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# HÖLDER CLASSES RELATIVE TO DEGENERATE ELLIPTIC OPERATORS AS INTERPOLATION SPACES

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The well known characterization of Hölder classes as interpolation spaces is here extended under suitable hypotheses to a class of spaces where the Hölder continuity is given in terms of an intrinsic distance relative to degenerate elliptic operators of Hörmander type.

## 1. Introduction and interpolation between Banach spaces.

In [9] we studied the generation of analytic semigroups by a proper degenerate elliptic operator $A$ and the aim was to apply that result to obtain optimal regularity for the corresponding evolution equation.

However a major difficulty arose: we could not completely characterize the interpolation spaces between the domain of the operator $D(A)$ and the Banach space $X$, mostly because of the lack of commutativity of the vectors fields $X_i$ involved in the definition of $A$.

Therefore in this paper we concentrate on the definition of Hölder spaces by means of interpolation and we prove some interpolation results for spaces of continuous functions.

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We recall that Prof. A. Lunardi in a recent paper ([16]) solved this problem in the framework of Heisenberg vector fields, using a method that could be applicable to more general subelliptic operators.

Let us introduce the general framework we will need in the following:

*1.1 Homogeneous Spaces.*

Let us be given a topological space $X$ with a distance $d$.

**Definition 1.1.** *We say that* $(X, d)$ *is a homogeneous space in the sense of Coifman e Weiss* (see [8]) *if*

a) *the balls* $B(r, x)$ *form a basis of open neighbourhoods of* $x$*;*
b) *there exists* $N \in \mathbb{N}$ *s.t.* $\forall x \in X$ *and* $\forall r > 0$ *the ball* $B(r, x)$ *contains at most* $N$ *points* $x_i$ *s.t.* $d(x_i, x_j) > \frac{r}{2}$ *(homogeneity property).*

**Remark 1.2.** A remarkable case where the homogeneity property is automatically verified is when there exists a Borel measure $\mu$ which satisfies the so called *doubling condition*

$$0 < \mu(B(r, x)) \leq A\mu(B(\frac{r}{2}, x)) < \infty, \tag{1.1}$$

where $A$ is an absolute constant.

Since the homogeneity property is usually verified showing (1.1), in the following by homogeneous space we will directly denote a set $X$ with a distance $d$ and a Borel measure $\mu$ which satisfies (1.1). Without entering too much into details, let us just say that homogeneous spaces have been recently used as a natural framework to study Poincaré inequality relative to Dirichlet forms (see [4], [5]). We recall also that it is possible to work with a pseudodistance instead of a distance. For the sake of simplicity here we prefer to consider only the case of a topological space $X$ with a distance $d$.

*1.2 Interpolation between Banach Spaces.*

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces s.t.

a) $Y \subset X$;
b) there exists an absolute constant $c > 0$ s.t. $\forall y \in Y \|y\|_X \leq c\|y\|_Y$.

Let now $t \in \mathbb{R}_+$ and consider the following real functional

$$K(t, x, X, Y) = \inf\{\|a\|_X + t\|b\|_Y : x = a + b, a \in X, b \in Y\}.$$

We then define

$$(X, Y)_{\theta,\infty} := \left\{x \in X : \sup_{t>0} \frac{1}{t^\theta} K(t, x, X, Y) < \infty\right\}$$

with $\theta \in (0, 1]$. It can be proved that $(X, Y)_{\theta,\infty}$ is a Banach space if it is endowed with the norm

$$\|x\|_{(X,Y)_{\theta,\infty}} = \sup_{t>0} \frac{1}{t^\theta} K(t, x, X, Y)$$

and it turns out that it is also an interpolation space beetween $X$ and $Y$ in the sense that for any linear operator $T : X \to X$ and $T : Y \to Y$ s.t. $T \in \mathcal{L}(X, X)$ and $T \in \mathcal{L}(Y, Y)$ we have $T \in \mathcal{L}((X, Y)_{\theta,\infty}, (X, Y)_{\theta,\infty})$.

