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A NOTE ON THE FABER-KRAHN INEQUALITY

TILAK BHATTACHARYA

In this work we study the well known Faber-Krahn inequality for planar domains. Let $u > 0$ be the first eigenfunction of the Laplacian on a bounded domain and λ_1 be the first eigenvalue. Let λ_1^* be the first eigenvalue for the symmetrized domain. We prove that a certain weighted L^1 integral of the isoperimetric deficiencies of the level sets of u may be bounded by the quantity $\lambda_1 - \lambda_1^*$. This leads to a sharper version of the Faber-Krahn inequality. It can be easily shown that this result also holds for more general divergence type equations.

1. Introduction.

This note is a continuation of [2], where we derived estimates on the symmetrized first eigenfunction of the Laplacian on bounded planar domains. We introduced a method for obtaining maximal solutions to the well known Talenti's inequality for the first eigenfunction [4], and derived upper bounds for the symmetrized eigenfunctions by studying the corresponding maximal solution. Our effort in this work will be to employ methods of [2] and quantify the Faber-Krahn inequality in terms of the perimeter of the level sets. Our basic conclusion is that the closer the first eigenvalue is to that of the disk of the same area the smaller the isoperimetric deficiency is in a weighted L^1 sense.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\partial\Omega$ be the boundary of Ω. Let u solve

$$\Delta u + \lambda_1 u = 0 \text{ in } \Omega, \text{ and } u|_{\partial\Omega} = 0, \tag{1.1}$$

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where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplacian. We refer to u as the first eigenfunction. It is well known that λ_1 is simple and u has one sign. From hereon, we will assume that $u > 0$ and $\sup u = 1$. For $0 \le t \le 1$ we let

$$\Omega_t = \{x \in \Omega : u(x) > t\}, \tag{1.2}$$

$$\mu(t) = |\Omega_t|, \tag{1.3}$$

where $|S|$ denotes the area of a set $S \subset \mathbb{R}^2$. Also let Ω^* be the disc centered at the origin with area equal to that of Ω. Let (x, y) denote the Cartesian coordinates of a point in Ω^*. Define

$$\begin{aligned} u^{\#}(a) &= \inf\{t > 0 : \mu(t) < a\}, \text{ and} \\ u^*(x, y) &= u^{\#}(\pi(x^2 + y^2)). \end{aligned} \tag{1.4}$$

The function u^* is called the Schwarz nonincreasing radial rearrangement of u. In this work, we take $|\Omega| = 1$; also set $\lambda_1^* = \lambda_1(\Omega^*)$.

We will now provide a formal derivation of Talenti's inequality for the positive solution of (1.1) (see Sections 4 and 5 in [4]. Also see [1]). This will make explicit the questions we will be studying in this work. Recall that u is locally analytic and thus by Sard's Theorem and the Coarea formula we have for $0 < t < 1$

$$\left(\int_{\partial\Omega_t} 1\right)^2 \le \int_{\partial\Omega_t} |Du| \int_{\partial\Omega_t} \frac{1}{|Du|}. \tag{1.5}$$

Let $L(\partial\Omega_t)$ denote the length (1-dimensional Hausdorff measure) of $\partial\Omega_t$. An application of the divergence theorem on (1.1), the classical isoperimetric inequality and (1.5) yield that for a. e. t

$$\begin{aligned} 4\pi\mu(t) \le L(\partial\Omega_t)^2 \le \int_{\partial\Omega_t} |Du| \int_{\partial\Omega_t} \frac{1}{|Du|} = \\ = \lambda_1 \int_{\partial\Omega_t} \frac{1}{|Du|} \int_{\Omega_t} u = -\lambda_1\mu'(t) \int_{\Omega_t} u, \text{ for a. e. } t, \end{aligned} \tag{1.6}$$

where $\mu'(t) = d\mu/dt$. Thus we have that

$$4\pi\mu(t) \le -\lambda_1\mu'(t)\left(\int_t^1 \mu(s)ds + t\mu(t)\right), \text{ for a. e. } t, \tag{1.7}$$

