty} \|y\|_k = n\|y\|_n + \sum_{k=n}^{\infty} \|y\|_k = |y - p_n|.$$

Hence

$$\sum_{k=0}^{n-1} \|y - q_n\|_k = n\|y\|_n.$$

On the other hand

$$\|y - q_n\|_0 \geq \|y - q_n\|_1 \geq \ldots \geq \|y - q_n\|_k \geq \|y - q_n\|_{n-1} \geq \|y\|_n,$$

hence $\|y - q_n\|_k = \|y\|_n$ $(k = 0, 1, \ldots, n-1)$. Consequently

$$\|y - q_n\|_0 = \|y\|_n = \|y - p_n\|.$$

Then, there exists $w \in X_0$ such that

$$\|y - q_n\|_0 = \|y - q_n - w\| = \|y - p_n\|.$$

Since p_n is the unique best $\|\cdot\|$-approximation to y from X_n, we obtain $q_n + w = p_n$; that is $q_n \in p_n + X_0$. □

Proposition 3. *The linear subspace* $\bigcup_{k=0}^{\infty} X_k$ *is* $|\cdot|$*-dense in* Y.

Proof. From Lemma 1 (5),

$$|y - p_n| = |y| + n\|y\|_n - \sum_{k=0}^{n-1} \|y\|_k. \tag{2.1}$$

Since $X_n \subset X_{n+1}$, the sequence $(|y - p_n|)$ is decreasing, hence is convergent to a certain $\lambda \geq 0$. From (2.1) taking into account that $\sum_{k=0}^{n-1} \|y\|_k$ is a partial sum of $|y|$, $\lambda = \lim n\|y|_n$. Since $\sum_{n=0}^{\infty} \|y\|_n$ is a convergent series and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\lambda = 0$. □

Corollary 4. *Let* $y \in Y$. *Let* q_n *be a best* $|\cdot|$*-approximation to* y *from* X_n, *for each* $n = 0, 1, 2, \ldots$. *Then*

$$y = |\cdot| - \lim q_n.$$

Proof. Taking into account that $|y - p_n| = |y - q_n|$, where p_n is the best $\|\cdot\|$-approximation to y from X_n and $\lim |y - p_n| = 0$. □

3. Certain Banach spaces.

Through this section, M is a closed subspace of X and $Z = M \cap Y$. Moreover, we shall use the following two conditions on Z:

($*$) *there exist* $\alpha > 0$ *such that, for every* $z \in Z$, $\alpha\|z\| \leq \|z\|_0$;
($**$) *given* $z_1 \in Z$ *and* $k = 0, 1, 2, \ldots$, *there exists* $\gamma_k > 0$, *depending on* z_1, *such that,*

for every $z_2 \in Z$, $\|p_k^1 - p_k^2\| \leq \gamma_k\|z_1 - z_2\|$, *where* p_k^i *is the best* $\|\cdot\|$-*approximation to* z_i *from* X_k.

The hypothesis ($*$) implies, for every $z \in Z$,

$$\alpha\|z\| \leq \|z\|_0 \leq |z|.$$

Note that this condition implies that $Z \cap X_0 = M \cap X_0 = \{0\}$. Hence $|\cdot|$ is a norm on Z.

The notion of equiconvergence of numerical series plays an important role in the characterization of the $|\cdot|$-convergent sequences and $|\cdot|$-Cauchy sequences.

Definition 5. Let $(S_j)_{j\in J}$ be a family of convergent numerical series

$$S_j = \sum_{k=0}^{\infty} a_{kj}\ .$$

We say that this family is *equiconvergent* if for each $\varepsilon > 0$, there exist $k_\varepsilon \in \mathbb{N}$ such that, for every $j \in J$,

$$\sum_{k=k_\varepsilon}^{\infty} a_{kj} < \varepsilon.$$

Lemma 6. *Let* $(y_n)_{n\in\mathbb{N}} \subset Y$ *and* $y \in Y$. *The following families of series are at same time equiconvergent*

(1) $(|y_n|)_{n\in\mathbb{N}}$,

(2) $(|y_n - y|)_{n\in\mathbb{N}}$,

(3) $(|y_n - y_m|)_{(n,m)\in\mathbb{N}x\mathbb{N}}$.

