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ONE-DIMENSIONAL MOTION OF A MATERIAL WITH A STRAIN THRESHOLD

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We consider the one-dimensional shearing motion of a material exhibiting elastic behaviour when the stress is below some threshold. The threshold represents a limit to the deformability, i.e. no further deformation can occur on increasing the stress. The mathematical formulation leads to a free boundary problem for the wave equation, whose structure depends on whether the stress (and the velocity) are continuous across the propagating interface for the strain threshold .

Local existence and uniqueness are proved for the continuous case (in which the interface propagation is subsonic). Some explicit solutions are calculated for another case (with a supersonic interface). It is shown that the model with strain threshold is never the limit of hyperelastic systems.

1. Introduction

In a recent paper [2] we have formulated a one-dimensional mathematical model for a material characterized by elastic behaviour when the applied shear stress is below some threshold, and which becomes inextensible beyond that threshold. The stress threshold corresponds to a maximum value of the strain. Materials which can be approximated as having a strain threshold exist in nature. Tendons

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and ligaments have two components (elastin and collagen) with contrasting mechanical properties resulting in an elastic behaviour with limited extensibility (see [1]).

As a limiting case we consider the one dimensional stress-strain relation portrayed in fig. 1 where ε is the strain and σ is the shear stress. When the applied stress exceeds τ_o (or equivalently when the strain has reached its threshold ε_o) the body is incapable of further deformation[1]. Thus, the constitutive law is given as an implicit[2] relation between the stress and the strain. The material is non-hyperelastic since the stress cannot be derived from a potential function usually referred to as stored elastic energy. Here the elastic energy is a function of both the stress and the strain. The constitutive law depicted in fig. 1 takes the implicit form

$$g(\sigma,\varepsilon)=\left(\sigma-\frac{d\varphi(\varepsilon)}{d\varepsilon}\right)[\theta(\sigma)-\theta(\sigma-\tau_o)]+\theta(\sigma-\tau_o)(\varepsilon-\varepsilon_o)=0, \quad (1)$$

where $\varphi(\varepsilon)$ is a given function such that $\varphi'(\varepsilon)>0$ for $\varepsilon>0$ and $\varphi(0)=0$, and $\theta(\sigma)=0$ for $\sigma<0$, $\theta(\sigma)=1$ for $\sigma\geq 0$. Accordingly the following generalized stored energy can be defined

$$\psi(\sigma,\varepsilon)=\begin{cases}\varphi(\varepsilon), & 0\leq\sigma<\tau_o,\\ \\ \varphi(\varepsilon_o)=const., & \sigma\geq\tau_o,\end{cases} \quad (2)$$

The specific motion examined here is the one of a layer with the bottom held fixed and top subject to a shear stress $\widehat{\sigma}(t)$, taking values beyond threshold. The deformable region ($\sigma<\tau_o$) and the fully strained one ($\sigma>\tau_o$) are separated by a moving interface $\mathscr{S}$. In any region that does not contain $\mathscr{S}$ we can write the non-dissipation equation for isothermal processes

$$\sigma\dot{\varepsilon}-\dot{\psi}=0. \quad (3)$$

Thus, in the unstrained region $\dot{\varepsilon}=0$ and, from (3) $\dot{\psi}=0$, while in the deformable region $\dot{\psi}=\sigma\dot{\varepsilon}$. However equation (3) does not necessarily hold in regions containing $\mathscr{S}$, since on such a surface we have entropy production and dissipation is not zero (see Section 2.4). A comparison of the problem with pure threshold ε_o with the limit of a hyperelastic material whose stiffness tends to infinity for $\varepsilon>\varepsilon_o$ is performed in Section 5. Open problems and future development are shortly illustrated in Section 6

[1] For the sake of simplicity in the deformation range $[0,\varepsilon_o]$ the stress-strain relation is supposed to be linear.

[2] A detailed discussion about implicit constitutive relationship within the general framework of continuum mechanics can be found in [3].

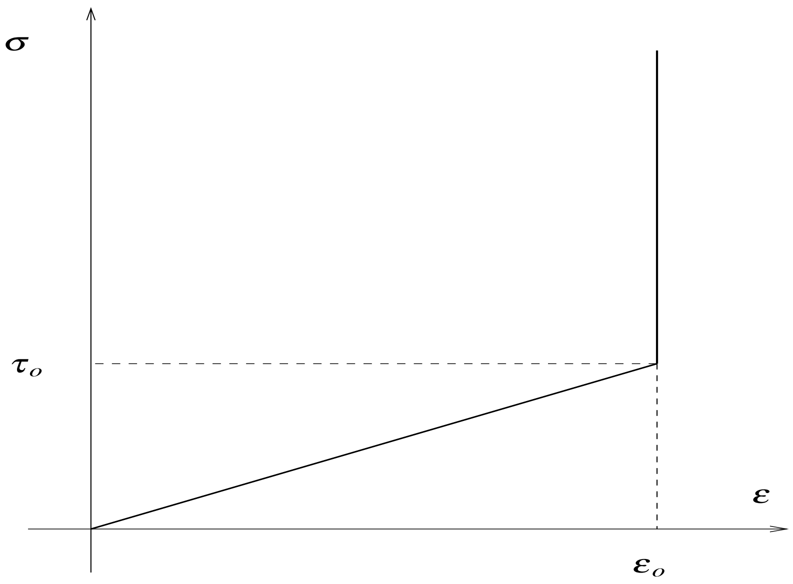

Figure 1: Stress vs deformation gradient. Implicit non-hyperelastic model

2. The mathematical model

We consider a homogeneous layer of material of thickness h. The top surface is loaded with a known shear stress $\widehat{\sigma}$. Let x and y be the axes respectively parallel and orthogonal to the layers forming the material. The Lagrangian and Eulerian coordinates are denoted by $\vec{X}=(X,Y,Z)$, $\vec{x}=(x,y,z)$ respectively. We consider the pure shear motion

$$\left\{\begin{array}{lcl} x & = & X+f\,(Y,t)\,, \\ y & = & Y, \\ z & = & Z, \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{lcl} x & = & X+f\,(y,t)\,, \\ y & = & Y, \\ z & = & Z, \end{array}\right.$$

$f\,(y,t)$ being the unknown displacement. We assume that the applied shear stress $\widehat{\sigma}$ takes values beyond τ_o. The surface $\mathscr{S}$ separating the fully strained and the deformable region is given by $y=s\,(t)$. Notice that the latter is not a material surface. The material is fully strained for $s\,(t)<y\leq h$ and deformable (elastic) for $0\leq y<s\,(t)$.

2.1. Deformable region

This is the region $0\leq y<s\,(t)$. The displacement is $f\,(y,t)\,\vec{e}_1$. The velocity and acceleration are given by $v\,(y,t)\,\vec{e}_1=f_t\,(y,t)\,\vec{e}_1$ and $a\,(y,t)\,\vec{e}_1=f_{tt}\,(y,t)\,\vec{e}_1$.