**Definition 1.3.** *Let $\alpha \in [0, 1]$ and $E$ a Banach space s.t. $Y \subset E \subset X$.*

*a) $E$ is said to belong to the class $J_\alpha(X, Y)$ if*

$$\|x\|_E \le C\|x\|_X^{1-\alpha}\|x\|_Y^\alpha \quad \forall x \in Y$$

*for some absolute constant $C > 0$; in this case we write $E \in J_\alpha(X, Y)$;*

*b) $E$ is said to belong to the class $K_\alpha(X, Y)$ if*

$$K(t, x, X, Y) \le kt^\alpha\|x\|_E \quad \forall x \in E, t > 0$$

*for some absolute constant $k > 0$; in this case we write $E \in K_\alpha(X, Y)$.*

For more details on general interpolation spaces and also on the spaces we are considering here, see for example [2], [3], [6] and [15]. We conclude this section, briefly presenting another interpolation method.

**Definition 1.4.** *For $0 \le \theta < 1$ and $1 \le p \le \infty$ set*

$$V(p, \theta, Y, X) = \Big\{u : \mathbb{R}_+ \mapsto X : t \mapsto u_\theta(t) = t^{\theta-1/p}u(t) \in L^p((0, \infty); Y),$$
$$t \mapsto v_\theta(t) = t^{\theta-1/p}u'(t) \in L^p((0, \infty); X)\Big\},$$
$$\|u\|_{V(p,\theta,Y,X)} = \|u_\theta\|_{L^p((0,\infty);Y)} + \|v_\theta\|_{L^p((0,\infty);X)}.$$

Then

**Proposition 1.5.** *For $(\theta, p) \in ]0, 1[\times[1, \infty] \cup \{(1, \infty)\}$, $(X, Y)_{\theta,p}$ is the set of traces at $t = 0$ of the functions in $V(p, 1-\theta, Y, X)$ and the norm*

$$\|x\|_{\theta,p}^T = \inf\{\|u\|_{V(p,1-\theta,Y,X)} : x = u(0), u \in V(p, 1-\theta, Y, X)\}$$

*is an equivalent norm in $(X, Y)_{\theta,p}$.*

*Proof.* See for example [15], Theorem 1.2.10. □

*1.3 Hörmander vector fields.*

Let us consider $C^\infty(\mathbb{R}^n)$ vector fields $X_i$, $i = 1, \dots, m$, that satisfy a Hörmander condition of order $K$: at any point the vectors and their commutators up to order $K$ span $\mathbb{R}^n$. In this case an intrinsic distance $d_X$ associated to the $X_i$ can be defined. Namely

$$d_X(x, y) = \inf \Big\{ \delta > 0 : \exists \Phi \text{ Lipschitz curve s.t.} \tag{1.2}$$

$$\Phi'(t) = \sum_{i=1}^{m} a_i(t) X_i(\Phi(t)) \text{ with } |a_i(t)| \le \delta \text{ and } \Phi(0) = x, \Phi(1) = y \Big\},$$

(see also [18]). It is well-known that $\forall x, y \in \mathbb{R}^n$ $d_X(x, y)$ satisfies the condition

$$\frac{1}{c}|x - y| \le d_X(x, y) \le c|x - y|^\varepsilon$$

where $|\cdot|$ stands for the usual euclidean norm, $\varepsilon = \frac{1}{K+1}$ and $c > 1$ is a suitable constant. We can then define balls in the usual way relying on the distance $d_X$. When referring to Hörmander vector fields in the following we will always deal with these so - called *intrinsic* balls.

If we denote with $m$ the Lebesgue measure in $\mathbb{R}^n$, in [18] it is proved that the following duplication property holds:

$$0 < m(B(2r, y)) \le c_0 m(B(r, y)) \tag{1.3}$$

for every ball with center at $y \in \mathbb{R}^n$ and radius $r < R_0$, with the constant $c_0$ possibly depending only on $R_0$. The validity of the duplication property is a fundamental step in the proof of the Poicaré inequality for vector fields (see [11] and [14]). Anyway we can say that the space $\mathbb{R}^n$ with distance $d_X$ and Lebesgue measure $m$ gets the structure of homogeneous space as discussed in *1.1*.