Proof. (1) ⇒ (2) Let $\varepsilon > 0$. By hypothesis there exist $k_1 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$,

$$\sum_{k=k_1}^{\infty} \|y_n\|_k < \frac{\varepsilon}{2}.$$

On the other hand, $|y| < \infty$ implies that there exists $k_2 \in \mathbb{N}$ such that

$$\sum_{k=k_2}^{\infty} \|y\|_k < \frac{\varepsilon}{2}.$$

Let $k_\varepsilon = \max\{k_1, k_2\}$. Then, for every $n \in \mathbb{N}$,

$$\sum_{k=k_\varepsilon}^{\infty} \|y_n - y\|_k \leq \sum_{k=k_\varepsilon}^{\infty} \|y_n\|_k + \sum_{k=k_\varepsilon}^{\infty} \|y\|_k < \varepsilon.$$

(2) ⇒ (3) Let $\varepsilon > 0$. Since $(|y_n - y|)$ is equiconvergent, there exists $k_\varepsilon \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$\sum_{k=k_\varepsilon}^{\infty} \|y_n - y\|_k < \frac{\varepsilon}{2}.$$

For every $m, n \in \mathbb{N}$, we obtain

$$\sum_{k=k_\varepsilon}^{\infty} \|y_m - y_n\|_k \leq \sum_{k=k_\varepsilon}^{\infty} \|y_m - y\|_k + \sum_{k=k_\varepsilon}^{\infty} \|y_n - y\|_k < \varepsilon.$$

(3) ⇒ (1) Since the family $(|y_m - y_n|)$ is equiconvergent, we have that $(|y_m - y_1\|)$ is equiconvergent. From the implication (1) ⇒ (2) results $(|y_n|)$ is equiconvergent. □

Proposition 7. *Assume that the hypothesis (*) holds. Then a sequence $(z_n) \subset Z$ is $|\cdot|$-Cauchy if and only if the two following conditions are satisfied:*

(1) *(z_n) is $\|\cdot\|$-Cauchy,*

(2) *the sequence of series $(|z_n|)$ is equiconvergent.*

Proof. Assume that (z_n) is a $|\cdot|$-Cauchy sequence.
(1) By the hypothesis $(*)$, there exists $\alpha > 0$ such that

$$\alpha\|z_m - z_n\| \leq \|z_m - z_n\|_0 \leq |z_m - z_n|.$$

Hence it is clear that (z_n) is $\|\cdot\|$-Cauchy.

(2) Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $n \geq n_\varepsilon$,

$$|z_n - z_{n_\varepsilon}| = \sum_{k=0}^{\infty} \|z_n - z_{n_\varepsilon}\|_k < \varepsilon.$$

On the other hand, for each $n = 1, 2, \ldots, n_\varepsilon - 1$, we have that $|z_n - z_{n_\varepsilon}|_k < \infty$. Then there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=k_\varepsilon}^{\infty} \|z_n - z_{n_\varepsilon}\|_k < \varepsilon,$$

for all $n \in \mathbb{N}$. That is, $(|z_n - z_{n_\varepsilon}|)$ is equiconvergent. Thus from Lemma 6 $(|z_n|)$ is equiconvergent.