Denoting by σ the shear stress acting on the layer located at y at time t, we have

$$\sigma(y,t) = \mu f_y(y,t).$$

The stored energy function is thus

$$\varphi(f_y(y,t)) = \frac{1}{2}\mu[f_y(y,t)]^2.$$

We require that [3]

$$\sigma\left(s^-,t\right) \le \tau_o \;\;\Leftrightarrow\;\; f_y\left(s^-,t\right) \le \frac{\tau_o}{\mu}. \tag{4}$$

Of course if $\sigma(s^-,t) = \tau_o$ then $\varphi = (\tau_o^2/2\mu)$.

Assuming that no pressure gradient is applied on the lateral sides of the body, i.e. $P(t) = 0$, the equation of motion in the elastic region obviously reduces to

$$f_{tt} - c^2 f_{yy} = 0, \tag{5}$$

where $c = \sqrt{(\mu/\rho)}$ is the speed of sound. The initial data for (5) are $f(y,0) = f_o(y)$ and $f_t(y,0) = f_1(y)$, $y \in [0,s_o]$. At the bottom surface $y = 0$ we assume that there is no displacement, i.e. $f(0,t) = 0$.

2.2. Fully strained region

This is the region $s(t) < y < h$. Velocity and acceleration are denoted here by $v^{(s)}(t)$ and $a^{(s)}(t)$ with $a^{(s)}(t) = (dv^{(s)}(t)/dt)$. At the interface $y = s(t)$ we impose continuity of the displacement

$$f\left(s^+,t\right) = f\left(s^-,t\right). \tag{6}$$

Hence we write

$$f(y,t) = f\left(s^-,t\right) + \frac{\tau_o}{\mu}(y-s), \quad \forall\, y \in [s,h], \tag{7}$$

since, in the fully strained region, the strain is uniformly equal to (τ_o/μ). Differentiating (7) w.r.t. time and recalling that $\sigma(s^-,t) = f_y(s^-,t)/\mu$ we get

$$[\![v]\!]_{\mathscr{S}} = -\frac{\tau_o - \sigma(s^-,t)}{\mu}\dot{s}, \tag{8}$$

$$[\![a]\!]_{\mathscr{S}} = -\frac{\tau_o - \sigma(s^-,t)}{\mu}\ddot{s} + \frac{\dot{s}}{\mu}\frac{d\sigma(s^-,t)}{dt} + \frac{\dot{s}}{\mu}\sigma_t\left(s^-,t\right), \tag{9}$$

[3] For a generic function $q(y,t)$, $q\left(s^-,t\right)$ and $q\left(s^+,t\right)$ denote $\lim_{y\to s^-} q(y,t)$ and $\lim_{y\to s^+} q(y,t)$, respectively.

where $[\![(\cdot)]\!]_{\mathscr{S}} = (\cdot)|_{y=s^+} - (\cdot)|_{y=s^-}$ denotes the jump of the quantity $(\cdot)$ across $\mathscr{S}$. The equation of motion is simply

$$\rho a^{(s)}(t) = \sigma_y. \tag{10}$$

Imposing the boundary condition $\sigma(h,t) = \widehat{\sigma}(t)$ we get

$$\sigma(y,t) = \rho a^{(s)}(t)(y-h) + \widehat{\sigma}(t). \tag{11}$$

We require that $\sigma(y,t) \geq \tau_o$, $\forall\ y \geq s(t)$. Equations (8), (9) are kinematic in nature. The dynamics comes into play if we consider the Rankine-Hugoniot condition on $\mathscr{S}$ for the conservation of the linear momentum (e.g. [4])

$$\rho [\![v]\!]_{\mathscr{S}}\, \dot{s} = -[\![\sigma]\!]_{\mathscr{S}}. \tag{12}$$

Relation (12) can also be deduced by combining the balance of linear momentum applied to the whole system and the one applied only to the fully stretched region.

2.3. Dynamics of the moving interface

Let us consider the domain

$$\Omega = \{x_o < x < x_o + 1,\ z_o < z < z_o + 1,\ \ell < y < h\},$$

for given x_o, z_o and $s(t) < \ell < h$. Conservation of linear momentum implies that

$$\frac{d}{dt}\left[\rho v^{(s)}(t)(h-\ell)\right] = \rho a^{(s)}(t)(h-\ell) = \widehat{\sigma} - \sigma(\ell,t), \tag{13}$$

where $\widehat{\sigma}(t) - \sigma(\ell,t)$ is the resultant of all external forces acting on Ω. Taking the limit $\ell \to s^+$ we get

$$\rho\, a^{(s)}(t)(h-s) = \widehat{\sigma} - \sigma\left(s^+,t\right). \tag{14}$$

Combining (8) and (12) we obtain that

$$\sigma\left(s^+,t\right) = \sigma\left(s^-,t\right) + \frac{\dot{s}^2}{c^2}\left(\tau_o - \sigma\left(s^-,t\right)\right), \tag{15}$$

which, when substituted into (14), yields

$$\rho\, a^{(s)}(t)(h-s) = \widehat{\sigma} - \left[\sigma\left(s^-,t\right) + \frac{\dot{s}^2}{c^2}\left(\tau_o - \sigma\left(s^-,t\right)\right)\right]. \tag{16}$$

Finally, recalling that $\sigma(s^-,t) = \mu f_y(s^-,t)$ we get the free boundary condition

$$\Big[(h-s)\left(f_y - \frac{\tau_o}{\mu}\right)\ddot{s} + \left((h-s) f_{yy} + \frac{\tau_o}{\mu} - f_y\right)\dot{s}^2 +$$

$$+ 2(h-s) f_{yt}\dot{s} + (h-s) f_{tt} + c^2 f_y\Big]_{y=s^-} = \frac{\widehat{\sigma}}{\rho}, \tag{17}$$

which has to be supplemented with the initial conditions

$$\begin{cases} s(0) = s_o, & 0 \leq s_o \leq h, \\ \dfrac{\dot{s}^2(0)}{c^2}\left(\tau_o - \sigma(s_o^-,0)\right) = [\![\sigma]\!]_{\mathscr{S}}\big|_{t=0} . \end{cases} \tag{18}$$

The second of (18) has been deduced from (15). We remark that when the stress is continuous the coefficient of $\ddot{s}$ in (17) vanishes and accordingly the second of (18) becomes an identity.