As a consequence of (1.3) there exists a constant $\nu = \log_2 c_0$ such that

$$m(B(r, y)) \le 2m(B(s, y)) \left(\frac{r}{s}\right)^\nu \tag{1.4}$$

for every $0 < s < r \le \frac{R_0}{2}$. On the other hand it is easy to see that

$$m(B(2r, y)) \ge c^* m(B(r, y))$$

(where $c^* > 1$ is a constant that depends only on $c_0$) and

$$m(B(s,y)) \le m(B(r,y))\left(\frac{s}{r}\right)^{\sigma} \tag{1.5}$$

where $0 < s < \frac{r}{2} < \frac{R_0}{2}$ and $\sigma = 1 - \frac{1}{c^*}$.

The numbers $\nu$ and $\sigma$ give an upper and a lower bound on the *intrinsic* dimension of $\mathbb{R}^n$ and are in general different. However there are special cases in which they can coincide and to these we will come back in Section 3. In any case the intrinsic dimension $\nu$ is usually different from $n$.

Finally, if we define

$$d_* := \sup\left\{\Phi(x) - \Phi(y) : \Phi \in C_0^\infty(\mathbb{R}^n), \sum_{i=1}^{m} |X_i(\Phi)|^2 \le 1\right\},$$

then $d_*$ is actually a distance and is equivalent to $d_X$ (see [12]). However in the following we will always refer to $d_X$.

Starting from his fundamental work on hypoelliptic operators (see [10]), Hörmander vector fields and more general subelliptic operators have been the subject of a tremendous amount of work (see [12] for the references updated until 1987, but much more has followed).

## 2. Interpolation between $C^0(X)$ and $C^{0,1}(X)$.

Let us consider a homogeneous space $(X, d, \mu)$, as explained in the previous section.

By $C^0(X)$ $(C^{0,\theta}(X), C^{0,1}(X))$ we denote the space of bounded and uniformly continuous functions (the space of bounded Hölder continuous functions, the space of bounded Lipschitz continuous functions, respectively) endowed with the usual norms.

The following interpolation result holds

**Proposition 2.1.** *Let $\theta \in (0,1)$. Then*

$$(C^0(X), C^{0,1}(X))_{\theta,\infty} = C^{0,\theta}(X).$$

To prove Proposition 2.1 we need this approximation Lemma.

**Lemma 2.2.** *Let $\theta \in (0, 1]$ and $\varphi \in C^{0,\theta}(X)$. Define for $t > 0$ and $x \in X$*

$$u(t, x) = \sup_{z\in X} \Big\{ \inf_{y\in X} [\varphi(y) + \frac{1}{2t} d^2(z, y)] - \frac{1}{t} d^2(z, x) \Big\}.$$

*Then $u(t, \cdot) \in C^{0,1}(X)$ and*

$$\|u(t, \cdot)\|_{C^0(X)} \le \|\varphi\|_{C^0(X)}, \tag{2.1}$$

$$0 \le \varphi(x) - u(t, x) \le (2^\theta |\varphi|_\theta^2)^{\frac{1}{2-\theta}} t^{\frac{\theta}{2-\theta}}, \tag{2.2}$$

$$\sup_{x\neq y} \frac{|u(t, x) - u(t, y)|}{d(x, y)} \le H (2^{3-\theta} |\varphi|_\theta)^{\frac{1}{2-\theta}} t^{\frac{\theta-1}{2-\theta}}, \tag{2.3}$$

*$\forall x \in X$ and $t > 0$ with $H$ proper absolute constant.*

*Proof.* By the definition of $u(t, x)$ we obtain

$$u(t, x) \le \varphi(x)$$

(so that the first part of (2.2) is immediately proved) and

$$u(t, x) \ge \inf_{y\in X} [\varphi(y) + \frac{1}{2t} d^2(y, x)].$$

Now fix $\varepsilon > 0$ and let $y_\varepsilon \in X$ be such that

$$u(t, x) + \varepsilon > \varphi(y_\varepsilon) + \frac{1}{2t} d^2(y_\varepsilon, x) \ge \varphi(y_\varepsilon).$$

Since $\varepsilon$ is arbitrary, we have $u(t, x) \ge -\|\varphi\|_{C^0(X)}$ and we have proved (2.1).