$$\mu(0) = 1 \text{ and } \mu(1) = 0.$$

In [2], a detailed study of (1.7) was carried out and a maximal solution $Z \geq \mu$ was constructed which lead to a better understanding of μ. In particular, we obtained estimates for u^* in terms of λ_1 and λ_1^*. In this work we derive a sharper version of the well known Faber-Krahn isoperimetric inequality, which says that $\lambda_1 \geq \lambda_1^*$ and $\lambda_1 = \lambda_1^*$ if and only if $\Omega = \Omega^*$. Prompted by (1.6) and (1.7), we define the following two quantities. Let $s(t)$ and $\sigma(t)$ be such that

$$\begin{aligned}(1.8)\qquad & 4\pi\{1+\sigma(t)\}\mu(t) = \\ &= L(\partial\Omega_t)^2 \text{ and } 4\pi\{1+s(t)\}\mu(t) = \int_{\partial\Omega_t} |Du| \int_{\partial\Omega_t} \frac{1}{|Du|},\end{aligned}$$

for all t's at which (1.7) is well defined. Clearly, $0 \leq \sigma(t) \leq s(t)$ and Ω_t is a disc whenever $\sigma(t) = 0$ (or $s(t) = 0$). We refer to the quantity $4\pi\sigma(t)$ as the *isoperimetric deficiency.* Our effort will be to try to quantify the Faber-Krahn inequality in terms of $\sigma(t)$ and $s(t)$. More precisely,

Theorem 1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* $|\Omega| = 1$. *Let* λ_1 *be the first eigenvalue of* Ω *and let* λ_1^* *be the first eigenvalue of* Ω^*. *Suppose* u *satisfies* (1.1) *with* $0 < u \leq 1$ *and* $\sup u = 1$. *Define* μ, σ *and* s *by* (1.3) *and* (1.8). *Then there exists an absolute positive constant* C *such that*

$$(1.9)\qquad 0 \leq \int_0^1 \sigma(t)\mu(t)dt \leq \int_0^1 s(t)\mu(t)dt \leq C(\lambda_1 - \lambda_1^*).$$

Remark 1.1. Theorem 1 holds for more general uniformly elliptic p.d.e's. Consider the eigenvalue problem,

$$-\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = \lambda_1 u, \text{ in } \Omega,\ u \in W_0^{1,2}(\Omega).$$

Here $a_{ij}(x)$ and $c(x) \in L^\infty(\Omega)$. We require $c(x) \geq 0$ and

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^2.$$

Let λ_1 be the first eigenvalue. It is well known that $\lambda_1 > 0$ and the first eigenfunction does not change sign. From [4], one sees that (1.7) continues to hold.

Remark 1.2. It is not clear whether or not the exponent 1 on the term $(\lambda_1 - \lambda_1^*)$ in (1.9), is sharp. We are also unable to determine whether a lower bound, in terms of $(\lambda_1 - \lambda_1^*)$, holds. It also seems to be unknown whether an estimate of the type (1.9) holds for the first eigenvalue for the p-Laplacian.

The proof of Theorem 1 is achieved by adapting the methods employed in [2].

We have divided our work as follows. Section 2 contains some preliminary results and the proof of Theorem 1 appears in Section 3.

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2. Preliminary Results.

We will first present a short proof of the Faber-Krahn inequality based on our previous study of Talenti's inequality in [2]. In order to prove the Faber-Krahn inequality we first prove that $\lambda_1 \geq \lambda_1^*$ (see Theorem 3.3 in [2] and the related Lemma 2.1 in this work). To prove that Ω is a disc when $\lambda_1 = \lambda_1^*$, we proceed as follows. We recall the definition of the maximal solution Z for (1.7). For any given $\lambda_1 > 0$, the function Z solves

$$\begin{aligned} &\frac{4\pi}{\lambda_1} Z(t) = -Z'(t)\left[\int_t^1 Z(s)ds + tZ(t)\right], \text{ for } 0 < t < 1, \\ &Z(0) = 1. \end{aligned} \tag{2.1}$$