2.4. Energy considerations and dissipation

We introduce the total kinetic energy

$$K = \int_0^h \frac{1}{2}\rho v^2 dy,$$

and the total stored energy

$$\Psi = \int_0^h \varphi dy.$$

Recalling that $\varphi = (\mu/2) f_y^2$ in $[0,s]$ and that $\varphi = (\tau_o^2/2\mu)$ in $[s,h]$, we have

$$\Psi = \int_0^s \frac{\mu}{2} f_y^2 dy + \int_s^h \frac{\tau_o^2}{2\mu} dy \quad \text{or} \quad \Psi = \int_0^s \frac{\sigma^2(y,t)}{2\mu} dy + \int_s^h \frac{\tau_o^2}{2\mu} dy. \tag{19}$$

The global energy balance for the system is given by (see [4])

$$\frac{d}{dt}(K+\Psi) = \mathscr{P}^{ext} - \mathscr{P}^{diss}, \tag{20}$$

where $\mathscr{P}^{diss}$ is the global power dissipated and $\mathscr{P}^{ext}$ is the power exerted by external forces, namely

$$\mathscr{P}^{ext} = \widehat{\sigma}\, v(h,t) = \widehat{\sigma}\, v^{(s)}(t).$$

After some straightforward calculation we find that

$$\frac{dK}{dt} = \mathscr{P}^{ext} - \int_0^s \sigma v_y dy - \frac{\rho}{2}[\![v^2]\!]_{\mathscr{S}}\,\dot{s} - [\![\sigma v]\!]_{\mathscr{S}}\,, \tag{21}$$

and

$$\frac{d\Psi}{dt} = \int_0^s \sigma v_y dy - \frac{1}{2\mu}\left(\tau_o^2 - \sigma\left(s^-,t\right)^2\right)\dot{s}\,. \tag{22}$$

Substituting of (21) and (22) into (20) gives

$$\mathscr{P}^{diss} = [\![\sigma v]\!]_{\mathscr{S}} + \frac{\rho\dot{s}}{2}[\![v^2]\!]_{\mathscr{S}} + \frac{\dot{s}}{2\mu}\left(\tau_o^2 - \sigma\left(s^-,t\right)^2\right). \tag{23}$$

The r.h.s. of (23) can be expressed in a more compact form. From (12)

$$[\![\sigma v]\!]_{\mathscr{S}} + \frac{1}{2}\rho\,\dot{s}[\![v^2]\!]_{\mathscr{S}} = \frac{\sigma\left(s^+,t\right) + \sigma\left(s^-,t\right)}{2}[\![v]\!]_{\mathscr{S}},$$

and the r.h.s. of (23) can be rewritten as

$$\frac{\sigma\left(s^+,t\right) + \sigma\left(s^-,t\right)}{2}[\![v]\!]_{\mathscr{S}} + \frac{\dot{s}}{2\mu}\left(\tau_o^2 - \sigma^2(s^-,t)\right).$$

Recalling (15) and (12) we finally get

$$\mathscr{P}^{diss} = \frac{[\![v]\!]_{\mathscr{S}}}{2}\left(\sigma\left(s^+,t\right) - \tau_o\right) = \frac{[\![\sigma]\!]_{\mathscr{S}}}{2\rho\,\dot{s}}\left(\tau_o - \sigma\left(s^+,t\right)\right). \tag{24}$$

So, $\mathscr{P}^{diss} = 0$ if and only if[4] $\tau_o = \sigma\left(s^+,t\right)$. We can prove that $\mathscr{P}^{diss}$ is non–negative. Indeed, introducing the total entropy S, the global form of the Clausius–Duhem inequality for system at uniform constant temperature is

$$\frac{dS}{dt} + \frac{Q}{\theta} \geq 0,$$

with θ absolute temperature and Q total heat exchanged with the exterior. For isothermal problems involving elastic solids $\theta S = E - \Psi$, with E total internal energy. Hence

$$\frac{d\left(S\theta\right)}{dt} + Q = \frac{dE}{dt} - \frac{d\Psi}{dt} + Q = \underbrace{\frac{d\left(E+K\right)}{dt}}_{\mathscr{P}^{ext}-Q} - \underbrace{\frac{d\left(\Psi+K\right)}{dt}}_{\mathscr{P}^{ext}-\mathscr{P}^{diss}} + Q = \mathscr{P}^{diss},$$

[4]Recall that $\tau_o = \sigma\left(s^+,t\right)$ does not imply, in general, $[\![\sigma]\!]_{\mathscr{S}} = 0$, while $[\![\sigma]\!]_{\mathscr{S}} = 0$ implies $\tau_o = \sigma\left(s^+,t\right)$.

where we have used (20) and the first principle of thermodynamics. We therefore conclude that $\mathscr{P}^{diss} \geq 0$ and, from (24), we have

$$\frac{[\![\sigma]\!]_{\mathscr{S}}}{2\rho\,\dot{s}}\left(\tau_o - \sigma\left(s^+,t\right)\right) \geq 0, \tag{25}$$

that requires $\dot{s} < 0$, since $[\![\sigma]\!]_{\mathscr{S}}\left(\tau_o - \sigma\left(s^+,t\right)\right) \leq 0$. Energy dissipation thus occurs only if the fully strained region is expanding. In any region that does not contain the interface $\mathscr{S}$, the local rate of dissipation vanishes. Such a result (consistent with our constitutive procedure) does not apply to any domain containing $\mathscr{S}$, since in this case we must account for entropy production due to the motion of $\mathscr{S}$. We have proved the following

Proposition 1. The dissipated energy $\mathscr{P}^{diss}$ is given by (24) and is necessarily non-negative. $\mathscr{P}^{diss} > 0$ is compatible only with $\dot{s} < 0$ and $\sigma(s^+,t) > \tau_o$ (a regression of the fully strained region cannot lead to dissipation and requires $\sigma(s^+,t) = \tau_o$).

2.5. The free boundary problem

In summary the free boundary problem in the deformable region is given by

$$\begin{cases}
f_{tt} - c^2 f_{yy} = 0, & 0 < y < s\left(t\right),\ t > 0, \\
f\left(0,t\right) = 0, & t > 0, \\
f_y\left(s\left(t\right),t\right) = \dfrac{\sigma\left(s^-,t\right)}{\mu}, \quad \text{with} \quad \sigma\left(s^-,t\right) \leq \tau_o, & t > 0, \\
\Big[\left(h-s\right)\left(f_y - \dfrac{\tau_o}{\mu}\right)\ddot{s} + \left(\left(h-s\right)f_{yy} + \dfrac{\tau_o}{\mu} - f_y\right)\dot{s}^2 + & \\
+\,2\left(h-s\right)f_{yt}\dot{s} + \left(h-s\right)f_{tt} + c^2 f_y\Big]_{y=s^-} = \dfrac{\widehat{\sigma}}{\rho}, & t > 0, \\
f\left(y,0\right) = f_o(y), & 0 \leq y \leq s_o, \\
f_t\left(y,0\right) = f_1\left(y\right), & 0 \leq y \leq s_o, \\
s(0) = s_o, & 0 \leq s_o \leq h, \\
\dfrac{\dot{s}^2\left(0\right)}{c^2}\left(\tau_o - \sigma\left(s_o^-,0\right)\right) = [\![\sigma]\!]_{\mathscr{S}}\big|_{t=0}\,, &
\end{cases} \tag{26}$$

In order to render the problem (26) meaningful, we need to specify $\sigma(s^-,t)$. This difficulty can be overcome by considering the following facts:

- When $|\dot{s}| \geq c$ the solution $f(y,t)$ can be obtained via d'Alembert representation formula. In this case $\sigma(s^-,t)$ is given by $\mu f_y(s,t)$ and the corresponding equality is actually not a condition and can be removed from (26).