Conversely, assume that $(|z_n|)$ satisfies (1) and (2). Note that, for any $h = 0, 1, 2, \ldots$,

$$|z_n - z_m| \leq h\|z_n - z_m\| + \sum_{k=h}^{\infty} \|z_n - z_m\|_k.$$

Let $\varepsilon > 0$. Since $(|z_n|)$ is equiconvergent $(|z_n - z_m|)$ is equiconvergent (Lemma 6). Hence there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=k_\varepsilon}^{\infty} \|z_n - z_m\|_k < \frac{\varepsilon}{2},$$

for every $m, n \in \mathbb{N}$. On the other hand, since $(\|z_n\|)$ is a $\|\cdot\|$-Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that, for $m, n \geq n_\varepsilon$, we have that

$$\|z_n - z_m\| < \frac{\varepsilon}{2k_\varepsilon}.$$

Then, for $m, n \geq n_\varepsilon$, we obtain

$$|z_n - z_m| \leq k_\varepsilon \|z_n - z_m\| + \sum_{k=k_\varepsilon}^{\infty} \|z_n - z_m\|_k < \varepsilon.$$

Thus (z_n) is a $|\cdot|$-Cauchy sequence. □

Proposition 8. *Assume that the hypothesis* $(*)$ *holds. Then a sequence* $(z_n) \subset Z$ *is* $|\cdot|$*-convergent to* $z \in Z$, $z = |\cdot|\text{-}\lim z_n$, *if and only if*

(1) $z = \|\cdot\|\text{-}\lim z_n$,

(2) *the sequence of series* $(|z_n|)$ *is equiconvergent.*

Proof. Assume $z = |\cdot|\text{-}\lim z_n$. By the hypothesis $(*)$, there exists $\alpha > 0$ such that, for any $n \in \mathbb{N}$,

$$\alpha\|z - z_n\| \leq \|z - z_n\|_0 \leq |z - z_n|.$$

It is clear that $z = \|\cdot\|\text{-}\lim z_n$. Also, applying Proposition 7, $(|z_n|)$ is equiconvergent.

Conversely, assume that (1) and (2) hold. By a similar argument to the second part of the proof of Proposition 7, we obtain $z = |\cdot|\text{-}\lim z_n$. □

Theorem 9. *Under the hypotheses* $(*)$ *and* $(**)$*, let* $(z_n) \subset Z$ *be a* $|\cdot|$*-bounded sequence; that is, for a certain* β *and for every* n, $|z_n| \leq \beta$. *If* (z_n) *is* $\|\cdot\|$*-convergent to* $z \in X$*, then* $|z| \leq \beta$*; hence* $z \in Z$.

Proof. Denote by p_k^n and p_k the best $\|.\|$-approximation to z_n and z, respectively, from X_k. Applying the condition $(**)$, we obtain

$$\|z\|_k = \|z - p_k\| \leq \|z - z_n\| + \|z_n - p_k^n\| + \|p_k^n - p_k\| \leq$$
$$\leq (1 + \gamma_k)\|z - z_n\| + \|z_n\|_k.$$

Given $\varepsilon > 0$, we choose

$$n_0 \leq n_1 \leq \ldots \leq n_k \leq \ldots$$

such that, for every $n \geq n_k$,

$$\|z - z_n\| < \frac{\varepsilon}{2^{k+1}(1 + \gamma_k)}.$$

Consequently, for $n \geq n_k$,

$$\|z\|_k < \frac{\varepsilon}{2^{k+1}} + \|z_n\|_k.$$

Given $h \in \mathbb{N}$, for any $n \geq n_h$, we have that

$$\sum_{k=0}^{h} \|z\|_k \leq \sum_{k=0}^{h} \|z_n\|_k + \sum_{k=0}^{h} \frac{\varepsilon}{2^{k+1}} \leq |z_n| + \varepsilon \leq \beta + \varepsilon,$$

for every $\varepsilon > 0$. Hence, for any h,

$$\sum_{k=0}^{h} \|z\|_k \leq \beta.$$

Consequently $|z| \leq \beta$. □

Theorem 10. *If the hypotheses* $(*)$ *and* $(**)$ *hold, then the subspace* Z *with the norm* $|\cdot|$ *is a Banach space.*