As was shown in Theorem 1.1 in [2], $\mu(t) \leq Z(t)$ in [0,1] (also see (2.13)), and the value of $Z(1)$ played a central role in obtaining estimates on μ and consequently for u^*. If $\lambda_1 = \lambda_1^*$ then Corollary 3.1 in [2] implies that $Z(1) = 0$. Now $\mu'(t)$ exists outside a set of zero measure. Thus from (2.1), (1.7) and the fact that $\mu \leq Z$, for a. e. t,

$$\begin{aligned} \mu'(t)/\mu(t) &\leq -\frac{4\pi}{\lambda_1^*}\left(\int_t^1 \mu(s)ds + t\mu(t)\right)^{-1} \leq \\ &\leq -\frac{4\pi}{\lambda_1^*}\left(\int_t^1 Z(s)ds + tZ(t)\right)^{-1} = Z'(t)/Z(t). \end{aligned} \tag{2.2}$$

Note that $-\log \mu(t)$ is nondecreasing and continuous in [0,1). Thus (2.2) yields,

$$\log\{\mu(t)/\mu(s)\} \leq \int_s^t \mu'(\tau)/\mu(\tau)\, d\tau \leq \int_s^t Z'(\tau)/Z(\tau)\, d\tau = \log\{Z(t)/Z(s)\}.$$

Clearly then,

$$1 \leq Z(s)/\mu(s) \leq Z(t)/\mu(t),\ 0 \leq s \leq t < 1. \tag{2.3}$$

Employing that $-\mu$ is increasing and Z is absolutely continuous, we see that

$$Z(t) = -\int_t^1 Z'(\tau)\,d\tau \ \text{ and }\ \mu(t) \geq -\int_t^1 \mu'(\tau)\,d\tau. \tag{2.4}$$

Now using (2.4), (2.1), (1.7) and the fact that both Z and μ are non-increasing, we obtain that

$$\frac{Z(t)}{\mu(t)} \leq \frac{\int_t^1 -Z'(\tau)\,d\tau}{\int_t^1 -\mu'(\tau)\,d\tau} \leq \tag{2.5}$$

$$\leq \left(\int_t^1 \frac{Z(\tau)}{\int_\tau^1 Z(\theta)\,d\theta + \tau Z(\tau)}\,d\tau\right) \Bigg/ \left(\int_t^1 \frac{\mu(\tau)}{\int_\tau^1 \mu(\theta)\,d\theta + \tau\mu(\tau)}\,d\tau\right)$$

$$\leq \left(\int_t^1 \frac{Z(\tau)}{\tau Z(\tau)}\,d\tau\right) \Bigg/ \left(\int_t^1 1\,d\tau\right) = \frac{\log(1/t)}{1-t} \to 1 \text{ as } t \to 1.$$

From (2.3) we see that $Z(t) = \mu(t)$, $\forall t \in [0,1]$. Thus equality holds in (1.7), $\forall t \in [0,1]$, and hence in the classical isoperimetric inequality in (1.6). Thus Ω_t is a disc for all t. Since $\Omega = \cup_{t>0}\Omega_t$, and Ω_t increases to Ω as t decreases to 0, it is clear that Ω is a disc. $\square$

In order to prove Theorem 1, we will use inequality (1.7) as follows. Recall the definition of $\sigma(t)$ from (1.8); thus μ satisfies,

$$\frac{4\pi}{\lambda_1}\{1+\sigma(t)\}\mu(t) \leq -\mu'(t)\left[\int_t^1 \mu(\tau)d\tau + t\mu(t)\right], \text{ for a.e. } t. \tag{2.6}$$

If we use $s(t)$ instead, we get

$$\frac{4\pi}{\lambda_1}\{1+s(t)\}\mu(t) = -\mu'(t)\left[\int_t^1 \mu(\tau)d\tau + t\mu(t)\right], \text{ for a.e. } t. \tag{2.7}$$

