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ALGEBRAIC AND GEOMETRIC AUTOMORPHISMS OF HYPERGROUPOIDS

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Two concepts of automorphism of a hypergroupoid are introduced; the first one preserve the algebraic structure of a hypergroupoid, the second one preserve the geometric structure that one can associate naturally to a hypergroupoid. The groups of such automorphisms are studied, in particular in the case that the geometric space associated to a hypergroupoid is a Steiner system.

1. Definitions and notations.

Definition 1.1. *A hypergroupoid* $(H, \circ)$ *is a non-empty set* H *equipped with a hyperoperation* $\circ$*, that is an application* $\circ : H \times H \to \mathcal{P}^*(H)$*, where* $\mathcal{P}^*(H)$ *is the set of non-empty subsets of* H*. If* $x, y \in H$*, we will denote by* $x \circ y$ *the hyperproduct of* x *and* y*.*

Definition 1.2. *A geometric space is a pair* $(H, \mathcal{B})$*, where* H *is a non-empty set, which elements are called points, and* $\mathcal{B}$ *is a family of non-empty subsets of* H*, called blocks.*

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If $(H, \circ)$ is a hypergroupoid, we say that the geometric space $(H, \mathcal{B})$ is associated to $(H, \circ)$ if and only if the elements of $\mathcal{B}$ are exactly the hyperproducts of two elements of H, that is:

$$\mathcal{B} = \{x \circ y\}_{x,y \in H}.$$

Conversely, if $(H, \mathcal{B})$ is a geometric space, many hypergroupoids $(H, \circ)$ exist such that the set of all hyperproducts $x \circ y$ is exactly $\mathcal{B}$; each one of these hypergroupoids is said to be associated to $(H, \mathcal{B})$.

We recall the following

Definition 1.3. *An automorphism of the geometric space* $(H, \mathcal{B})$ *is a bijective application* $\varphi : H \to H$ *such that:*

$$\forall B \in \mathcal{B}, \quad \varphi(B) \in \mathcal{B}.$$

Now we can introduce the following notion

Definition 1.4. *Let* $(H, \circ)$ *be a hypergroupoid. We say that* $\varphi : H \to H$ *is a geometric automorphism of* $(H, \circ)$ *if it is an automorphism of the geometric space* $(H, \mathcal{B})$ *associated to* $(H, \circ)$*, that is:*

$$\forall x, y \in H, \quad \exists u, v \in H : \quad \varphi(x \circ y) = \varphi(u) \circ \varphi(v).$$

We will denote by $\mathcal{A}ut_G(H, \circ)$ *(or simply* $\mathcal{A}ut_G H$*) the group of all geometric automorphisms of* $(H, \circ)$.

Definition 1.5. *An automorphism of a hypergroupoid* $(H, \circ)$ *is a bijective application* $f : H \to H$ *such that:*

$$\forall x, y \in H\ ,\ f(x \circ y) = f(x) \circ f(y).$$

We will call these automorphisms, algebraic automorphisms of $(H, \circ)$ *and we will denote by* $\mathcal{A}ut_A(H, \circ)$ *(or simply* $\mathcal{A}ut_A H$*) the group of all algebraic automorphisms of* $(H, \circ)$.

Remark 1.1. From the previous definitions it follows immediately that for any hypergroupoid $(H, \circ)$ we have that:

$$\mathcal{A}ut_A H \leq \mathcal{A}ut_G H,$$

that is $\mathcal{A}ut_A H$ is a subgroup of $\mathcal{A}ut_G H$.

In general $\mathcal{A}ut_A H$ is proper a subgroup of $\mathcal{A}ut_G H$ in fact we have the following

Example 1.1. Let $(H, \circ)$ be the hypergroupoid defined by:

$\circ$	1	2	3
1	1, 2	1, 3	1, 3
2	1, 2	1, 3	1, 3
3	2, 3	1, 2	1, 2

The geometric space associated to $(H, \circ)$ is $(H, \mathcal{B})$, where:

$$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Clearly, any permutation on the set H is a geometric automorphism, that is:

$$\mathcal{A}ut_G H = S_3;$$

while:

$$\mathcal{A}ut_A H = \{1_H\}.$$

In fact we have that:

$$f_1 \equiv (23) \notin \mathcal{A}ut_A H, \text{ since } f_1(1 \circ 1) = \{1, 3\} \neq \{1, 2\} = f_1(1) \circ f_1(1);$$

$$f_2 \equiv (13) \notin \mathcal{A}ut_A H, \text{ since } f_2(1 \circ 1) = \{2, 3\} \neq \{1, 2\} = f_2(1) \circ f_2(1);$$

$$f_3 \equiv (12) \notin \mathcal{A}ut_A H, \text{ since } f_3(1 \circ 1) = \{1, 2\} \neq \{1, 3\} = f_3(1) \circ f_3(1);$$

$$f_4 \equiv (123) \notin \mathcal{A}ut_A H, \text{ since } f_4(1 \circ 1) = \{2, 3\} \neq \{1, 3\} = f_4(1) \circ f_4(1);$$

$$f_5 \equiv (132) \notin \mathcal{A}ut_A H, \text{ since } f_5(1 \circ 1) = \{1, 3\} \neq \{1, 2\} = f_5(1) \circ f_5(1).$$

Remark 1.2. Let $(H, \mathcal{B})$ and $(H, \mathcal{B}')$ be two geometric structures such that $\mathcal{B}'$ do not contains singletons and $\mathcal{B} = \mathcal{B}' \cup \mathcal{S}$ where $\mathcal{S}$ is the set of all singletons of H. In [3] is proved that $(H, \mathcal{B})$ and $(H, \mathcal{B}')$ have the same group of automorphisms and so they are geometrically equivalent. If $\mathcal{B}'$ do not contains singletons then we say that $(H, \mathcal{B}')$ is the canonical representation of all structures $(H, \mathcal{B})$ geometrically equivalent to $(H, \mathcal{B}')$ obtained from $(H, \mathcal{B}')$ by union of singletons.

