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FUNCTIONAL RELATIONS INVOLVING GENERALIZED H-FUNCTION

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A number of papers have appeared in literature in which certain functional relations associated with hypergeometric functions and Digamma function $\Psi(z)$ are derived. In order to unify and extend the existing results due to Kalla [7]. Kalla and Ross [9], Al-Saqabi and Kalla [1], Nishimoto and Saxena [12], Srivastava and Nishimoto [17] and Pandey and Srivastava [14] etc., the author establishes two functional relations between $\Psi(z)$ and generalized H-function due to Inayat-Hussain [6]. The results obtained are of general character and provide extension and unification of functional relations of various hypergeometric functions available in literature. Existence conditions and computable representation of the $\overline{H}$-function are also investigated.

1. Introduction and preliminaries.

In an attempt to evaluate certain Feynman integrals in two different ways which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain ([6], p. 4126) introduced a

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generalization of Fox's H-function in the form:

$$\overline{H}(z) = \overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[z \left| \begin{matrix} (\alpha_j, A_j, a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j, b_j)_{m+1,q} \end{matrix} \right. \right] = \tag{1.1}$$
$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \circleddash(s) z^s ds,$$

where

$$\circleddash(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} \left\{\Gamma(1 - \alpha_j + A_j s)\right\}^{a_j}}{\prod_{j=m+1}^{q} \left\{\Gamma(1 - \beta_j + B_j s)\right\}^{b_j} \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j s)} \tag{1.2}$$

which contains fractional powers of some of the Γ-functions. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; p, q, m and n are integers such that $1 \leq m \leq q$, $0 \leq n \leq p$; $A_j > 0$ $(j = 1, \ldots, p)$, $B_j > 0$ $(j = 1, \ldots, q)$ and $\alpha_j (j = 1, \ldots, p)$ and $\beta_j (j = 1, \ldots, q)$ are complex parameters. The exponents $a_j (j = 1, \ldots, n)$ and $b_j (j = m + 1, \ldots, q)$ can take on non-integer values. The poles of the integrand of (1.1) are assumed to be simple. The contour in (1.1) is presumed to be the imaginary axis $\mathrm{Re}(s) = 0$, which is suitably indented in order to avoid the singularities of the gamma functions and to keep these singularities at appropriate sides.

It has been shown by Buschman and Srivastava ([3], p. 4708) that the sufficient condition for absolute convergence of the contour integral (1.1) is given by

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0. \tag{1.3}$$

This condition provides exponential decay of the integrand in (1.1), and region of absolute convergence of (1.1) is

$$|\arg z| < \frac{1}{2}\pi\Omega. \tag{1.4}$$

For further details about $\overline{H}$-function the reader is referred to the original paper of Buschman and Srivastava [3] and Inayat-Hussain [6].

Abelian theorems and complex inversion theorem for distributional $\overline{H}$-function transformation are established by Saxena and Gupta ([15], [16]).

When the exponents $a_j = b_j = 1$, $\forall i, j$, the $\overline{H}$-function reduces to the familiar Fox's H-function defined by Fox [5]; see also Mathai and Saxena [11]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{matrix}\right] = \tag{1.5}$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} X(s) z^s \, ds,$$

where

$$\chi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j + A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j + B_j s) \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j s)} \tag{1.6}$$

an empty product is interpreted as unity; the integer m, n, p, q satisfy the inequalities $0 \le n \le p$ and $1 \le m \le q$, the coefficients $A_j > 0$ $(j = 1, \ldots, p)$ and $B_j > 0$ $(j = 1, \ldots, q)$ and the complex parameters α_j and β_j are such that the poles of the integrand are simple and $\mathcal{L}$ is a suitable contour of Mellin-Barnes type in complex s-plane separating the poles of $\Gamma(\beta_j - B_j s)$ for $j = 1, \ldots, m$ from those of $\Gamma(1 - \alpha_j + A_j s)$ for $j = 1, \ldots, n$. The integral in (1.1) converges absolutely and defines the H-function, analytic in the sector

$$|\arg z| < \frac{1}{2}\pi\lambda^*, \tag{1.7}$$

where

$$\lambda^* = \sum_{j=1}^{m} B_j - \sum_{j=1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0, \tag{1.8}$$

the point $z = 0$ being tacitly excluded.
A detailed account of the H-function is available from the monograph of Mathai and Saxena [11]. Existence conditions, analytic continuation and asymptotic expansions of the H-function have been discussed by Braaksma [2].