In either case, $\mu(0) = 1$ and $\mu(1) = 0$. We now derive an easy weighted L^1 estimate for $s(t)$. Observe that $F(t) = \left(\int_t^1 \mu(s)ds + t\mu(t)\right) = \left(\int_{\Omega_t} u\right)$ is continuous and decreasing in t. Thus $F(t)\mu(t)$ is also continuous and decreasing. Hence,

$$F(1)\mu(1) - F(0)\mu(0) \leq \int_0^1 (F(t)\mu(t))'\,dt \leq 0.$$

Simplifying, we obtain

$$0 \le -\int_0^1 \mu'(t)F(t)\,dt - \int_0^1 \mu(t)F'(t)\,dt \le F(0) = \int_0^1 \mu(t)\,dt \le 1.$$

Combining this with (2.7) yields

$$\frac{4\pi}{\lambda_1}\int_0^1 \{1+s(t)\}\mu(t)\,dt \le \int_0^1 \mu(t)\,dt \le 1. \tag{2.8}$$

Also noting that $-\log \mu(t)$ is nondecreasing, continuous and

$$\mu'(t)/\mu(t) = -\frac{4\pi}{\lambda_1}(1+s(t))\left(\int_t^1 \mu(\tau)d\tau + t\mu(t)\right)^{-1} \quad \text{for a.e. } t,$$

we obtain that for $t \in (0, 1)$,

$$0 < \mu(t) \le \exp\left[-\frac{4\pi}{\lambda_1}\int_0^t \frac{1+s(\tau)}{\int_\tau^1 \mu(\theta)d\theta + \tau\mu(\tau)}\,d\tau\right] \le 1. \tag{2.9}$$

We now construct a maximal solution $G(t)$ to (2.9) as follows (see proof of Theorem 1.1 in [2]). For $n = 1, 2, \ldots$, set

$$G_n(t) = \exp\left[-\frac{4\pi}{\lambda_1}\int_0^t \frac{1+s(\tau)}{\int_\tau^1 G_{n-1}(\theta)\,d\theta + \tau G_{n-1}(\tau)}\,d\tau\right], \tag{2.10}$$

where $G_0(t) = 1$ on [0,1]. Now $G_n(0) = 1$, $\forall n = 1, 2 \ldots$. Using (2.9), (2.10) and $\mu \le 1$, we have

$$G_1(t) = \exp\left[-\frac{4\pi}{\lambda_1}\int_0^t (1+s(\tau))\,d\tau\right] \ge \mu(t).$$

If $G_n(t) \ge \mu(t)$ for some n, then (2.9) and (2.10) imply that $G_{n+1}(t) \ge \mu(t)$. Again $G_1(t) \le G_0(t)$, and if $G_n(t) \le G_{n-1}(t)$, for some n, then (2.10) implies that $G_{n+1}(t) \le G_n(t)$. Thus, arguing by induction, we see $\{G_n(t)\}$ is a decreasing sequence of decreasing functions, bounded below by $\mu(t)$ and bounded above by 1. Thus taking limits in (2.10) and setting $G(t) = \lim_{n\to\infty} G_n(t)$, we obtain

$$0 < \mu(t) \le G(t) = \exp\left[-\frac{4\pi}{\lambda_1}\int_0^t \frac{1+s(\tau)}{\int_\tau^1 G(\theta)\,d\theta + \tau G(\tau)}\,d\tau\right] \le 1, \tag{2.11}$$
$$G(0) = 1.$$

Furthermore, $G(t)$ is absolutely continuous on $[0, a]$, $\forall a < 1$. By differentiating (2.11), we see that for a. e. t,

$$\frac{4\pi}{\lambda_1}\{1+s(t)\}G(t) = -G'(t)\left(\int_t^1 G(\tau)\,d\tau + tG(t)\right), \tag{2.12}$$

$$G(0) = 1.$$

By taking $\sigma(t)$ in (2.10), in place of $s(t)$, we would generate a maximal solution, say $\bar{G}(t) \geq \mu(t)$, to (2.6), i. e.,

$$\frac{4\pi}{\lambda_1}\{1+\sigma(t)\}\bar{G}(t) = -\bar{G}'(t)\left(\int_t^1 \bar{G}(\tau)\,d\tau + t\bar{G}(t)\right).$$