2. Steiner systems, Steiner hypergroupoids and quasi-Steiner hypergroupoids.

In this section we study the *k-Steiner systems*, that is those geometric spaces $(H, \mathcal{L})$, where H is a non-empty set, which elements are called points, and $\mathcal{L}$ is a family of non-empty subsets of H, which elements are called lines, such that every line has k points and any two distinct points are contained in a unique line.

In [6] the following notion was introduced:

Definition 2.1. *A hypergroupoid $(H, \circ)$ is said to be a k-Steiner hypergroupoid $(k \geq 2)$ (or simply Steiner hypergroupoid), if and only if the following conditions are satisfied:*

1) $\forall x, y \in H, \quad \{x, y\} \subseteq x \circ y;$
2) $\forall x, y \in H, \quad |x \circ y| = \begin{cases} 1 & \textit{if} \quad x = y \\ k & \textit{if} \quad x \neq y \end{cases};$
3) *the associativity holds for any triple of points not all distinct.*

If $(H, \mathcal{L})$ is a k-Steiner system, then one can define a hyperoperation on H in the following way:

$$\forall x, y \in H, \quad x \circ y = \begin{cases} l_{xy} & \text{if} \quad x \neq y \\ \{x\} & \text{if} \quad x = y \end{cases}$$

where l_{xy} denote the unique line trough x and y. In [6] it was proved that $(H, \circ)$ is a k-Steiner hypergroupoid, that we will call associated to $(H, \mathcal{L})$.

Conversely, if $(H, \circ)$ is a k-Steiner hypergroupoid and if $(H, \mathcal{L}')$ is the geometric structure associated to $(H, \circ)$, then the canonical representation $(H, \mathcal{L})$ of $(H, \mathcal{L}')$ is a k-Steiner system. So, the notions of k-Steiner hypergroupoid and k-Steiner system are equivalent.

Now we introduce the following

Definition 2.2. *A hypergroupoid $(H, \circ)$ is said to be a k-quasi-Steiner hypergroupoid $(k \geq 2)$ (or simply quasi-Steiner hypergroupoid), if and only if the following conditions are satisfied:*

1) $\forall x, y \in H, \quad x \neq y, \quad \{x, y\} \subseteq x \circ y;$
2) $\forall x, y \in H, \quad |x \circ y| = k;$
3) $\forall x, y, z, t \in H, \quad |x \circ y \cap z \circ t| > 1 \Rightarrow x \circ y = z \circ t.$

Remark 2.1. The geometric structure associated to a k-quasi-Steiner hypergroupoid is always canonically represented.

Theorem 2.1. *If $(H, \circ)$ is a k-quasi-Steiner hypergroupoid, then we have:*

$$\forall x \in H, \quad \exists (u, v) \in H^2, \quad u \neq v : \; x \circ x = u \circ v.$$

Proof. Let $x \in H$; by 2) it follows that:

$$|x \circ x| = k \geq 2$$

and therefore

$$\exists u, v \in x \circ x, \quad u \neq v.$$

By 1) we have that:

$$\{u, v\} \subseteq u \circ v;$$

and so:

$$|x \circ x \cap u \circ v| \supseteq \{u, v\};$$

by 3), it follows now that:

$$x \circ x = u \circ v. \qquad \square$$

Theorem 2.2. *Any quasi-Steiner hypergroupoid $(H, \circ)$ is commutative.*

Proof. By 1) it follows that:

$$\forall x, y \in H, \quad x \neq y, \quad y \circ x \supseteq \{x, y\} \subseteq x \circ y;$$

and therefore, by 3), we have that:

$$x \circ y = y \circ x. \qquad \square$$

Let $(H, \mathcal{L})$ be a k-Steiner system; we define on H a hyperoperation by setting:

$$\forall x, y \in H, \quad x \circ y = \begin{cases} l_{xy} & \text{if} \quad x \neq y \\ l_x & \text{if} \quad x = y \end{cases}$$

where l_{xy} denote the unique line trough x and y and l_x is an arbitrary line of $\mathcal{L}$. Clearly, $(H, \circ)$ is a k-quasi-Steiner hypergroupoid that we will call associated to $(H, \mathcal{L})$.

Conversely, if $(H, \circ)$ is a k-quasi-Steiner hypergroupoid and if we consider the family:

$$\mathcal{L} = \{x \circ y\}_{(x,y) \in H \times H},$$

then $(H, \mathcal{L})$ is a k-Steiner system. In fact, from 2) it follows that every element of $\mathcal{L}$ has cardinality k; from 1) it follows that any pair of distinct points are contained in a line; from 3) it follows that any pair of distinct points are contained in one and only one line. So, the notions of k-Steiner system and k-quasi-Steiner hypergroupoid are equivalent.

Remark 2.2. If $(H, \mathcal{L})$ is a Steiner system, with $|H| = v$, $|\mathcal{L}| = b$, then the number of quasi-Steiner hypergroupoids associated to $(H, \mathcal{L})$ is b^v.