Regarding the second part of (2.2), recalling that $\varphi \in C^{0,\theta}(X)$, we have

$$\varphi(x) - u(t, x) < \varphi(x) - \varphi(y_\varepsilon) - \frac{1}{2t} d^2(y_\varepsilon, x) + \varepsilon,$$

that is

$$\varphi(x) - u(t, x) < |\varphi|_\theta d^\theta(x, y_\varepsilon) - \frac{1}{2t} d^2(x, y_\varepsilon) + \varepsilon.$$

Since the left-hand side above is nonnegative, we find

$$d(x, y_\varepsilon) \le M_\varepsilon,$$

where $M_\varepsilon$ is the greatest positive number s.t.

$$M^2 \leq 2t(|\varphi|_\theta M^\theta + \varepsilon).$$

Then we have

$$\varphi(x) - u(t,x) \leq |\varphi|_\theta M_\varepsilon^\theta + \varepsilon \quad \forall x \in X$$

and the second part of (2.2) follows as

$$\lim_{\varepsilon \downarrow 0} M_\varepsilon = (2t|\varphi|_\theta)^{\frac{1}{2-\theta}}.$$

To conclude the proof we have to show the validity of (2.3). For this purpose, let $\varepsilon > 0$ and $x \in X$ be fixed and let $z_{\varepsilon,x} \in X$ be such that

$$u(t,x) < \inf_{y \in X}[\varphi(y) + \frac{1}{2t}d^2(z_{\varepsilon,x}, y)] - \frac{1}{t}d^2(z_{\varepsilon,x}, x) + \varepsilon. \tag{2.4}$$

Then

$$\frac{1}{t}d^2(z_{\varepsilon,x}, x) < \varepsilon + \varphi(x) - u(t,x) + \frac{1}{2t}d^2(z_{\varepsilon,x}, x).$$

From (2.2) we get

$$\frac{1}{2t}d^2(z_{\varepsilon,x}, x) \leq \varepsilon + (2^\theta |\varphi|_\theta^2)^{\frac{1}{2-\theta}} t^{\frac{\theta}{2-\theta}}.$$

Now consider $x' \in X$. From (2.4) we have

$$u(t,x) - u(t,x') < \varepsilon + \frac{1}{t}[d^2(z_{\varepsilon,x}, x') - d^2(z_{\varepsilon,x}, x)] = \tag{2.5}$$

$$= \varepsilon + \frac{1}{t}[d(z_{\varepsilon,x}, x') - d(z_{\varepsilon,x}, x)][d(z_{\varepsilon,x}, x') + d(z_{\varepsilon,x}, x)].$$

From the triangle inequality $d(z_{\varepsilon,x}, x') \leq [d(z_{\varepsilon,x}, x) + d(x', x)]$ we have

i) $d(z_{\varepsilon,x}, x') - d(z_{\varepsilon,x}, x) \leq d(x, x')$,

ii) $d(z_{\varepsilon,x}, x') + d(z_{\varepsilon,x}, x) \leq 2d(z_{\varepsilon,x}, x) + d(x, x')$.

Then (2.5) becomes

$$u(t,x) - u(t,x') \leq \varepsilon + \frac{1}{t}d(x,x')[2d(z_{\varepsilon,x}, x) + d(x,x')] =$$

$$= \varepsilon + \frac{1}{t}d^2(x,x') + \frac{2}{t}d(x,x')d(z_{\varepsilon,x}, x)$$

and also

$$u(t,x) - u(t,x') \le \varepsilon + \frac{1}{t} d^2(x,x') + \frac{2}{t} d(x,x')[2t(\varepsilon + C_\theta t^{\frac{\theta}{2-\theta}})]^{1/2},$$