Dropping the term $s(t)$ entirely, in (2.10), we would get back the maximal solution $Z(t)$ to (1.7). By comparing the iterates Z_n, G_n, $\bar{G}_n$ and $\mu(t)$ (e. g. $\mu(t) \leq G_1(t) \leq \bar{G}_1(t) \leq Z_1(t)$) and employing an argument, similar to the one used above in the proof of the existence of $G(t)$, we find that

$$0 \leq \mu(t) \leq G(t) \leq \bar{G}(t) \leq Z(t) \leq 1, \ \forall t \in [0, 1]. \tag{2.13}$$

We now prove an identity for $G(t)$ which will help us in deriving the estimate in Theorem 1 (see Theorem 3.1 in [2]).

Lemma 2.1. *Let $G(t)$ be as in* (2.11). *Then the following identity holds, namely,*

$$G(1)J_2\left(\sqrt{\frac{\lambda_1 G(1)}{\pi}}\right) + \sqrt{\frac{4\pi}{\lambda_1}}\int_0^1 s(t)\sqrt{G(t)}J_1\left(\sqrt{\frac{\lambda_1 G(t)}{\pi}}\right)dt =$$

$$= -J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 G(t)\,dt,$$

where J_0, J_1 and J_2 are the Bessel functions of order 0, 1 and 2 respectively.

Proof. For $m = 0, 1, 2, \ldots$, multiply (2.12) by $G(t)^m$ and integrate the right side by parts to obtain

$$\frac{4\pi}{\lambda_1}\int_0^1 \{1+s(t)\}G(t)^{m+1}\,dt = \tag{2.14}$$

$$= -\frac{1}{m+1}\int_0^1 (G(t)^{m+1})'\left(\int_t^1 G(\tau)d\tau + tG(t)\right)dt =$$

$$= -\frac{1}{m+1}\left\{G(t)^{m+1}\left(\int_t^1 G(\tau)\,d\tau + tG(t)\right)\Big|_0^1 - \int_0^1 tG'(t)G(t)^{m+1}\,dt\right\} =$$

$$= -\frac{1}{m+1}\left\{G(1)^{m+2} - \int_0^1 G(t)\,dt - \right.$$

$$\left. - \frac{1}{m+2}\left(tG(t)^{m+2}\Big|_0^1 - \int_0^1 G(t)^{m+2}\,dt\right)\right\} =$$

$$= -\frac{1}{m+1}\left\{\frac{m+1}{m+2}G(1)^{m+2} - \int_0^1 G(t)\,dt + \frac{1}{m+2}\int_0^1 G(t)^{m+2}\,dt\right\}.$$

Thus, taking $m = 0$ and $m = 1$ we get

$$\frac{4\pi}{\lambda_1}\int_0^1 \{1 + s(t)\}G(t)\,dt = -\frac{1}{2}G(1)^2 + \int_0^1 G(t)\,dt - \frac{1}{2}\int_0^1 G(t)^2\,dt$$

and

$$\frac{4\pi}{\lambda_1}\int_0^1 \{1 + s(t)\}G(t)^2\,dt = -\frac{1}{3}G(1)^3 + \frac{1}{2}\int_0^1 G(t)\,dt - \frac{1}{2\cdot 3}\int_0^1 G(t)^3\,dt.$$

Combining the above equations we obtain that

$$\left(1 - \frac{\lambda_1}{4\pi} + \left(\frac{\lambda_1}{4\pi}\right)^2 \frac{1}{2\cdot 2}\right)\int_0^1 G(t)\,dt + \int_0^1 s(t)\left(G(t) - \frac{\lambda_1}{4\pi}\frac{1}{2}G(t)^2\right)dt =$$

$$= \left(-\frac{\lambda_1}{4\pi}\frac{G(1)^2}{2} + \left(\frac{\lambda_1}{4\pi}\right)^2 \frac{1}{1\cdot 2\cdot 3}G(1)^3\right) + \left(\frac{\lambda_1}{4\pi}\right)^2 \frac{1}{1\cdot 2\cdot 2\cdot 3}\int_0^1 G(t)^3\,dt.$$