- When $|\dot{s}| < c$, we have $[\![\sigma]\!]_{\mathscr{S}} = 0$ (see Proposition 2 below), then, from (15), $\sigma(s^-,t) = \tau_o$.

The following proposition plays a basic role in the theory relating the continuity or discontinuity of σ with the sub- or supersonic behaviour of the interface.

Proposition 2. Let (f,s,σ) be a solution of problem (26) and $\widehat{\sigma}(t) > \tau_o$, for some $t \geq 0$, then:

1. If $[\![\sigma]\!]_{\mathscr{S}} = 0$, i.e. $\sigma(s^-,t) = \sigma(s^+,t) = \tau_o$, then relation (15) is identically satisfied irrespective of $\dot{s}$.

2. If $[\![\sigma]\!]_{\mathscr{S}} > 0$, then $[\![v]\!]_{\mathscr{S}} \neq 0$ (if $\dot{s} \neq 0$) and $\tau_o > \sigma(s^-,t)$. The necessary physical requirement $\sigma(s^+,t) \geq \tau_o$ leads to
$$|\dot{s}| \geq c.$$

Conversely

3. If $|\dot{s}| < c$, then $[\![\sigma]\!]_{\mathscr{S}} = 0$.

4. If $|\dot{s}| = c$, then $\sigma(s^+,t) = \tau_o$ (and thus $[\![\sigma]\!]_{\mathscr{S}} \geq 0$).

5. If $|\dot{s}| > c$, then either $\sigma(s^-,t) = \tau_o = \sigma(s^+,t)$ (i.e. $[\![\sigma]\!]_{\mathscr{S}} = 0$) or $\sigma(s^-,t)$ $< \tau_o < \sigma(s^+,t)$.

Proof. Equation (15) may be rewritten as

$$\sigma\left(s^+,t\right) - \tau_o = \left(\tau_o - \sigma\left(s^-,t\right)\right)\left(\frac{\dot{s}^2}{c^2} - 1\right), \tag{27}$$

from which all the results 1-5 follow immediately.

We shall see that the case $[\![\sigma]\!]_{\mathscr{S}} = 0$ occurs for very peculiar initial data of problem (26), while $[\![\sigma]\!]_{\mathscr{S}} > 0$ is physically more interesting.

2.6. The non-dimensional mathematical formulation

Problem (26) can be rewritten for the unknown

$$w(y,t) = f_y(y,t), \tag{28}$$

(i.e. for $\sigma = \mu w$) whose inverse is given by

$$f(y,t) = \int_0^y w(\xi,t)d\xi. \tag{29}$$

Introducing $w_o(y) = f_o'(y)$, $w_1(y) = f_1'(y)$ and the rescaling $y = h\tilde{y}$, $t = (h/c)\tilde{t}$, problem (26) is transformed to

$$\begin{cases} \tilde{w}_{\tilde{t}\tilde{t}} - \tilde{w}_{\tilde{y}\tilde{y}} = 0, & 0 < \tilde{y} < \tilde{s}(\tilde{t}), \tilde{t} > 0, \\ \tilde{w}_{\tilde{y}}(0,\tilde{t}) = 0, & \tilde{t} > 0, \\ \tilde{w}(\tilde{y},0) = \tilde{w}_o(\tilde{y}), & 0 < \tilde{y} < \tilde{s}_o, \\ \tilde{w}(\tilde{s},\tilde{t}) = \dfrac{\sigma\left(\tilde{s}^-,\tilde{t}\right)}{\mu}, \text{ with } \sigma\left(\tilde{s}^-,\tilde{t}\right) \le \tau_o, & \tilde{t} > 0, \\ \Big[(1-\tilde{s})\left(\tilde{w} - \dfrac{\tau_o}{\mu}\right)\ddot{\tilde{s}} + \left((1-s)\tilde{w}_{\tilde{y}} + \dfrac{\tau_o}{\mu} - \tilde{w}\right)\dot{\tilde{s}}^2 + & \\ +2(1-\tilde{s})\tilde{w}_{\tilde{t}}\dot{\tilde{s}} + (1-\tilde{s})\tilde{w}_{\tilde{y}} + \tilde{w}\Big]_{\tilde{y}=\tilde{s}^-} = \dfrac{\widehat{\sigma}}{\mu}, & \tilde{t} > 0, \\ \tilde{s}(0) = \tilde{s}_o, & 0 \le \tilde{s}_o \le 1, \\ \dot{\tilde{s}}^2(0)\left(\tau_o - \sigma\left(\tilde{s}_o^-,0\right)\right) = [\![\sigma]\!]_{\mathscr{S}}|_{\tilde{t}=0}\,. & \end{cases} \tag{30}$$

where

$$\tilde{w}(\tilde{y},\tilde{t}) = w\left(h\tilde{y}, \frac{h}{c}\tilde{t}\right), \ \tilde{w}_o(\tilde{y}) = w_o(h\tilde{y}),$$

$$\tilde{w}_1(\tilde{y}) = \frac{h}{c} w_1(h\tilde{y}), \ \tilde{s}(\tilde{t}) = \frac{s(t)}{h}, \ \tilde{s}_o = \frac{s_o}{h}.$$

For the case $[\![\sigma]\!]_{\mathscr{S}} \equiv 0$ $(30)_4$, $(30)_5$ must be substituted by

$$\begin{cases} \tilde{w}(\tilde{s},\tilde{t}) = \dfrac{\tau_o}{\mu}, & \tilde{t} > 0,, \\ (1-\tilde{s})\left[\tilde{w}_{\tilde{t}}\dot{\tilde{s}} + \tilde{w}_{\tilde{y}}\right]_{\tilde{y}=\tilde{s}^-} = \dfrac{\widehat{\sigma} - \tau_o}{\mu}, & \tilde{t} > 0, \end{cases} \tag{31}$$

and the initial datum for $\dot{s}$ is no longer required. Here and in the sequel, to simplify the notation, we will omit the "tildas". Note that in the rescaled variables the slope of the characteristic is ± 1.