where we have set $C_\theta = (2^\theta |\varphi|_\theta^2)^{\frac{1}{2-\theta}}$. Therefore, letting $\varepsilon \downarrow 0$ we finally obtain

$$|u(t,x) - u(t,x')| \le \frac{1}{t} d^2(x,x') + 2d(x,x')\sqrt{2C_\theta} t^{\frac{\theta-1}{2-\theta}}$$

or

$$\frac{|u(t,x) - u(t,x')|}{d(x,x')} \le \frac{1}{t} d(x,x') + 2\sqrt{2C_\theta} t^{\frac{\theta-1}{2-\theta}}$$

and by standard argument this implies

$$|u(t,\cdot)|_1 \le 2\sqrt{2C_\theta} t^{\frac{\theta-1}{2-\theta}}, \quad \forall t > 0, \forall x \in X,$$

which is exactly (2.3) once we set $H = 2$. $\square$

**Remark 2.3.** It was already observed in [13] that this kind of regularization works in general metric spaces even if no precise estimates were given. Here we have adopted an argument due to [7]. What is interesting is that in [7] everything is done in Hilbert spaces, but the actual properties used are just the metric ones. We can now prove Proposition 2.1. As usual in this kind of results, we divide the proof in two parts. In the following the symbol $\hookrightarrow$ will denote continuous imbedding.

a) $(C^0(X), C^{0,1}(X))_{\theta,\infty} \hookrightarrow C^{0,\theta}(X)$.
Let $\varphi \in (C^0(X), C^{0,1}(X))_{\theta,\infty}$. Then for any $t > 0$ and any $\varepsilon > 0$ there exist $f_{t,\varepsilon} \in C^0(X)$ and $g_{t,\varepsilon} \in C^{0,1}(X)$ s.t.

$$\varphi(x) = f_{t,\varepsilon}(x) + g_{t,\varepsilon}(x), \quad \forall x \in X$$

and

$$\|f_{t,\varepsilon}\|_{C^0(X)} + t\|g\|_{C^{0,1}(X)} \le t^\theta \|\varphi\|_{C^0(X),C^{0,1}(X))_{\theta,\infty}} + \varepsilon,$$

where the $(C^0(X), C^{0,1}(X))_{\theta,\infty}$ norm has been defined in the previous section. Therefore

$$\varphi(x) - \varphi(y) = f_{t,\varepsilon}(x) - f_{t,\varepsilon}(y) + g_{t,\varepsilon}(x) - g_{t,\varepsilon}(y)$$

and also

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\le 2\|f_{t,\varepsilon}\|_{C^0(X)} + |g_{t,\varepsilon}|_1 d(x,y) \le \\ &\le 2Ct^\theta + Ct^{\theta-1} d(x,y) + \frac{\varepsilon}{t} d(x,y) + 2\varepsilon, \end{aligned}$$

$\forall x, y \in X, t > 0$ and some constant $C > 0$. Now, letting $\varepsilon \downarrow 0$ and taking $t = d(x, y)$, we are done.

b) $C^{0,\theta}(X) \hookrightarrow (C^0(X), C^{0,1}(X))_{\theta,\infty}$ .

Let $\varphi \in C^{0,\theta}(X)$ and let $u(t, x)$ as above in Lemma 2.2. For every $t > 0$ and $x \in X$ we set

$$f_t(x) = \varphi(x) - u(t^{2-\theta}, x),$$
$$g_t(x) = u(t^{2-\theta}, x).$$

Then for some proper constant $C$ we have

$$\|f_t\|_{C^0(X)} \leq Ct^{\theta}, \quad |g_t|_1 \leq Ct^{\theta-1}.$$

Since $\|g_t\|_{C^0(X)} \leq \|\varphi\|_{C^0(X)}$, we obtain

$$\|f_t\|_{C^0(X)} + t(\|g_t\|_{C^0(X)} + |g_t|_1) = \|f_t\|_{C^0(X)} + t\|g_t\|_{C^{0,1}(X)} \leq Ct^{\theta}$$

and so $K(t, \varphi, C^0(X), C^{0,1}(X)) \leq Ct^{\theta}$ for any $t \in (0, 1)$. Since this last inequality is trivial for $t \geq 1$, the conclusion follows. □