The well-known result in the theory of Digamma function [4], $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ proved by Kalla and Ross [9] by means of Riemann-Liouville fractional differintegral operator, is

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{\Gamma(\lambda+n)} = \Psi(\lambda) - \Psi(\lambda-\nu), \tag{1.9}$$

where $\operatorname{Re}(\lambda-\nu) > 0, \lambda \neq 0, -1, -2.$
(1.9) can be put in the convenient form

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{(\nu)_n}{(\lambda)_n} = \Psi(\lambda) - \Psi(\lambda-\nu);\ \operatorname{Re}(\lambda-\nu) > 0. \tag{1.10}$$

where $(\nu)_n$ denotes the Pochammer's symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1, & (n=0) \\ \nu(\nu+1)\cdots(\nu+n-1), & (n \in N). \end{cases}$$

A number of generalizations of (1.8) have appeared in literature during the last one decade.

A systematic analysis of various infinite series which were evaluated by Riemann-Liouville operator of Fractional Calculus together with the discussion of more general classes of infinite series available in literature with historical details have been presented by Nishimoto and Srivastava [13]. A systematic account of certain family of infinite series whose sums can be expressed in terms of Digamma functions together with their relevant unifications and generalizations have been given by Al-Saqabi et al. [1].

The result (1.10) has been extended to H-function by Nishimoto and Saxena [12] in the form

$$\sum_{K=1}^{\infty}\frac{(\sigma)_K}{K} H^{m,n+1}_{p+1,q+1}\left[ax\,\middle|\,\begin{matrix}(1-\rho,1),(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q),(1-\rho-K,1)\end{matrix}\right] = \tag{1.11}$$

$$= \sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(ax)^{s}\chi(s)}{(\nu)!B_h}\Big[\Psi(\rho+s) - \Psi(\rho-\sigma+s)\Big],$$

where $\operatorname{Re}(\rho + \min \beta_j/B_j) > 0$ for $j = 1, \ldots, m, \lambda^* > 0$, $\operatorname{Re}(\sigma) \geq 0$, $|\arg x| < \frac{1}{2}\pi\lambda^*$, $u^* = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \leq 0$; $\chi(s)$ and λ^* are defined respectively in (1.6) and (1.8).

Generalizations of the result (1.11) have been given by Srivastava and Nishimoto [17] and Pandey and Srivastava [14]. The object of this paper is to derive two new functional relations involving $\overline{H}$-function which provide unification and extension of the functional relations involving special functions of one variable available in literature. The importance of the results obtained in this paper further lies in the fact that besides the special functions which are special cases of the Fox's H-function, the $\overline{H}$-function of Inayat-Hussain also contains the Polylogarithm of a complex order and the exact partition function of the Gaussian Model in Statistical Mechanics.
A relation connecting $L^{\nu}(z)$, the polylogarithm of complex order ν and the $\overline{H}$-function is the following:

$$(1.12) \qquad L^{\nu}(z) = \overline{H}^{1,1:\nu}_{1,2:\nu-1}\left[-z \middle| \begin{matrix} (1,1:\nu) \\ (0,1),(0,1:\nu-1) \end{matrix}\right],$$

which readily follows on comparing their contour integral definitions. An account of $L^{\nu}(z)$, the polylogarithm of complex order ν is available from the book by Marichev [10].