3. Analytical results

3.1. The case $[\![\sigma]\!]_{\mathscr{S}} \equiv 0$

We consider

$$T < \min\{s_o, 1 - s_o\}. \tag{32}$$

and we assume the following hypotheses:

H.1 $0 < s_o < 1$ and $[\![v]\!]_{\mathscr{S}}|_{t=o} = 0$, which implies $[\![\sigma]\!]_{\mathscr{S}}|_{t=o} = 0$.

H.2 Extending the initial data $w_o(y)$ and $w_1(y)$ as even functions in $[-s_o, 0]$, there exist two positive constants W_1 and W_2, such that

$$W_1 \leq w_o^{'}(y) - w_1(y) \leq W_2, \text{ for all } y \in [-s_o, s_o].$$

H.3 $w_o(y)$ and $w_1(y)$ are such that $w_o(s_o) = (\tau_o/\mu)$, $w_o^{'}(s_o) = 0$, $w_o^{'}(0) = 0$. [5].

H.4 Setting $K(t) = (\widehat{\sigma}(t) - \tau_o)/(\mu)$ we assume that

$$\frac{\sup_{t\in[0,T]} K(t)}{2(1-s_o)W_1} < 1, \qquad \frac{\inf_{t\in[0,T]} K(t)}{W_2} > 1 \tag{33}$$

requiring the compatibility condition $W_2 < 2(1-s_o)W_1$.

Let us suppose that the problem defined through (30) with $(30)_4$, $(30)_5$ replaced by (31) has a solution and that $|\dot{s}(t)| < 1$ for $t \in [0,T]$ (a subsonic free boundary). We consider the domains

$$\begin{aligned} D_{s,T} &= \left\{(y,t) \in \mathbb{R}^2 : 0 < y < s(t),\, 0 < t < T\right\}, \\ D_{s,T}^{(I)} &= \{(y,t) \in D_{s,T} : 0 < y < s_o - t,\, 0 < t < T\}, \\ D_{s,T}^{(II)} &= \{(y,t) \in D_{s,T} : s_o - t < y < s(t),\, 0 < t < T\}, \end{aligned}$$

Following ([5]) w can be expressed as

$$w(y,t) = \begin{cases} \dfrac{1}{2}\left[w_o(y-t) + w_o(y+t)\right] + \dfrac{1}{2}\displaystyle\int_{y-t}^{y+t} w_1(\xi)\,d\xi, & \text{if } (y,t) \in D_{s,T}^{(I)}, \\[2ex] \dfrac{\tau_o}{\mu} + \dfrac{1}{2}\left[w_o(y-t) - w_o(s(t^*) - t^*)\right] + \\ +\dfrac{1}{2}\displaystyle\int_{y-t}^{s(t^*)-t^*} w_1(\xi)\,d\xi, & \text{if } (y,t) \in D_{s,T}^{(II)} \end{cases} \tag{34}$$

with $t^* = t^*(y,t)$ solution of $s(t^*) + t^* = y + t$. In practice $t^*(y,t) < t$ is the time at which the characteristic coming from (y,t), with slope -1, meets the free boundary. The condition $|\dot{s}(t)| < 1$, $\forall t \in [0,T]$, guarantees the existence and uniqueness of t^*. From (34) the derivatives w_y, w_t, w_{yy} and w_{tt} can be explicitly computed. Of course their expressions involve $\dot{s}$ and $\ddot{s}$. In particular, computing w_t, w_y from (34), one can prove that $(31)_2$ can be rewritten as

$$\dot{s}(t) = 1 - \frac{K(t)}{(1-s)\left[w_o^{'}(s-t) - w_1(s-t)\right]} \tag{35}$$

[5]Notice that under these hypotheses $W_1 \leq -w_1(0) \leq W_2$, $W_1 \leq -w_1(s_o) \leq W_2$. In particular, the former, because $f = 0$ on $y = 0$, implies that the initial velocity is negative in a neighborhood of $y = 0$

Remark 1. If we evaluate the derivative of w along the characteristics Σ_α : $y+t=\alpha$, with $\alpha \geq 0$, we get $\left(w|_{\Sigma_\alpha}\right)_t < 0$. Recalling that $w_o(s_o) = (\tau_o/\mu)$ we conclude that w remains below τ_o/μ in the domain $D_{s,T}$.

The following proposition can be proved (see [2])

Proposition 3. Let H.1–H.4 hold true. Then a time θ can be computed such that a unique solution (w,s,σ) to problem (30) with $(30)_4$, $(30)_5$ replaced by (31) exists for $t \in [0,\theta)$, with the property $-1 < \dot{s} < 0$ (hence $[\![\sigma]\!]_{\mathscr{S}} \equiv 0$).

The above proposition provides a sufficient condition for the local existence of a decreasing subsonic interface. The condition $|\dot{s}| < 1$ is guaranteed by $K > 0$ and assumption H.2. A particular case is the one in which $s(t)$ is stationary, i.e. $s(t) = \widehat{s}$ with $\widehat{s} \in (0,1)$. This occurs if

$$w_o^{'}(y) - w_1(y) = \beta \;\Leftrightarrow\; hf_o^{''}(y) - \frac{h}{c} f_1^{'}(y) = \beta, \;\; \forall\, y \in [0,\widehat{s}], \tag{36}$$

with β positive constant and $\widehat{s} = 1 - K\beta^{-1} < 1$, $K = (\widehat{\sigma} - \tau_o)/\mu$ with $\widehat{\sigma}$ constant and greater than τ_o. For $t \in \left[0,\widehat{t}\right]$, with $\widehat{t} \leq \widehat{s}$, (36) guarantees $s(t) = \widehat{s}$. At time $\widehat{s}$

$$w(y,\widehat{s}) = \frac{\tau_o}{\mu} + \int_{-(\widehat{s}-y)}^{\widehat{s}-y} w_1(\xi)\, d\xi, \;\; \forall\, y \in [0,\widehat{s}].$$

So, depending on the specific form of w_1, w may exceed the threshold τ_o/μ, that is the system has became fully strained before $\widehat{s}$. On the contrary, i.e. if $w(y,t) < \tau_o/\mu$, $\forall\ (y,t) \in [0,\widehat{s}) \times [0,\widehat{s}]$ (this happens if, for instance, w_1 is negative), we may analyze the problem for subsequent times (i.e. for $t \in [\widehat{s}, 2\widehat{s}]$), considering the new initial data $w(y,\widehat{s})$, and $w_t(y,\widehat{s})$. These data reproduce a stationary interface if

$$w_y(y,\widehat{s}) - w_t(y,\widehat{s}) = \beta, \;\; \forall\, y \in [-\widehat{s},\widehat{s}], \tag{37}$$

where one can see that

$$w_y(y,\widehat{s}) - w_t(y,\widehat{s}) = w_o'(-(\widehat{s}-y)) - w_1(-(\widehat{s}-y)) = -w_o'(\widehat{s}-y) - w_1(\widehat{s}-y)$$

since w_o' is an odd function. We conclude that (37) is fulfilled if

$$w_o'(y) + w_o'(\widehat{s}-y) = w_1(y) - w_1(\widehat{s}-y), \;\; \forall\, y \in [-\widehat{s},\widehat{s}]. \tag{38}$$

The extension of the stationary solution for later time is not trivial, since once again we have to ensure that w does not exceed the threshold τ_o/μ within the deformable region. However, the stationary solution $y = \widehat{s}$ cannot be maintained by the system for an infinite time. Indeed evaluating the acceleration of the fully strained region (which is equal to the one for the deformable region evaluated on $\mathscr{S}$) we have $a^{(s)}(t) = \beta > 0$. We thus deduce that the whole system becomes fully strained in a finite time.