3. Remarkable groups of automorphisms in a Steiner system.

We already know that, if $(H, \circ)$ is a hypergroupoid, then $\mathcal{A}ut_A H$ is a subgroup of $\mathcal{A}ut_G H$ and, in general, a proper subgroup; but in some cases these groups are coincident; in fact we have

Theorem 3.1. *Let $(H, \mathcal{L})$ be a Steiner system, $(H, \circ)$ be the associated Steiner hypergroupoid. Then we have:*

$$\mathcal{A}ut_A H = \mathcal{A}ut_G H.$$

Proof. We must show that:

$$\mathcal{A}ut_G H \subseteq \mathcal{A}ut_A H.$$

Let $\varphi \in \mathcal{A}ut_G H$, $(x, y) \in H^2$. By definition of Steiner hypergroupoid it follows that:

$$\{x, y\} \subseteq x \circ y;$$

since φ preserve the incidences, we have that:

$$\varphi(x) \in \varphi(x \circ y) \ni \varphi(y). \tag{3.1}$$

On the other hand, by definition of Steiner hypergroupoid, we have that:

$$\varphi(x) \in \varphi(x) \circ \varphi(y) \ni \varphi(y). \tag{3.2}$$

Finally, we have that:

if $x = y$, then:

$$\varphi(x \circ x) = \varphi(x) = \varphi(x) \circ \varphi(x);$$

if $x \neq y$, then from (3.1) e (3.2) it follows that:

$$(\varphi(x \circ y)) \supseteq \{\varphi(x), \varphi(y)\} \subseteq (\varphi(x) \circ \varphi(y)),$$

that is $\varphi(x \circ y)$ and $\varphi(x) \circ \varphi(y)$ are two lines in a Steiner system that meet in at least two points, and so they must to be equal, that is:

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y).$$

This shows that $\varphi \in \mathcal{A}ut_G H$. □

Now we will prove that, unlike Steiner hypergroupoids, the group $\mathcal{A}ut_A H$ of all algebraic automorphisms of an arbitrary quasi-Steiner hypergroupoid $(H, \circ)$ is a proper subgroup of $\mathcal{A}ut_G H$.

Theorem 3.2. *Let* $(H, \mathcal{L})$ *be a Steiner system,* $(H, \circ)$ *be a quasi-Steiner hypergroupoid associated to* $(H, \mathcal{L})$*; then we have:*

$$\mathcal{A}ut_A H \neq \mathcal{A}ut_G H.$$

Proof. We suppose that $\mathcal{A}ut_A H = \mathcal{A}ut_G H$. We put $\mathcal{L}' = \{x \circ x\}_{x \in H}$. First of all we note that, in this case:

$$\mathcal{L}' = \mathcal{L}, \tag{3.3}$$

in fact, if $\exists l \in \mathcal{L} \setminus \mathcal{L}'$, then:

$$\forall x \in H, \quad x \circ x \neq l,$$

and so the geometric automorphisms that send a line of $\mathcal{L}'$ in the line l are not algebraic and so $\mathcal{A}ut_A H \neq \mathcal{A}ut_G H$.

From (3.3) it follows that, if we put $v = |H|$, $b = |\mathcal{L}|$, we have that:

$$b \leq v.$$

On the other hand, in a Steiner system we have always that $v \leq b$, and so:

$$b = v,$$

and consequently:

$$\forall x, y \in H, \quad x \neq y \quad \Leftrightarrow \quad x \circ x \neq y \circ y. \tag{3.4}$$

Now let $z \in H$ and $f \in \mathcal{A}ut_A H$ such that the line $z \circ z$ is fixed by f; we have that:

$$f(z) \circ f(z) = f(z \circ z) = z \circ z,$$

and therefore from (3.4) it follows that $f(z) = z$; so, if an algebraic automorphism fixes the line $z \circ z$, then it must fixe the point z. So, the geometric automorphisms that fixe the line $z \circ z$, but do not fixe the point z are not algebraic, that is absurd. $\square$

Theorem 3.3. *Let* $(H, \mathcal{L})$ *be a Steiner system,* $(H, \circ)$ *be a quasi-Steiner hypergroupoid associated to* $(H, \mathcal{L})$*. Then we have that:*

$$\mathcal{A}ut_A H = \{\varphi \in \mathcal{A}ut_G H : \varphi(x \circ x) = \varphi(x) \circ \varphi(x), \forall x \in H\}.$$

Proof. It is enough to observe that, as in Theorem 3.1, we have that:

$$\forall \varphi \in \mathcal{A}ut_G H, \quad \forall x, y \in H, \quad x \neq y \quad \Rightarrow \quad \varphi(x \circ y) = \varphi(x) \circ \varphi(y),$$

but, in general, we have that:

$$\varphi(x \circ x) \neq \varphi(x) \circ \varphi(x). \qquad \square$$

Corollary 3.4. *Let $(H, \mathcal{L})$ be a Steiner system and $l \in \mathcal{L}$. Let $(H, \circ_l)$ be the quasi-Steiner hypergroupoid defined by:*

$$\forall x \in H, \quad x \circ_l x = l.$$

Then we have:

$$\mathcal{A}ut_A H = \{\varphi \in \mathcal{A}ut_G H : \varphi(l) = l\}$$

that is $\mathcal{A}ut_A H$ is the subgroup of $\mathcal{A}ut_G H$ that fixes the line l.

Proof. It suffices to observe that:

$$\forall x \in H, \begin{cases} \varphi(x \circ_l x) = \varphi(l) \\ \varphi(x) \circ_l \varphi(x) = l \end{cases}.$$

The corollary follows now from the previous theorem. $\square$

Corollary 3.5. *Let $A = AG(2, q)$, $P = PG(2, q)$; let l be a line in $PG(2, q)$. Let $(A, \circ)$, $(P, \circ_l)$ be respectively the Steiner hypergroupoid associated to $AG(2, q)$ and the quasi-Steiner hypergroupoid associated to $PG(2, q)$, where $\circ_l$ is defined as in the previous corollary. Then we have:*

$$\mathcal{A}ut_A A \cong \mathcal{A}ut_A P.$$

Proof. It is enough to observe that:

$$\mathcal{A}ut_A A = \mathcal{A}ut_G A,$$

by Theorem 3.1;

$$\mathcal{A}ut_G A \cong (\mathcal{A}ut_G P)_l,$$

where $(\mathcal{A}ut_G P)_l$ is the subgroup of $\mathcal{A}ut_G P$ that fixe the line l, since the automorphisms of an affine plane are (up to isomorphisms) those geometric ones that fixe a line in the projective plane in which it is imbedded;

$$(\mathcal{A}ut_G P)_l = \mathcal{A}ut_A P,$$

by the previous corollary. □

Example 3.1. Let $A \equiv AG(2, 2)$ and $P \equiv PG(2, 2)$. We consider the Steiner hypergroupoid $(A, \circ)$ associated to $AG(2, 2)$, that is:

$\circ$	1	2	3	4
1	1	1, 2	1, 3	1, 4
2	1, 2	2	2, 3	2, 4
3	1, 3	2, 3	3	3, 4
4	1, 4	2, 4	3, 4	4

and the quasi-Steiner hypergroupoid $(P, \circ_l)$ associated to $PG(2, 2)$, that is:

$\circ_l$	1	2	3	4	5	6	7
1	1, 2, 3	1, 2, 3	1, 2, 3	1, 4, 7	1, 5, 6	1, 5, 6	1, 4, 7
2	1, 2, 3	1, 2, 3	1, 2, 3	2, 4, 6	2, 5, 7	2, 4, 6	2, 5, 7
3	1, 2, 3	1, 2, 3	1, 2, 3	3, 4, 5	3, 4, 5	3, 6, 7	3, 6, 7
4	1, 4, 7	2, 4, 6	3, 4, 5	1, 2, 3	3, 4, 5	2, 4, 6	1, 4, 7
5	1, 5, 6	2, 5, 7	3, 4, 5	3, 4, 5	1, 2, 3	1, 5, 6	2, 5, 7
6	1, 5, 6	2, 4, 6	3, 6, 7	2, 4, 6	1, 5, 6	1, 2, 3	3, 6, 7
7	1, 4, 7	2, 5, 7	3, 6, 7	1, 4, 7	2, 5, 7	3, 6, 7	1, 2, 3

where $l = \{1, 2, 3\}$; from the last corollary it follows that:

$$\mathcal{A}ut_A A \cong \mathcal{A}ut_A P.$$

Theorem 3.6. *Let $(H, \mathcal{L})$ be a (non-degenerated) Steiner system; let $l \in \mathcal{L}$, $p \in l$; let $(H, \circ)$ be a quasi-Steiner hypergroupoid associated to $(H, \mathcal{L})$ such that:*

$$\begin{cases} x \circ x = l, & \forall x \in (H \setminus l) \cup \{p\}; \\ x \circ x \neq l, & \textit{otherwise}. \end{cases}$$

Then we have:

$$\mathcal{A}ut_A H \leq (\mathcal{A}ut_G H)_{l,p},$$

where $(\mathcal{A}ut_G H)_{l,p}$ *denote the subgroup of* $\mathcal{A}ut_G H$ *that fixes* l *and* p*.*

Proof. Let $f \in \mathcal{A}ut_A H$; first of all we observe that, because the Steiner system is not degenerated, then:

$$|(H \setminus l) \cup \{p\}| > \frac{|H|}{2}.$$

So, since f is bijective, we have that:

$$\exists x, y \in (H \setminus l) \cup \{p\} : \quad f(x) = y.$$

We have that:

$$\begin{aligned} f(x \circ x) &= f(l); \\ f(x) \circ f(x) &= y \circ y = l; \end{aligned}$$

and by Theorem 2.3:

$$f(l) = l. \tag{3.5}$$

Now we prove that $f(p) = p$. By hypothesis we have that:

$$f(p \circ p) = f(l),$$

and from (3.5) it follows that:

$$f(p \circ p) = l.$$

By Theorem 2.3, we have that:

$$f(p) \circ f(p) = l,$$

and therefore, from the hypothesis on $\circ$, it follows that:

$$f(p) \in (H \setminus l) \cup \{p\}.$$

But $f(l) = l$ is equivalent to $f(H \setminus l) = H \setminus l$, and therefore:

$$f(p) \notin H \setminus l,$$

that is:

$$f(p) = p. \qquad \square$$

In general if a quasi-Steiner hypergroupoid satisfies the conditions of the previous theorem, then $\mathcal{A}ut_A H$ is not equal to $(\mathcal{A}ut_G H)_{l,p}$, as we can see in the following

Example 3.2. Let $(H, \mathcal{L}) \equiv AG(2,3)$. Let $(H, \circ)$ be the quasi-Steiner hypergroupoid associated to $AG(2,3)$, defined by the following table:

$\circ$	1	2	3	4	5	6	7	8	9
1	1, 2, 3	1, 2, 3	1, 2, 3	1, 4, 7	1, 5, 9	1, 6, 8	1, 4, 7	1, 6, 8	1, 5, 9
2	1, 2, 3	4, 5, 6	1, 2, 3	2, 4, 9	2, 5, 8	2, 6, 7	2, 6, 7	2, 5, 8	2, 4, 9
3	1, 2, 3	1, 2, 3	7, 8, 9	3, 4, 8	3, 5, 7	3, 6, 9	3, 5, 7	3, 4, 8	3, 6, 9
4	1, 4, 7	2, 4, 9	3, 4, 8	1, 2, 3	4, 5, 6	4, 5, 6	1, 4, 7	3, 4, 8	2, 4, 9
5	1, 5, 9	2, 5, 8	3, 5, 7	4, 5, 6	1, 2, 3	4, 5, 6	3, 5, 7	2, 5, 8	1, 5, 9
6	1, 6, 8	2, 6, 7	3, 6, 9	4, 5, 6	4, 5, 6	1, 2, 3	2, 6, 7	1, 6, 8	3, 6, 9
7	1, 4, 7	2, 6, 7	3, 5, 7	1, 4, 7	3, 5, 7	2, 6, 7	1, 2, 3	7, 8, 9	7, 8, 9
8	1, 6, 8	2, 5, 8	3, 4, 8	3, 4, 8	2, 5, 8	1, 6, 8	7, 8, 9	1, 2, 3	7, 8, 9
9	1, 5, 9	2, 4, 9	3, 6, 9	2, 4, 9	1, 5, 9	3, 6, 9	7, 8, 9	7, 8, 9	1, 2, 3