Assume that for some $N > 0$

$$\left\{\sum_{m=0}^{N}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\left(\frac{1}{m!}\right)^2\right\}\int_0^1 G(t)\,dt + \tag{2.15}$$

$$+ \int_0^1 s(t)\left\{\sum_{m=0}^{N-1}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m \frac{G(t)^{m+1}}{(m!)^2(m+1)}\right\}dt =$$

$$= \left\{\sum_{m=1}^{N}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m \frac{G(1)^{m+1}}{(m-1!)^2 m(m+1)}\right\} +$$

$$+ (-1)^N\left(\frac{\lambda_1}{4\pi}\right)^N \frac{1}{(N!)^2(N+1)}\int_0^1 G(t)^{N+1}\,dt.$$

We employ (2.14) with $m = N$ to find that

$$\int_0^1 G(t)^{N+1}\,dt = -\frac{\lambda_1}{4\pi}\frac{G(1)^{N+2}}{N+2} + \frac{\lambda_1}{4\pi}\frac{1}{N+1}\int_0^1 G(t)\,dt - \\ - \int_0^1 s(t)G(t)^{N+1}\,dt - \frac{\lambda_1}{4\pi}\frac{1}{(N+1)(N+2)}\int_0^1 G(t)^{N+2}\,dt.$$

Substituting this formula in (2.15) we find that

$$\left\{\sum_{m=0}^{N}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\left(\frac{1}{m!}\right)^2 + (-1)^{N+1}\left(\frac{\lambda_1}{4\pi}\right)^{N+1}\left(\frac{1}{N+1!}\right)^2\right\}\cdot \\ \cdot\int_0^1 G(t)\,dt + \int_0^1 s(t)\left\{\sum_{m=0}^{N-1}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\frac{G(1)^{m+1}}{(m!)^2(m+1)} + \right. \\ \left. +(-1)^N\left(\frac{\lambda_1}{4\pi}\right)^N\frac{G(t)^{N+1}}{(N!)^2(N+1)}\right\}dt = \\ = \left\{\sum_{m=1}^{N}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\frac{G(1)^{m+1}}{(m-1!)^2m(m+1)} + \right. \\ \left. + (-1)^{N+1}\left(\frac{\lambda_1}{4\pi}\right)^{N+1}\frac{G(1)^{N+2}}{(N!)^2(N+1)(N+2)}\right\} + \\ + (-1)^{N+1}\left(\frac{\lambda_1}{4\pi}\right)^{N+1}\frac{1}{(N+1!)^2(N+2)}\int_0^1 G(t)^{N+2}\,dt.$$

Clearly formula (2.15) holds for $N = 1$ and $N = 2$. Thus (2.15) holds for every $N = 1, 2, \ldots$. Observing that $G(t) \le 1$ and taking limits in (2.15) we obtain that

$$\left(\sum_{m=0}^{\infty}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\left(\frac{1}{m!}\right)^2\right)\int_0^1 G(t)\,dt + \\ + \int_0^1 s(t)\left(\sum_{m=0}^{\infty}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\frac{G(t)^{m+1}}{(m!)^2(m+1)}\right)dt = \\ = \sum_{m=1}^{\infty}(-1)^m\left(\frac{\lambda_1}{4\pi}\right)^m\frac{G(1)^{m+1}}{(m-1!)^2m(m+1)}\,.$$

Comparing formulas for J_0, J_1 and J_2 [5], we see

$$J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 G(t)\,dt + \sqrt{\frac{4\pi}{\lambda_1}}\int_0^1 s(t)\sqrt{G(t)}J_1\left(\sqrt{\frac{\lambda_1 G(t)}{\pi}}\right)dt =$$

$$= -G(1)J_2\left(\sqrt{\frac{\lambda_1 G(1)}{\pi}}\right).$$

Rewriting we get,

$$G(1)J_2\left(\sqrt{\frac{\lambda_1 G(1)}{\pi}}\right) + \sqrt{\frac{4\pi}{\lambda_1}}\int_0^1 s(t)\sqrt{G(t)}J_1\left(\sqrt{\frac{\lambda_1 G(t)}{\pi}}\right)dt = \tag{2.16}$$

$$= -J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 G(t)\,dt.$$

Similarly we can show

$$\bar{G}(1)J_2\left(\sqrt{\frac{\lambda_1 \bar{G}(1)}{\pi}}\right) + \sqrt{\frac{4\pi}{\lambda_1}}\int_0^1 \sigma(t)\sqrt{\bar{G}(t)}J_1\left(\sqrt{\frac{\lambda_1 \bar{G}(t)}{\pi}}\right)dt =$$

$$= -J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 \bar{G}(t)\,dt. \qquad \square$$