We remark that the solutions obtained under the restriction $[\![\sigma]\!]_{\mathscr{S}} = 0$ require data which look rather artificial. Natural physical situations refer mostly to the case $[\![\sigma]\!]_{\mathscr{S}} \neq 0$.

3.2. The case $[\![\sigma]\!]_{\mathscr{S}} \neq 0$

Let us suppose that the system is initially at rest, that is $f_o(y) \equiv f_1(y) \equiv 0$. We consider two cases: either the stress $\widehat{\sigma}$ increases in time, i.e. $\widehat{\sigma}'(t) > 0$ from $\widehat{\sigma}(0) = 0$, reaching the threshold τ_o at some time $t_o < 1$, or $\widehat{\sigma}$ is greater than the threshold from the beginning. In both cases the stress σ exhibits jump across $\mathscr{S}$, with the interface traveling faster than the speed of sound. For $0 \leq t \leq t_o$, the dynamics is given by the following problem (already written in dimensionless form)

$$\begin{cases} f_{tt} - f_{yy} = 0, & 0 < y < 1,\ 0 \leq t < t_o, \\ f(0,t) = 0, & 0 \leq t < t_o, \\ f_y(1,t) = \dfrac{\widehat{\sigma}(t)}{\mu}, & 0 \leq t < t_o, \\ f(y,0) = 0, & 0 \leq y \leq 1, \\ f_t(y,0) = 0, & 0 \leq y \leq 1, \end{cases}$$

whose solution has the explicit expression

$$f(y,t) = \begin{cases} 0, & \text{if } (y,t) \in D^{(I)} \cap \{t < t_o\}, \\ \\ \dfrac{1}{\mu} \int_{1-t}^{y} \widehat{\sigma}(\xi + t - 1)\, d\xi, & \text{if } (y,t) \in D^{(II)} \cap \{t < t_o\}, \end{cases} \tag{39}$$

where

$$D^{(I)} = \{0 < y < 1, 0 < t < 1 - y\}, \tag{40}$$

$$D^{(II)} = \{1 - t < y < 1, 0 < t < 1\}. \tag{41}$$

At time t_o we have that $w_y(y,t_o) - w_t(y,t_o) \equiv 0$.

Proposition 4. Let (f, σ, s) be a solution for $t > t_o$. Then $s(t) < 1 - (t - t_o)$.

Proof. Let us first assume that, for $t \geq t_o$, the curve $\sigma = \tau_o$ is the characteristic Σ_{1+t_o} : $y + t = 1 + t_o$,. Exploiting (39) we can compute $w(y,t)$ for $t > t_o$ and conclude that $w(1+t_o-t,t)=\tau_o/\mu$, implying that $\sigma(s^-,t)=\tau_o$. Thus $[\![\sigma]\!]_{\mathscr{S}}=0$ and $[\![v]\!]_{\mathscr{S}}=0$. Therefore

$$\left.\frac{\partial f}{\partial t}\right|_{\Sigma}=v^{(s)}(t)=\widehat{\sigma}(t_o),\quad \forall t>t_o \;\Rightarrow\; a^{(s)}(t)=\frac{dv^{(s)}(t)}{dt}=0. \tag{42}$$

At this point (14) leads to a contradiction, unless $\widehat{\sigma}(t)=\tau_o$, $\forall t\geq t_o$. Thus s cannot coincide with Σ_{1+t_o}.

Next, suppose that $\mathscr{S}$ is located to the right of Σ_{1+t_o}, namely

$$s(t)>1-(t-t_o),\; t\geq t_o.$$

We consider the domain $D_\delta=\{1-(t-t_o)<y<s(t),\; t_o<t<t_o+\delta\}$, for some positive (and "small") δ and the following Goursat problem

$$\begin{cases} w_{tt}-w_{yy}=0, & (y,t)\in D_\delta,\\[2mm] w|_{\Sigma}=\dfrac{\tau_o}{\mu}, & t_o\leq t<t_o+\delta,\\[2mm] w|_{\mathscr{S}}=\dfrac{\tau_o}{\mu}, & t_o\leq t<t_o+\delta, \end{cases}$$

whose unique solution is $w=\tau_o/\mu$, $\forall\,(y,t)\in D_\delta$. Hence $a^{(s)}(t)=0$, $\forall\,t\geq t_o$. Moreover $\sigma(s^-,t)=\tau_o$, and from Proposition 2, $\sigma(s^+,t)=\tau_o$ for those t such that $\dot{s}(t)>-1$. Thus

$$\underbrace{(1-s)\,a^{(s)}(t)}_{=0}\;=\;\frac{\widehat{\sigma}(t)-\tau_o}{\mu},$$

which is an evident contradiction. The same argument shows that in a neighborhood of $t=t_o$ there cannot be any intersection between Σ_{1+t_o} and the characteristic. We conclude that $s(t)<1-(t-t_o)$, $\forall\,t>t_o$ (supersonic interface).

□

As a consequence of Proposition 4, $\sigma(s^-,t)$ can be computed in term of $f(y,t_o)$ and the unknown $s(t)$, by using d'Alembert formula.

4. Computing some explicit solutions for some particular applied shear $\widehat{\sigma}$

We confine ourselves to three specific cases for which we can provide the explicit solutions. The problem with generic data remains open.