**Remark 2.4.** As a matter of fact, the implication $(C^0(X), C^{0,1}(X))_{\theta,\infty} \hookrightarrow C^{0,\theta}(X)$ holds not only for spaces defined on $(X, d, \mu)$ metric and homogeneous, but also for $(X, d, \mu)$ generally homogeneous. In fact it is the second inclusion that requires $d$ to be a distance and not just a pseudodistance. For a general characterization of Lipschitz functions in this more general setting, see [17].

## 3. The Gagliardo completion.

Consider now the case of $C^{\infty}$ vector fields $X_i$ that satisfy a Hörmander condition of order $K$. As we said in Section 1, $\mathbb{R}^n$ endowed with the intrinsic distance and the Lebesgue measure is a homogeneous space of the kind considered in Section 2; we can therefore specialize its situation to this new one.

We denote by $C^0_X(\mathbb{R}^n)$ the class of bounded and uniformly continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ and set

$$C^{0,\theta}_X(\mathbb{R}^n) := \left\{ f \in C^0_X(\mathbb{R}^n) : |f|_{\theta} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^{\theta}(x, y)} < \infty \right\}, \quad \theta \in (0, 1),$$

$$C_X^{0,1}(\mathbb{R}^n) := \left\{ f \in C_X^0(\mathbb{R}^n) : |f|_1 := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\},$$

exactly as in the previous section, where for the sake of simplicity we write $d$ instead of $d_X$ from here on. It is obvious that Proposition 2.1 immediately applies here so that we can conclude that $(C_X^0(\mathbb{R}^n), C_X^{0,1}(\mathbb{R}^n))_{\theta,\infty} = C_X^{0,\theta}(\mathbb{R}^n)$.

As a matter of fact, the presence of a differential structure in this context allows us to say a little bit more.

First of all, let us define

$$C_X^1(\mathbb{R}^n) := \{ f \in C_X^0(\mathbb{R}^n) : X_i(f) \in C_X^0(\mathbb{R}^n), \quad \forall i = 1, \dots, m \}.$$

It is immediate to see that if $f \in C_X^1(\mathbb{R}^n)$, then $f \in C_X^{0,1}(\mathbb{R}^n)$. In fact take $x_0$ and $x_1$ in $\mathbb{R}^n$. Then there certainly exists $\Phi$ such that

$$u(x_1) - u(x_0) = u(\Phi(1)) - u(\Phi(0)) = \int_0^1 \sum_{i=1}^m a_i(t) X_i(u(\Phi(t)))\, dt.$$

Since $X_i(u) \in C_X^0(\mathbb{R}^n)$, $\forall i = 1, \dots, m$, they are all bounded in $\mathbb{R}^n$. Therefore

$$|u(x_0) - u(x_1)| \leq C d(x_0, x_1)$$

where the constant $C$ does not depend on $x_0$ or $x_1$ and we conclude that

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x, y)} \leq C < \infty.$$

One is now naturally confronted with the following.

**Problem 3.1.** *Can we strengthen the result of the previous section and conclude that actually*

$$(C_X^0(\mathbb{R}^n), C_X^1(\mathbb{R}^n))_{\theta,\infty} = C_X^{0,\theta}(\mathbb{R}^n),$$

*with $\theta \in (0, 1)$?*

Under this point of view we have the following general result due to Gagliardo (see for example [1] and [2]).