Before getting to the proof of Theorem 1, we make an observation regarding the Bessel function J_1.

Proposition 2.1. *Let ν_0, α_0 and β_0 denote the smallest positive zeros of J_0, J_1 and J_2. Let $\theta_0 = (\alpha_0 + \nu_0)/2$, then there exists an absolute positive constant K such that*

$$0 < K \le \frac{J_1(\theta)}{\theta} \le \frac{1}{2}, \ \forall\theta \in (0, \theta_0]. \tag{2.17}$$

Proof. We first recall the following formula [5]

$$\frac{d}{d\theta}\left(\frac{J_1(\theta)}{\theta}\right) = -\frac{J_2(\theta)}{\theta}.$$

Then $J_1(\theta)/\theta$ is decreasing in $(0, \beta_0]$, and consequently $J_1(\theta)/\theta > 0$ in $(0, \alpha_0]$. Since $\nu_0 < \theta_0 < \alpha_0$, we see that $J_1(\theta_0)/\theta_0 \le J_1(\theta)/\theta$, in $(0, \theta_0]$. It is easy to verify that $J_1(\theta)/\theta \to 1/2$ as $\theta \to 0^+$. Thus by setting $K = J_1(\theta_0)/\theta_0$ we obtain (2.17). $\square$

3. Proof of Theorem 1.

Let ν_0, α_0 and β_0 be as defined in the statement of Proposition 2.1. By the interlacing property of the zeros of the Bessel functions, $0 < \nu_0 < \alpha_0 < \beta_0$. Now $\lambda_1 \geq \lambda_1^* = \nu_0^2\pi$. Assume that $\lambda_1 \leq \pi\left((\nu_0+\alpha_0)/2\right)^2$. Since $0 \leq \mu(t) \leq G(t) \leq \bar{G}(t) \leq Z(t) \leq 1$, for $t \in [0, 1]$, it follows that $0 \leq \sqrt{\lambda_1 G(1)/\pi} \leq (\nu_0 + \alpha_0)/2 \leq \alpha_0 \leq \beta_0$, and consequently, $J_2\left(\sqrt{\lambda_1 G(1)/\pi}\right) \geq 0$ and $J_1\left(\sqrt{\lambda_1 G(t)/\pi}\right) > 0, \forall t \in (0, 1)$. Thus

$$\sqrt{\frac{4\pi}{\lambda_1}}\int_0^1 s(t)\sqrt{G(t)}J_1\left(\sqrt{\frac{\lambda_1 G(t)}{\pi}}\right)dt \leq -J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 G(t)\,dt. \tag{3.1}$$

We now use the estimate in Proposition 2.1, Lemma 2.1 and (3.1). Recall that G is nonincreasing and $0 \leq G(t) \leq 1$. Hence

$$0 \leq \sqrt{\frac{\lambda_1 G(t)}{\pi}} \leq \sqrt{\frac{\lambda_1}{\pi}} \leq \frac{(\alpha_0+\nu_0)}{2} = \theta_0.$$

If K is as in (2.17), then

$$K \leq J_1\left(\sqrt{\frac{\lambda_1 G(t)}{\pi}}\right)\Bigg/\sqrt{\frac{\lambda_1 G(t)}{\pi}} \leq \frac{1}{2},\ t \in [0, 1]. \tag{3.2}$$

Inserting (3.2) in (3.1), we find that

$$2K\int_0^1 s(t)G(t)\,dt \leq -J_0\left(\sqrt{\frac{\lambda_1}{\pi}}\right)\int_0^1 G(t)\,dt. \tag{3.3}$$