4.0.1. The applied load increases linearly in time

This is the case when

$$\widehat{\sigma} = mt, \text{ with } m = \frac{\tau_o}{t_o}, \quad t_o < 1. \tag{43}$$

By virtue of (39), the solution of problem (26) is

$$f(y,t) = \frac{m}{2\mu}(y+t-1)^2, \quad t_o < t, \ 0 < y < s(t). \tag{44}$$

We have

$$a^{(s)}(t) = \frac{m}{\mu}(\dot{s}+1)^2 + \frac{\ddot{s}}{\mu}\left[m(s+t-1) - \tau_o\right],$$

and $(26)_4$ can be rewritten as

$$(1-s)\left\{m(\dot{s}+1)^2 + \ddot{s}\left[m(s+t-1) - \tau_o\right]\right\} =$$

$$= mt - \left[m(s+t-1)\left(1-\dot{s}^2\right) + \dot{s}^2\tau_o\right]. \tag{45}$$

Looking for a solution to (45) of the form $s(t) = 1 - \alpha(t-t_o)$, with $\alpha > 1$, we get

$$m\alpha(t-t_o)(1-\alpha)^2 = m\alpha(t-t_o)\left(\alpha+1-\alpha^2\right), \ \Rightarrow \ \alpha = \frac{3}{2}. \tag{46}$$

For any m, the free boundary $\mathscr{S}$ and the jump $[\![\sigma]\!]_{\mathscr{S}}$ are

$$s(t) = 1 - \frac{3}{2}(t-t_o), \qquad [\![\sigma]\!]_{\mathscr{S}} = \frac{9}{8}m(t-t_o). \tag{47}$$

Moreover, from (11), (24) and (47), the following expression for $\mathscr{P}^{diss}$ (normalized by $\tau_o^2/\rho c$) can be obtained

$$\mathscr{P}^{diss} = \left(\frac{3}{8}\right)^2\left(\frac{t}{t_o} - 1\right)^2,$$

4.0.2. Linear growth of $\widehat{\sigma}$, with $\widehat{\sigma} = \tau_o$ for $t \geq t_o$

This is the case when

$$\widehat{\sigma} = \begin{cases} mt, \ \text{with } m = \dfrac{\tau_o}{t_o}, & 0 \leq t < t_o < 1, \\ \tau_o, & t \geq t_o. \end{cases} \tag{48}$$

Using the same argument as that for the previous case we get

$$m\alpha\,(t-t_o)\,(1-\alpha)^2 = m\,(t-t_o)\left[\alpha\left(\alpha+1-\alpha^2\right)-1\right]$$
$$\Downarrow$$
$$(2\alpha+1)\,(\alpha-1)^2 = 0,$$

whose only meaningful solution is $\alpha = 1$. In such a case $\mathscr{S}$ coincides with the characteristic Σ_{1+t_o}, and $[\![\sigma]\!]_{\mathscr{S}} = 0$, implying $\mathscr{P}^{diss} = 0$.

4.0.3. Constant load greater than the threshold

This is the case when $\widehat{\sigma}$ is constant in time and greater than τ_o. We have $t_o = 0$ and $f(y,t) \equiv 0$ in the domain $\{0 < y < s(t),\ 0 < t < 1\}$. Thus $a^{(s)}(t) = -\dfrac{\tau_o}{\mu}\ddot{s}$ and the problem for s becomes

$$\begin{cases} -(1-s)\ \tau_o\ddot{s} = \widehat{\sigma} - \dot{s}^2\tau_o, \\ s(0) = 1, \\ \dot{s}^2(0) = \dfrac{\widehat{\sigma}}{\tau_o}. \end{cases} \quad . \tag{49}$$

Setting $\widehat{\sigma} = \gamma^2\tau_o$, with $\gamma^2 > 1$, the solution turns out to be $s(t) = 1 - \gamma t$ (again a supersonic interface). Concerning dissipation (still normalized to $\tau_o^2/\rho c$), we have

$$\mathscr{P}^{diss} = \frac{\gamma\left(\gamma^2-1\right)}{2},$$

which corresponds to half of the power supplied to the system.

5. Comparison with a hyperelastic model

Here we want to show that the dynamics of the fully strained model cannot be obtained as a limit case of a piece-wise hyperelastic model[6]. This is in agreement with the fact that in general our model is dissipative, whereas any hyperelastic model is not. We consider the following stress–strain relation, depending on the parameter λ

$$\sigma = \begin{cases} \mu\,\varepsilon, & 0 \le \varepsilon \le \dfrac{\tau_o}{\mu}, \\ \mu\lambda^2\left(\varepsilon - \dfrac{\tau_o}{\mu}\right) + \tau_o, & \varepsilon > \dfrac{\tau_o}{\mu}, \end{cases} \quad \text{with } \lambda^2 > 1, \tag{50}$$

[6]For the sake of simplicity here we consider a piecewise linear hyperelastic model.

where $\varepsilon = f_y$ and whose inverse is

$$\varepsilon = \begin{cases} \dfrac{\sigma}{\mu}, & 0 \le \sigma \le \tau_o, \\ \\ \dfrac{\sigma - \tau_o}{\mu\lambda^2} + \dfrac{\tau_o}{\mu}, & \sigma > \tau_o. \end{cases} \tag{51}$$

Model (50) is hyperelastic and the stress can be derived from the following elastic energy

$$\psi(\varepsilon) = \begin{cases} \dfrac{\mu}{2}\varepsilon^2, & 0 \le \varepsilon \le \varepsilon_o, \\ \\ \dfrac{\mu\lambda^2}{2}(\varepsilon - \varepsilon_o)^2 + \dfrac{\mu}{2}\left(2\varepsilon\,\varepsilon_o - \varepsilon_o^2\right), & \varepsilon > \varepsilon_o, \end{cases} \quad \text{with } \varepsilon_o = \frac{\tau_o}{\mu}, \tag{52}$$

which, in terms of σ, can be rewritten as

$$\psi(\sigma) = \begin{cases} \dfrac{\sigma^2}{2\mu}, & 0 \le \sigma \le \tau_o, \\ \\ \dfrac{(\sigma - \tau_o)^2}{2\mu\lambda^2} + \dfrac{\tau_o}{\mu\lambda^2}(\sigma - \tau_o) + \dfrac{\tau_o^2}{2\mu}, & \sigma > \tau_o. \end{cases} \tag{53}$$

Once again considering the problem of a layer subject to a known shear $\widehat{\sigma}$ applied on the top surface, we divide the domain $\{0 < y < 1,\ t > t_o\}$ into two sub–domains $D_1 = \{0 < y < s(t),\ t > t_o\}$ and $D_2 = \{s(t) < y < 1,\ t > t_o\}$, separated by the curve $y = s(t)$, still denoted by $\mathscr{S}$. In D_1 the governing equation in dimensionless form is $f_{tt} - f_{yy} = 0$. In D_2 the governing equation is $f_{tt} - \lambda^2 f_{yy} = 0$. Besides the relations

$$\text{kinematic} \quad : \quad f\left(s^-, t\right) = f\left(s^+, t\right), \ \Rightarrow [\![v]\!]_{\mathscr{S}} = -\dot{s}[\![f_y]\!]_{\mathscr{S}}, \tag{54}$$

$$\text{dynamic} \quad : \quad \dot{s}[\![v]\!]_{\mathscr{S}} = -\frac{1}{\mu}[\![\sigma]\!]_{\mathscr{S}}, \tag{55}$$

now we also have energy conservation across $\mathscr{S}$

$$\dot{s}\left(\frac{1}{2}[\![v^2]\!]_{\mathscr{S}} + \frac{1}{\mu}[\![\psi]\!]_{\mathscr{S}}\right) = -\frac{1}{\mu}[\![\sigma v]\!]_{\mathscr{S}}. \tag{56}$$