Such hypergroupoid satisfies all conditions of the previous theorem with $l = \{1, 2, 3\}$, $p = 1$; we have that $\mathcal{A}ut_A H \neq (\mathcal{A}ut_G H)_{l,p}$; more precisely, we have that:

$$\mathcal{A}ut_A H = \left\{ \begin{array}{c} 1_H, (456)(798), (465)(789), \\ (32)(47)(59)(68), (32)(485769), (32)(496758) \end{array} \right\}.$$

Theorem 3.7. *Let $(H, \mathcal{L}) \equiv \prod_q$ be a projective plane of order q; let $l \in \mathcal{L}$, $p \notin l$; let $(H, \circ)$ be a quasi-Steiner hypergroupoid associated to $\prod_q$ such that:*

$$\begin{cases} x \circ x = l, & \forall x \in l \cup \{p\}; \\ x \circ x \neq l, & \text{otherwise.} \end{cases}$$

Then we have that:

$$\mathcal{A}ut_A H \leq (\mathcal{A}ut_G H)_{l,p},$$

where $(\mathcal{A}ut_G H)_{l,p}$ denote the subgroup of $\mathcal{A}ut_G H$ fixing l and p.

Proof. Let $f \in \mathcal{A}ut_A H$ and $l = \{b_1, b_2, \ldots, b_{q+1}\}$; first of all we prove that:

$$\exists x, y \in l \cup \{p\} : \quad f(x) = y. \tag{3.6}$$

We suppose that:

$$\forall x, y \in l \cup \{p\}, \quad f(x) \neq y$$

that is:

$$\forall x \in l \cup \{p\}, \quad f(x) \notin l \cup \{p\}.$$

So, by definition of $\circ$, it follows that:

$$\forall x \in l \cup \{p\}, \quad f(x) \circ f(x) = f(x \circ x) = f(l);$$

but:

$$f(x) \notin l \cup \{p\} \quad \Rightarrow \quad f(x) \circ f(x) \neq l.$$

So, it follows that:

$$f(l) \neq l. \tag{3.7}$$

But f is also a geometric automorphism; therefore we have that $f(l) \in \mathcal{L}$; let $f(l) = \{c_1, c_2, \ldots, c_{q+1}\}$. Now $l \cap f(l) \neq \emptyset$ because $(H, \mathcal{L})$ is a projective plane. This means that:

$$\exists i, j \in \{1, 2, \ldots, q+1\} : \quad b_i = c_j.$$

From this we obtain:

$$c_j \circ c_j = b_i \circ b_i = l. \tag{3.8}$$

On the other hand, from $c_j \in f(l)$ it follows that:

$$\exists b_k \in l : \quad c_j = f(b_k);$$

therefore we have that:

$$c_j \circ c_j = f(b_k) \circ f(b_k) = f(b_k \circ b_k) = f(l),$$

and so, from (3.8), we have that:

$$f(l) = l,$$

that is absurd, by (3.7); this shows that (3.6) is true.

So, we have that:

$$\begin{aligned} f(x \circ x) &= f(l); \\ f(x) \circ f(x) &= y \circ y = l; \end{aligned}$$

and therefore, by Theorem 2.3, it follows that:

$$f(l) = l.$$

We observe now that:

$$f(p) \circ f(p) = f(p \circ p) = f(l) = l;$$

so, by definition of $\circ$, we have that:

$$f(p) \in l \cup \{p\};$$

but from $f(l) = l$, it follows necessarily:

$$f(p) \notin l,$$

that is:

$$f(p) = p. \qquad \square$$

Example 3.3. Let $(H, \mathcal{L}) \equiv PG(2.2)$. We consider the quasi-Steiner hypergroupoid $(H, \circ)$ associated to $PG(2, 2)$, defined by the following table:

$\circ$	1	2	3	4	5	6	7
1	1, 2, 3	1, 2, 3	1, 2, 3	1, 4, 7	1, 5, 6	1, 5, 6	1, 4, 7
2	1, 2, 3	1, 2, 3	1, 2, 3	2, 4, 6	2, 5, 7	2, 4, 6	2, 5, 7
3	1, 2, 3	1, 2, 3	1, 2, 3	3, 4, 5	3, 4, 5	3, 6, 7	3, 6, 7
4	1, 4, 7	2, 4, 6	3, 4, 5	2, 5, 7	3, 4, 5	2, 4, 6	1, 4, 7
5	1, 5, 6	2, 5, 7	3, 4, 5	3, 4, 5	1, 4, 7	1, 5, 6	2, 5, 7
6	1, 5, 6	2, 4, 6	3, 6, 7	2, 4, 6	1, 5, 6	1, 2, 3	3, 6, 7
7	1, 4, 7	2, 5, 7	3, 6, 7	1, 4, 7	2, 5, 7	3, 6, 7	3, 4, 5