Now $J_0\left(\sqrt{\lambda_1/\pi}\right) = J_0\left(\sqrt{\lambda_1/\pi}\right) - J_0\left(\sqrt{\lambda_1^*/\pi}\right) \leq \bar{C}(\lambda_1 - \lambda_1^*)$, where $\bar{C}$ is again an absolute positive constant. Thus, (3.3) yields

$$0 \leq \int_0^1 \sigma(t)\mu(t)\,dt \leq \int_0^1 s(t)\mu(t)\,dt \leq \int_0^1 s(t)G(t)\,dt \leq C(\lambda_1 - \lambda_1^*),$$

where $C > 0$ is absolute. $\square$

Remark 3.1. A natural question, one could ask, is what happens when $G(1) = 0$. It can be easily shown from (2.13) that $\mu(t) = G(t)$. Thus equality holds in the second inequality (from the left) in (2.9). It is not clear to us whether or not Ω has to be a disk.

We now make a few brief remarks about the quantities s and σ.

Remark 3.2. If Ω is a disc then it is well known that the first eigenfunction u, in (1.1), is radial and thus $s(t) = \sigma(t) = 0$. Conversley, if $s(t) = 0$ (or $\sigma(t) = 0$) at some $t \in (0, 1)$, then Ω_t is a disk. One can then conclude from Proposition 3.1, below, that Ω is a disk.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Let u be the first eigenfunction of (1.1) on Ω. Suppose that* $0 < u \leq 1$ *and* $\sup u = 1$. *If Ω_t is a ball for some* $t \in (0, 1)$ *then Ω is a ball.*

Proof. Since $\Omega_t \subset \Omega$, it follows that $\lambda_1(\Omega_t) > \lambda_1(\Omega)$. Also, if $B_R(0)$, the ball of radius R, is such that $\lambda_1(B_R(0)) = \lambda_1(\Omega)$ then $vol(B_R(0)) > vol(\Omega_t)$. Let $\bar{R}$ be such that $vol(\Omega_t) = \omega_n \bar{R}^n$, then $\bar{R} < R$. For each $\eta \in \mathbb{R}$, set J_η to be the Bessel function of order η. Let $P \in \Omega_t$ be the center of $\partial\Omega_t$. Setting $r = |P - Q|$, $Q \in \Omega_t$, define

$$v(r) = t\left(\frac{\bar{R}}{r}\right)^{(n-2)/2} \frac{J_{(n-2)/2}(\sqrt{\lambda_1}r)}{J_{(n-2)/2}(\sqrt{\lambda_1}\bar{R})}.$$

Furthermore, if ν_{n-2} is the first positive zero of $J_{(n-2)/2}$ then $\lambda_1 = \lambda_1(\Omega) = \nu_{n-2}^2/R^2$. Now $\Delta v + \lambda_1 v = 0$, in Ω_t, $v(\bar{R}) = t$ and $v(R) = 0$. If $w = u - v$, then $\Delta w + \lambda_1 w = 0$, in Ω_t, and $w = 0$ on $\partial\Omega_t$. Since $\lambda_1 < \lambda_1(\Omega_t)$, it follows that $w = 0$ in Ω_t. By unique continuation, $u = v$ in $\Omega \cap B_R(0)$. We claim $\Omega = B_R(0)$. Suppose there are points of $\partial B_R(0)$ inside Ω, then $u = v$ will vanish somewhere in Ω. This contradicts that $u > 0$ in Ω. Similarly, one can show that there are no points of $\partial\Omega$ inside $B_R(0)$. Hence the claim. $\square$

Remark 3.3. Finally, if $s(t) = \sigma(t)$ for some t, then we may again conclude that Ω is a disc. First observe that (1.6) and (1.8) imply that $|Du| = C_t$ on $\partial\Omega_t$, for some constant $C_t > 0$. A celebrated result of Serrin's [3] then implies that Ω_t is a disc. This togetherwith Proposition 3.1 implies that Ω has to be a disc.

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Indian Statistical Institute,
7, S.J.S. Sansanwal Marg,
New Delhi 110016 (INDIA)