The latter implies that the stress is continuous across $s(t)$. Indeed using (54) and (55), we get from (56)

$$[\![\psi]\!]_{\mathscr{S}} = \left(\frac{\sigma(s^+, t) + \sigma(s^-, t)}{2}\right)[\![f_y]\!]_{\mathscr{S}}. \tag{57}$$

From (51) and (53) we find that

$$[\![f_y]\!]_{\mathscr{S}} = \frac{\sigma(s^+,t)-\tau_o}{\mu\lambda^2} + \frac{\tau_o - \sigma(s^-,t)}{\mu}, \tag{58}$$

$$[\![\psi]\!]_{\mathscr{S}} = \frac{\sigma^2(s^+,t)-\tau_o^2}{2\mu\lambda^2} - \frac{\sigma^2(s^-,t)-\tau_o^2}{2\mu} \tag{59}$$

Substituting (58) and (59) into (57) we get

$$(\tau_o - \sigma(s^-,t))(\sigma(s^+,t)-\tau_o)(1-\lambda^2) = 0. \tag{60}$$

Thus either $\sigma(s^-,t) = \tau_o$ or $\sigma(s^+,t) = \tau_o$. From (54) and (55) $\dot{s}^2\mu[\![f_y]\!]_{\mathscr{S}} = [\![\sigma]\!]_{\mathscr{S}}$, that is

$$\dot{s}^2(\sigma(s^+,t)-\tau_o) + \lambda^2\dot{s}^2(\tau_o - \sigma(s^-,t)) = \lambda^2(\sigma(s^+,t)-\sigma(s^-,t)). \tag{61}$$

If $\sigma(s^+,t) = \tau_o$, then, from (61),

$$\dot{s}^2(\tau_o - \sigma(s^-,t)) = (\tau_o - \sigma(s^-,t)),$$

implying $\sigma(s^-,t) = \tau_o$ or $\dot{s}^2 = 1$. It is easy to check that $\dot{s}^2 = 1 \Longleftrightarrow \sigma(s^-,t) = \tau_o$. On the other hand, if $\sigma(s^-,t) = \tau_o$ (and consequently $\dot{s}^2 = 1$), then, from (61)

$$(\sigma(s^+,t)-\tau_o) = \lambda^2(\sigma(s^+,t)-\tau_o)),$$

which yields $\sigma(s^+,t) = \tau_o$, since $\lambda^2 > 1$. Thus we conclude that in the bi-elastic model $s(t) = 1-(t-t_o)$.

We now prove that the fully strained model cannot be obtained taking the limit $\lambda \to \infty$. For the sake of simplicity we assume $\widehat{\sigma}$ as in (43). In D_1 the displacement f is given by (44), while in D_2 it is given by

$$f(y,t) = f(s^-,t) + \int_{s(t)}^{y} w(\xi,t)\,d\xi = \frac{mt_o^2}{2\mu} + \int_{s(t)}^{y} w(\xi,t)\,d\xi,$$

where w solution of the Goursat problem

$$\begin{cases} w_{tt} - \lambda^2 w_{yy} = 0, & 1-(t-t_o) < y < 1,\ t_o < t, \\ w(\Sigma_{1+t_o}) = \dfrac{\tau_o}{\mu}, & t_o < t, \\ w(1,t) = \dfrac{1}{\mu\lambda^2}(mt-\tau_o) + \dfrac{\tau_o}{\mu}, & t_o < t, \end{cases} \tag{62}$$

i.e.

$$w(y,t) = \frac{\tau_o}{\mu} - \frac{m}{\mu\lambda^2}(1-y-t+t_o),$$

Let us now consider the limit $\lambda \to +\infty$. We have:

- $\mathscr{S}$ coincides with Σ_{1+t_o}, so that $\sigma(s^-,t) = \tau_o$.
- In D_1, f is still given by (44).
- In D_2, $w(y,t) \longrightarrow \dfrac{\tau_o}{\mu}$ and $\sigma(y,t) = \tau_o - m(1-y-t+t_o)$, i.e. the stress solves

$$\begin{cases} \sigma_{yy} = 0, \\ \sigma(s^+,t) = \tau_o, \\ \sigma(1,t) = mt. \end{cases}$$

In such a limit the dynamics breaks down. Indeed, $[\![\sigma]\!]_{\mathscr{S}} = 0$ entails, by virtue of (55), $[\![v]\!]_{\mathscr{S}} = 0$, i.e. $v^{(s)} = (mt_o/\mu) = \tau_o/\mu$. On the other hand, $v^{(s)}(t)$ is obtained solving the equation of motion

$$a^{(s)} = \frac{\sigma_y}{\mu}, \;\Rightarrow\; v^{(s)}(t) = v^{(s)}(0) + \frac{m}{\mu}(t-t_o),$$

which contradicts $v^{(s)} = \dfrac{\tau_o}{\mu}$.

The reason why the model (51) fails to represent the response of the fully strained model in the limit $\lambda \to +\infty$ lies essentially in the fact that the limit tends to preserve its non-dissipative character. Indeed it is easy to check that equation (56) implies $\mathscr{P}^{diss} = 0$ for any λ (details are given in [2]).

6. Open problems

As we said, the problem for a generic load $\widehat{\sigma}$ with $[\![\sigma]\!]_{\mathscr{S}} \neq 0$ is open. It would also be interesting to inquire the possibility of introducing dissipation in the sequence of models in such a way to approach the solution of the problem with strain threshold (this could also be a constructive tool to prove existence). We stress the fact that the present contribution is a first step in the investigation of the dynamics of a number of materials possessing constitutive laws of the implicit type (e.g. materials with yield stress, self blocking materials, etc.), which the authors want to pursue in the next future.

REFERENCES

[1] Y.C. Fung, *Biomechanics: Mechanical properties of living tissue*, Springer-Verlag 1993.

[2] A. Farina - A. Fasano - L. Fusi - K.R. Rajagopal, *Modelling Materials With a Stretching Threshold*, submitted to M3AS.

[3] K.R. Rajagopal, *On implicit constitutive theories for fluids*, J. Fluid Mech. **550** (2006), 243-249.

[4] J. Coirier, *Mécanique des milieux continus*, Dunod 1997.

[5] L. Fusi - A. Farina, *An Extension of the Bingham model to the case of an elastic core*, Adv. Math. Sci. Appl. **13** (2003), 113-163.

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