Such hypergroupoid satisfies all conditions of the previous theorem with $l = \{1, 2, 3\}$ and $p = 6$; in such case we have that:

$$\mathcal{A}ut_A H = (\mathcal{A}ut_G H)_{l,p}.$$

In fact any geometric automorphism fixing the line $\{1, 2, 3\}$ and the point 6 is also algebraic. To prove this, it is enough to show that if $f \in (\mathcal{A}ut_G H)_{l,p}$, then:

$$\forall x \notin l \cup \{p\}, \quad f(x \circ x) = f(x) \circ f(x),$$

because for $x \in l \cup \{p\}$ the preceding equality holds, since for these x we have $x \circ x = l$ and f fixes $l \cup \{p\}$. In other words, we must show that:

$$f(x \circ x) = f(x) \circ f(x), \quad \forall x \in \{4, 5, 7\}.$$

First of all we observe that:

$$\left|(\mathcal{A}ut_G H)_{l,p}\right| = 6.$$

We have that:

$$f_1 \equiv (23)\,(47) \in \mathcal{A}ut_A H$$

because:

$$\begin{aligned}
f_1(4 \circ 4) &= f_1\{2, 5, 7\} = \{3, 4, 5\} = 7 \circ 7 = f_1(4) \circ f_1(4);\\
f_1(5 \circ 5) &= f_1\{1, 4, 7\} = \{1, 4, 7\} = 5 \circ 5 = f_1(5) \circ f_1(5);\\
f_1(7 \circ 7) &= f_1\{3, 4, 5\} = \{2, 5, 7\} = 4 \circ 4 = f_1(7) \circ f_1(7).
\end{aligned}$$

Moreover we have:

$$f_2 \equiv (13)\,(57) \in \mathcal{A}ut_A H$$

because:

$$\begin{aligned}
f_2(4 \circ 4) &= f_2\{2, 5, 7\} = \{2, 5, 7\} = 4 \circ 4 = f_2(4) \circ f_2(4);\\
f_2(5 \circ 5) &= f_2\{1, 4, 7\} = \{3, 4, 5\} = 7 \circ 7 = f_2(5) \circ f_2(5);\\
f_2(7 \circ 7) &= f_2\{3, 4, 5\} = \{1, 4, 7\} = 5 \circ 5 = f_2(7) \circ f_2(7).
\end{aligned}$$

Finally we have:

$$f_3 \equiv (12)\,(45) \in \mathcal{A}ut_A H$$

because:

$$\begin{aligned}
f_3(4 \circ 4) &= f_3\{2, 5, 7\} = \{1, 4, 7\} = 5 \circ 5 = f_3(4) \circ f_3(4);\\
f_3(5 \circ 5) &= f_3\{1, 4, 7\} = \{2, 5, 7\} = 4 \circ 4 = f_3(5) \circ f_3(5);\\
f_3(7 \circ 7) &= f_3\{3, 4, 5\} = \{3, 4, 5\} = 7 \circ 7 = f_3(7) \circ f_3(7).
\end{aligned}$$

Therefore, since $1_H \in \mathcal{A}ut_A H$, we obtain that:

$$|\mathcal{A}ut_A H| \geq 4$$

and so:

$$\mathcal{A}ut_A H = (\mathcal{A}ut_G H)_{l,p}.$$

The following example shows that, if a quasi-Steiner hypergroupoid satisfies all conditions of the previous theorem, in general we have that $\mathcal{A}ut_A H$ is a proper subgroup of $(\mathcal{A}ut_G H)_{l,p}$.

Example 3.4. Let $(H, \mathcal{L}) \equiv PG(2, 2)$. We consider the following quasi-Steiner hypergroupoid $(H, \circ)$ associated to $PG(2, 2)$, defined by:

$\circ$	1	2	3	4	5	6	7
1	1, 2, 3	1, 2, 3	1, 2, 3	1, 4, 7	1, 5, 6	1, 5, 6	1, 4, 7
2	1, 2, 3	1, 2, 3	1, 2, 3	2, 4, 6	2, 5, 7	2, 4, 6	2, 5, 7
3	1, 2, 3	1, 2, 3	1, 2, 3	3, 4, 5	3, 4, 5	3, 6, 7	3, 6, 7
4	1, 4, 7	2, 4, 6	3, 4, 5	3, 4, 5	3, 4, 5	2, 4, 6	1, 4, 7
5	1, 5, 6	2, 5, 7	3, 4, 5	3, 4, 5	2, 5, 7	1, 5, 6	2, 5, 7
6	1, 5, 6	2, 4, 6	3, 6, 7	2, 4, 6	1, 5, 6	1, 2, 3	3, 6, 7
7	1, 4, 7	2, 5, 7	3, 6, 7	1, 4, 7	2, 5, 7	3, 6, 7	1, 4, 7

As in the previous example $(H, \circ)$ is a quasi-Steiner hypergroupoid associated to $PG(2, 2)$ satisfying the conditions of the theorem, with $l = \{1, 2, 3\}$, $p = 6$; but in this case $g \equiv (12)(45)$ is a geometric automorphism fixing l and p that is not algebraic. So, in this case:

$$\mathcal{A}ut_A H \neq (\mathcal{A}ut_G H)_{l,p}.$$

Theorem 3.8. *Let $(H, \mathcal{L})$ be a projective plane (finite of order $q \neq 2^h$, $\forall h \in \mathbb{N}$ or infinite); let Γ be a conic in $(H, \mathcal{L})$, $(H, \circ_\Gamma)$ the quasi-Steiner hypergroupoid defined by:*

$$\forall x \in H, \quad x \circ_\Gamma x = l_{x,\Gamma},$$

where $l_{x,\Gamma}$ denote the polar line of x in respect to the conic Γ. Then we have:

$$\mathcal{A}ut_A H = \{f \in \mathcal{A}ut_G H : f(l_{x,\Gamma}) = l_{f(x),\Gamma}, \forall x \in H\}.$$

Proof. This follows immediately from Theorem 2.3, since in this case we have:

$$\begin{aligned} f(x \circ_\Gamma x) &= f(l_{x,\Gamma}) \\ f(x) \circ_\Gamma f(x) &= l_{f(x),\Gamma}. \qquad \square \end{aligned}$$

Theorem 3.9. *Let $(H, \mathcal{L})$ be a projective plane (finite of order $q \neq 2^h$, $\forall h \in \mathbb{N}$ or infinite); let Γ be a conic, $(H, \circ_\Gamma)$ the quasi-Steiner hypergroupoid defined by:*

$$\forall x \in H, \quad x \circ_\Gamma x = l_{x,\Gamma},$$

where $l_{x,\Gamma}$ denote the polar line of x in respect to the conic Γ. Then we have:

$$\mathcal{A}ut_A H = (\mathcal{A}ut_G H)_\Gamma,$$

where $(\mathcal{A}ut_G H)_\Gamma$ denote the group of all geometric automorphisms fixing the conic Γ.