**Proposition 3.2.** *Consider a couple of Banach spaces $A_0$ and $A_1$ and the real functional $K(t, f; A_0, A_1)$ defined in Section 2. We have*

$$K(t, f; A_0, A_1) = K(t, f; A_0 + \infty A_1, A_1 + \infty A_0) = \\ = K(t, f; A_0, A_1 + \infty A_0), \; t > 0 \tag{3.1}$$

*for all* $f \in A_0 + A_1$ *where*

*a) an element* $f \in A_0 + A_1$ *is said to belong to* $A_0 + \infty A_1$ *if there is a sequence* $\{g_\varepsilon\} \subseteq A_0$ *for which*

$$\sup_{\varepsilon} \|g_\varepsilon\|_{A_0} < \infty, \quad \lim_{\varepsilon \to 0} \|f - g_\varepsilon\|_{A_0 + A_1} = 0;$$

*b) analogous statement holds for* $A_1 + \infty A_0$ *with obvious interchange of the roles.*
*In this case we say that* $A_1 + \infty A_0$ *is the* Gagliardo completion *of* $A_1$ *in* $A_0 + A_1$.

Referrring to the previous notation, if we take $A_0 = C_X^0(\mathbb{R}^n)$ and $A_1 = C_X^1(\mathbb{R}^n)$, it is clear that

$$A_0 + A_1 = C_X^0(\mathbb{R}^n), \quad A_0 \cap A_1 = C_X^1(\mathbb{R}^n).$$

In the classical case, i.e. $X_i = \frac{\partial}{\partial x_i} (m = n)$, we have $C^1(\mathbb{R}^n) + \infty C^0(\mathbb{R}^n) = C^{0,1}(\mathbb{R}^n)$; in this situation the answer to Problem 3.1 is immediate, since we obtain

$$\begin{aligned} &K(f, t; C^0(\mathbb{R}^n), C^{0,1}(\mathbb{R}^n)) = K(f, t; C^0(\mathbb{R}^n), C^1(\mathbb{R}^n)) \Rightarrow \\ &\quad \Rightarrow (C^0(\mathbb{R}^n), C^{0,1}(\mathbb{R}^n))_{\theta,\infty} = (C^0(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta,\infty} = C^{0,\theta}(\mathbb{R}^n). \end{aligned}$$

Is the same true in the case of general Hörmander vector fields? What we were able to prove is the following

**Proposition 3.3.** *Suppose that the (upper bound of the) intrinsic dimension* $\nu$ *is such that*

$$C_1 r^\nu \le m(B(r, x)) \le C_2 r^\nu, \tag{3.2}$$

*where* $C_1$ *and* $C_2$ *are two constants that do not depend on* $x$ *or* $r$. *Then for any function* $f \in C_X^{0,1}(\mathbb{R}^n)$ *there exists a sequence* $\{f_\varepsilon\} \subseteq C_X^1(\mathbb{R}^n)$ *s.t.*

*a)* $f_\varepsilon \xrightarrow{C_X^0(\mathbb{R}^n)} f$ *as* $\varepsilon \to 0$,

*b)* $\|f_\varepsilon\|_{C_X^1(\mathbb{R}^n)} \le C$ *for any* $\varepsilon > 0$,

*which is the same to say that* $C_X^{0,1}(\mathbb{R}^n)$ *is the Gagliardo completion of* $C_X^1(\mathbb{R}^n)$ *in* $C_X^0(\mathbb{R}^n)$.

*Proof.* Take the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

and for $\varepsilon > 0$ fixed, set

$$a_\varepsilon(x) = \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \Phi\Big(\frac{d(x,y)}{\varepsilon}\Big)\, dy.$$

Relying on (3.2) it is then easy to see that with $\nu$ as in the hypotheses

1) $0 < C_1 \leq |a_\varepsilon| \leq C_2$;
2) $|X_i(a_\varepsilon)| \leq \frac{C_3}{\varepsilon}$.

Given now $f \in C_X^{0,1}(\mathbb{R}^n)$, define

$$f_\varepsilon(x) := \frac{1}{a_\varepsilon(x)} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^N} \Phi\Big(\frac{d(x,y)}{\varepsilon}\Big) f(y)\, dy.$$

It is not difficult to see that $f_\varepsilon \in C_X^1(\mathbb{R}^n)$. In particular, if we take $f^* \equiv 1$, we have

$$f_\varepsilon^* = \frac{1}{a_\varepsilon(x)} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \Phi\Big(\frac{d(x,y)}{\varepsilon}\Big)\, dy \equiv 1.$$