Proof. Let $(z_n) \subset Z$ a $|\cdot|$-Cauchy sequence. Applying Proposition 7 results that $(|z_n|)$ is an equiconvergent sequence of series and (z_n) is a $\|\cdot\|$-Cauchy sequence. Since M is $\|\cdot\|$-closed, there exists $z \in M$ such that $z = \|\cdot\|\text{-}\lim z_n$. Because (z_n) is a $|\cdot|$-bounded sequence, from Theorem 9 we obtain $z \in Z$. By Proposition 8, results $z = |\cdot|\text{-}\lim z_n$. □

4. Examples.

1. Let $X = C[a, b]$ with the uniform norm $\|\cdot\|_\infty$. Let X_k be the subspace of all polynomials of degree k at most. For each $f \in C[a, b]$ the so called *minimaxes* $\|f\|_k$ and the so called *minimax series* $|f|$ are considered (see [2] and [4]).

The subspaces X_k are Chebyshev subspaces ([3], Theorem 6.3-5); consequently, the best polynomial approximations q_n to a function $f \in Y$ with the norm $|\cdot|$ agree with the best approximation p_n with the uniform norm unless an additive constant

$$q_n \in p_n + X_0 = \{p_n + c : c \in \mathbb{C}\}.$$

If t_0 is a fixed point of $[a, b]$ we define $M = \{f \in X : f(t_0) = 0\}$ and

$$Z = \{f \in X : f(t_0) = 0 \text{ and } |f| < \infty\}.$$

The fundamental hypothesis $(*)$ holds:

$$\frac{\|f\|_\infty}{2} \leq \|f\|_0 \leq \|f\|_\infty,$$

for every $f \in Z$. Moreover, the condition $(**)$ is the well known Freud Theorem [1; p. 82]. Consequently, Z is a Banach space with the norm $\|\cdot\|$. The space $(Z, |\cdot|)$ has been considered in a previous paper of the second author et al. [4]. In this sense this paper is a generalization of [4].

2. Let $X = \ell^1 = \{(\alpha_i) \subset \mathbb{R} : \sum_{i=0}^{\infty} |\alpha_i| < \infty\}$. We consider the subspaces

$$X_k = \{(\alpha_i) \in \ell^1 : i \geq k \text{ implies } \alpha_i = 0\},$$

for $k = 0, 1, 2, \ldots$. Take $M = X$. Since $X_0 = \{0\}$, the fundamental hypothesis $(*)$ holds. The condition $(**)$ is satisfied for $\gamma_k = 1$. Note that if $x = (\alpha_i) \in X$ then $\|x\|_k = \sum_{i=k}^{\infty} |\alpha_i|$. Then, for $x \in Y$,

$$|x| = \sum_{k=0}^{\infty} (k+1)|\alpha_k|.$$

Consequently

$$Y = Z = \left\{ x = (\alpha_k) \in \ell^1 : \sum_{k=1}^{\infty} k|\alpha_k| < \infty \right\}$$

is a Banach space with the norm $|\cdot|$.

3. Let X be a Hilbert space. The inner product is denoted by $< \cdot, \cdot >$. Assume that the subspaces X_k have dimension k, hence they are Chebyshev subspaces and the condition $(*)$ is satisfied. Since in Hilbert spaces the nearest point mapping of a subspace coincides with the orthogonal projection to the subspace (and this projection is linear of norm 1), we have immediately

$$\|p_k^x - p_k^y\| \leq \|x - y\|.$$

This means that $(**)$ holds with $\gamma_k = 1$. Moreover if $\{y_1^k, y_2^k, \ldots, y_k^k\}$ is an orthogonal basis of X_k, then

$$Y = \left\{ x \in H : \sum_{k=1}^{\infty} \|x - \sum_{i=1}^{k} < x, y_i^k > y_i^k\| < \infty \right\}.$$

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