Proof. First of all we observe that, from the geometric properties of the polar line, it follows that:

$$\forall x \in H, \quad x \in x \circ_\Gamma x \quad \Leftrightarrow \quad x \in \Gamma \quad \Leftrightarrow \quad x \circ_\Gamma x \cap \Gamma = \{x\}; \tag{3.9}$$

$$\forall x, y \in H, \quad x \in y \circ_\Gamma y \quad \Leftrightarrow \quad y \in x \circ_\Gamma x. \tag{3.10}$$

Let $f \in \mathcal{A}ut_A H$; we suppose that $f \notin (\mathcal{A}ut_G H)_\Gamma$, that is:

$$\exists x \in \Gamma, \quad \exists y \notin \Gamma : \quad f(x) = y.$$

From $x \in \Gamma$, it follows that $x \in x \circ_\Gamma x$; and so:

$$y = f(x) \in f(x \circ_\Gamma x).$$

From $y \notin \Gamma$, it follows that $y \notin y \circ_\Gamma y$, that is:

$$y \notin f(x) \circ_\Gamma f(x).$$

So, we have that:

$$f(x \circ_\Gamma x) \neq f(x) \circ_\Gamma f(x),$$

that is absurd.

Now let $f \in (\mathcal{A}ut_G H)_\Gamma$, that is $f(\Gamma) = \Gamma$; by the previous theorem, it suffices to show that:

$$\forall x \in H, \quad f(x \circ_\Gamma x) = f(x) \circ_\Gamma f(x).$$

If $x \in \Gamma$ then it is easily proved that:

$$f(x \circ_\Gamma x) \cap \Gamma = \{f(x)\}.$$

So, from (3.9), it follows that $f(x \circ_\Gamma x)$ is the polar line of $f(x)$, that is:

$$f(x \circ_\Gamma x) = f(x) \circ_\Gamma f(x).$$

If $x \notin \Gamma$ we distinguish two cases:

$$\begin{aligned} &\text{a)} \quad (x \circ_\Gamma x) \cap \Gamma \neq \emptyset; \\ &\text{b)} \quad (x \circ_\Gamma x) \cap \Gamma = \emptyset. \end{aligned}$$

In case a), since $|(x \circ_\Gamma x) \cap \Gamma| = 2$, we can set $(x \circ_\Gamma x) \cap \Gamma = \{p_x, q_x\}$; so, it follows that:

$$x \circ_\Gamma x = p_x \circ_\Gamma q_x$$

and therefore:

$$f(x \circ_\Gamma x) = f(p_x \circ_\Gamma q_x). \tag{3.11}$$

Recalling the proof of Theorem 2.3, we have that:

$$f(p_x \circ_\Gamma q_x) = f(p_x) \circ_\Gamma f(q_x). \tag{3.12}$$

Now, since $\{p_x, q_x\} \subseteq x \circ_\Gamma x$, from (3.10) it follows that:

$$x \in p_x \circ_\Gamma p_x \cap q_x \circ_\Gamma q_x;$$

and therefore:

$$f(x) \in f(p_x \circ_\Gamma p_x) \cap f(q_x \circ_\Gamma q_x).$$

Moreover, since $\{p_x, q_x\} \subseteq \Gamma$, it follows that $f(p_x),\ f(q_x) \in \Gamma$, and so, by the first case, it follows that:

$$f(p_x \circ_\Gamma p_x) = f(p_x) \circ_\Gamma f(p_x) \quad \text{and} \quad f(q_x \circ_\Gamma q_x) = f(q_x) \circ_\Gamma f(q_x);$$

and therefore:

$$f(x) \in f(p_x) \circ_\Gamma f(p_x) \quad \text{and} \quad f(x) \in f(q_x) \circ_\Gamma f(q_x);$$

and so:

$$f(p_x),\ f(q_x) \in f(x) \circ_\Gamma f(x) \quad \text{with} \quad f(p_x) \neq f(q_x).$$

Therefore:

$$f(p_x) \circ_\Gamma f(q_x) = f(x) \circ_\Gamma f(x). \tag{3.13}$$

Finally, from (3.11), (3.12) and (3.13), it follows now that:

$$f(x \circ_\Gamma x) = f(x) \circ_\Gamma f(x).$$

In case b) let $p_x \in \Gamma$, since $(H, \mathcal{L})$ is a projective plane, then we can set:

$$p'_x = (p_x \circ_\Gamma p_x) \cap (x \circ_\Gamma x). \tag{3.14}$$

From $p_x \in p'_x \circ_\Gamma p'_x$, it follows that:

$$(p'_x \circ_\Gamma p'_x) \cap \Gamma \neq \emptyset;$$

and so, by case a), it follows that:

$$f(p'_x \circ_\Gamma p'_x) = f(p'_x) \circ_\Gamma f(p'_x). \tag{3.15}$$

On the other hand, from $x \in p'_x \circ_\Gamma p'_x$ it follows that:

$$f(x) \in f(p'_x \circ_\Gamma p'_x) = f(p'_x) \circ_\Gamma f(p'_x).$$

and so:

$$f(p'_x) \in f(x) \circ_\Gamma f(x). \tag{3.16}$$

Now we choose another point $q_x \in \Gamma$, and analogously we set:

$$q'_x = (q_x \circ_\Gamma q_x) \cap (x \circ_\Gamma x).$$

We can suppose $p'_x \neq q'_x$. So, we have that:

$$f(q'_x) \in f(x) \circ_\Gamma f(x). \tag{3.17}$$

Consequently, from (3.16) and (3.17), we have that:

$$f(x) \circ_\Gamma f(x) = f(p'_x) \circ_\Gamma f(q'_x). \tag{3.18}$$

On the other hand we have that:

$$p'_x, q'_x \in x \circ_\Gamma x$$

and so:

$$x \circ_\Gamma x = p'_x \circ_\Gamma q'_x.$$

Therefore:

$$f(x \circ_\Gamma x) = f(p'_x \circ_\Gamma q'_x);$$

so, since $p'_x \neq q'_x$, we have:

$$f(p'_x \circ_\Gamma q'_x) = f(p'_x) \circ_\Gamma f(q'_x)$$

and so:

$$f(x \circ_\Gamma x) = f(p'_x) \circ_\Gamma f(q'_x).$$

Comparing this equality with (3.18), we have that:

$$f(x \circ_\Gamma x) = f(x) \circ_\Gamma f(x).$$

The proof is now complete. □

4. A general result about algebraic automorphisms.

Theorem 4.1. *Let $(H, \circ)$ be a hypergroupoid associated to a geometric space $(H, \mathcal{B})$ and let $\varphi \in \mathcal{A}ut_G(H, \circ)$; let $\bullet$ be the hyperoperation on H defined by:*

$$\forall x, y \in H, \quad x \bullet y = \varphi(x \circ y).$$

Then we have:

$$\mathcal{A}ut_A(H, \circ) = \mathcal{A}ut_A(H, \bullet) \quad \Leftrightarrow \quad \varphi \in \mathcal{C}_{\mathcal{A}ut_G(H,\circ)}(\mathcal{A}ut_A(H, \circ)),$$

where $\mathcal{C}_{\mathcal{A}ut_G(H,\circ)}(\mathcal{A}ut_A(H, \circ))$ denote the centralizer of $\mathcal{A}ut_A(H, \circ)$ in $\mathcal{A}ut_G(H, \circ)$.

Proof. $\Leftarrow$) Let $f \in \mathcal{A}ut_A(H, \circ)$; by definition of $\bullet$ it follows immediately that:

$$\begin{cases} f(x \bullet y) = f(\varphi(x \circ y)) \\ f(x) \bullet f(y) = \varphi(f(x) \circ f(y)) = \varphi(f(x \circ y)). \end{cases} \tag{4.1}$$

Then, since $\varphi \in \mathcal{C}_{\mathcal{A}ut_G(H,\circ)}(\mathcal{A}ut_A(H, \circ))$, we have that:

$$\forall z \in H, \quad f(\varphi(z)) = \varphi(f(z))$$

and, from (4.1), it follows that:

$$\forall x, y \in H, \quad f(x \bullet y) = f(x) \bullet f(y)$$

that is:

$$f \in \mathcal{A}ut_A(H, \bullet).$$

So, we have that:

$$\mathcal{A}ut_A(H, \circ) \subseteq \mathcal{A}ut_A(H, \bullet).$$

Now let $g \in \mathcal{A}ut_A(H, \bullet)$; from (4.1) it follows that:

$$\forall x, y \in H, \quad g(\varphi(x \circ y)) = \varphi(g(x \circ y)).$$

Such property holds for φ^{-1}; in fact, we have:

$$\forall x, y \in H, \quad \varphi^{-1}(g(x \circ y)) = \varphi^{-1}(g\varphi\varphi^{-1}(x \circ y)) =$$
$$= \varphi^{-1}(\varphi g\varphi^{-1}(x \circ y)) = \varphi^{-1}\varphi(g\varphi^{-1}(x \circ y)) = g(\varphi^{-1}(x \circ y)).$$

From this it follows now that:

$$\begin{cases} g(x \circ y) = g(\varphi^{-1}(x \bullet y)) \\ g(x) \circ g(y) = \varphi^{-1}(g(x) \bullet g(y)) = \varphi^{-1}(g(x \bullet y)) \end{cases}$$

that is:

$$g \in \mathcal{A}ut_A(H, \circ).$$

So, we have that

$$\mathcal{A}ut_A(H, \bullet) \subseteq \mathcal{A}ut_A(H, \circ).$$

$\Rightarrow$) We will show that:

$$\varphi \notin \mathcal{C}_{\mathcal{A}ut_G(H,\circ)}(\mathcal{A}ut_A(H, \circ)) \quad \Rightarrow \quad \mathcal{A}ut_A(H, \circ) \neq \mathcal{A}ut_A(H, \bullet).$$

Since $\varphi \notin \mathcal{C}_{\mathcal{A}ut_G(H,\circ)}(\mathcal{A}ut_A(H,\circ))$, we have that:

$$\exists h \in \mathcal{A}ut_A(H,\circ): \quad h\varphi \neq \varphi h$$

that is:

$$\exists v \in H: \quad h\varphi(v) \neq \varphi h(v).$$

Since φ is a geometric automorphism, we have that:

$$\exists B \in \mathcal{B}: \quad h\varphi(B) \neq \varphi h(B),$$

and therefore, since $(H,\circ)$ is associated to $(H,\mathcal{B})$:

$$\exists x, y \in H: \quad h\varphi(x \circ y) \neq \varphi h(x \circ y).$$

So, from (4.1) it follows that:

$$h(x \bullet y) \neq h(x) \bullet h(y)$$

that is:

$$h \notin \mathcal{A}ut_A(H,\bullet). \qquad \square$$

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