Therefore

$$0 = X_i(f_\varepsilon^*) = X_i\left(\frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \frac{\Phi(\frac{d(x,y)}{\varepsilon})}{a_\varepsilon(x)}\, dy\right) = \tag{3.3}$$
$$= \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} X_i\left(\frac{\Phi(\frac{d(x,y)}{\varepsilon})}{a_\varepsilon(x)}\right) dy.$$

Moreover

$$\|f - f_\varepsilon\|_{C_X^0(\mathbb{R}^n)} = \sup_{\mathbb{R}^n} \left| \frac{1}{a_\varepsilon(x)} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \Phi(\frac{d(x,y)}{\varepsilon}) f(y)\, dy - f(x) \right| =$$
$$= \sup_{\mathbb{R}^n} \left| \frac{1}{a_\varepsilon(x)} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \Phi(\frac{d(x,y)}{\varepsilon})(f(y) - f(x))\, dy \right|$$
$$\leq \left( \frac{1}{|a_\varepsilon(x)|} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \Phi(\frac{d(x,y)}{\varepsilon})\, dy \right) \left( \sup_{x \neq y} \frac{|f(y) - f(x)|}{d(x,y)} \right) \varepsilon = L\varepsilon,$$

where we have taken into account that $\Phi(\frac{t}{\varepsilon}) = 0$ when $|t| \geq \varepsilon$ and $L$ is the Lipschitz constant of $f$.

$$\begin{aligned}
\|X_i(f_\varepsilon)\|_{C^0_X(\mathbb{R}^n)} &= \sup_{\mathbb{R}^n} |X_i(f_\varepsilon)| = \\
&= \sup_{\mathbb{R}^n} \left| X_i \left( \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \frac{1}{a_\varepsilon(x)} \Phi(\frac{d(x,y)}{\varepsilon}) f(y)\, dy \right) \right| \leq \\
&\leq \sup_{\mathbb{R}^n} \frac{1}{\varepsilon^\nu} \int_{\mathbb{R}^n} \left| X_i \left( \frac{\Phi(\frac{d(x,y)}{\varepsilon})}{a_\varepsilon(x)} \right) \right| |f(x) - f(y)|\, dy
\end{aligned}$$

where we have taken into acount (3.3). Moreover

$$\begin{aligned}
\left| X_i \left( \frac{\Phi(\frac{d(x,y)}{\varepsilon})}{a_\varepsilon(x)} \right) \right| &= \left| \frac{1}{a^2_\varepsilon(x)} X_i(a_\varepsilon(x)) \Phi(\frac{d(x,y)}{\varepsilon}) + \right. \\
&\left. + \frac{1}{a_\varepsilon(x)} \frac{\Phi'(d(x,y))}{\varepsilon} X_i(d(x,y)) \right| \leq \frac{K_1}{\varepsilon} + \frac{K_2}{\varepsilon} = \frac{K}{\varepsilon}.
\end{aligned}$$

Therefore considering that $\Phi^{(m)}(\frac{t}{\varepsilon}) = 0\ \forall t \geq \varepsilon$ and $m \in \mathbb{N} \cup \{0\}$, we conclude that $\|X_i(f_\varepsilon)\|_{C^0_X(\mathbb{R}^n)} \leq K$, where $K$ *does not* depend on $\varepsilon$ or $i$ and we are done. $\square$

**Remark 3.4.** Actually a kind of regularization as the one used here is considered in [20], where however the aim is more in proving the existence of regular cut - off functions relative to a suitable quasi-distance, defined in terms of the exponential mapping.

**Remark 3.5.** The essential hypothesis assumed above is (3.2); as a matter of fact, due to the particular nature of the function $\Phi$, we do not really need (3.2) for any $r > 0$: it is enough that it holds for $r < 1$ and in fact we could have assumed it directly in the hypotheses. Therefore, even if it does not hold for general vector fields, as it remarked in Section 1, it is however verified in a wide range of situations, as it is the case of $X_i$ associated to polynomial groups (see [19], Proposition 4.9).

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