2. Existence conditions for the $\overline{H}$-function.

It can be established by following the procedure adopted by Braaksma ([2], pp. 278–279), that the function $\overline{H}(z)$ makes sense and defines an analytic function of z in the following two cases:

I. $\mu > 0$, and $0 < |z| < \infty$,
where

$$(2.1) \qquad \mu = \sum_{j=1}^{m} |B_j| + \sum_{j=m+1}^{q} |B_j\beta_j| - \sum_{j=1}^{n} |A_j\alpha_j| - \sum_{j=n+1}^{p} |A_j|.$$

II. $\mu = 0$ and $0 < |z| < \tau^{-1}$ holds,

$$(2.2) \qquad \tau = \left\{\prod_{j=1}^{m}(B_j)^{-B_j}\right\}\left\{\prod_{j=1}^{n}(A_j)^{A_j\alpha_j}\right\}\left\{\prod_{j=n+1}^{p}(A_j)^{A_j}\right\}\left\{\prod_{j=m+1}^{q}(B_j)^{-B_j\beta_j}\right\}.$$

3. Computable representation for the $\overline{H}$-function.

By calculating the residues at the poles of $\Gamma(\beta_j - B_j s)$ for $j = 1, \ldots, m$ in (1.1), we obtain the following representation of the $\overline{H}$-function in a computable form as

$$\overline{H}(z) = \overline{H}_{p,q}^{m,n}[z] = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(\nu)!} \frac{\circleddash(\zeta) z^{\zeta}}{B_h}, \tag{3.1}$$

where $\zeta = (\beta_h + \nu)/B_h$. (3.1) exists for $0 < |z| < \infty$, if $\mu < 0$ or $\mu = 0$ and $0 < |z| < \tau^{-1}$, where τ^{-1} is defined in (2.2), μ in (2.1) and $\circleddash(\cdot)$ in (1.2); $B_h(\beta_j + \nu_1) \neq B_j(\beta_h + \nu_2)$ for $j \neq h$; $j, h = 1, \ldots, m$; $\nu_1, \nu_2 = 0, 1, 2, \ldots$.

4. Two functional relations for the $\overline{H}$-function.

The following functional relations will be established here:

(i)

$$\sum_{K=1}^{\infty} \frac{(\sigma)_K}{K} \overline{H}_{p+1,q+1}^{m,n+1} \cdot \left[z \left| \begin{matrix} (1-\rho, C), (\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}, (1-\rho-K, C) \end{matrix} \right. \right] =$$
$$= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \circleddash(\zeta) z^{\zeta}}{(\nu)! B_h} \Big[\Psi(\rho + C\zeta) - \Psi(\rho - \sigma + C\zeta) \Big], \tag{4.1}$$

where $C > 0$, $\mathrm{Re}(\rho - \sigma) > 0$, $\rho \neq 0, -1, -2$; $\mathrm{Re}(\rho + \min \beta_j/B_j) > 0$, $j = 1, \ldots, m$; $\mathrm{Re}(\sigma) > 0$, $|\arg z| < \frac{1}{2}\pi\Omega$, $\Omega > 0$; Ω *is defined in* (1.3)*; the conditions* (2.1) *and* (2.2) *hold*;

$$\zeta = (\beta_h + \nu)/B_h \ (h = 1, \ldots, m;\ \nu \in \mathbb{N}_0). \tag{4.2}$$

(ii)

$$\sum_{K=1}^{\infty} \frac{1}{K} \overline{H}_{p+1,q+1}^{m,n+1} \left[z \left| \begin{matrix} (1-\sigma-K, C), (\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}, (1-\rho-K, C) \end{matrix} \right. \right] = \tag{4.3}$$

$$= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(\nu)!} \frac{\Gamma(\sigma + C\zeta) \circledS (\zeta) z^{\zeta}}{B_h \Gamma(\rho + C\zeta)} \cdot$$

$$\cdot \Big[\Psi(\rho + C\zeta) - \Psi(\rho - \sigma) \Big] \quad (C > 0),$$

where $\mathrm{Re}(\rho - \sigma) > 0$, $(\rho \neq 0, -1, -2, \ldots)$; $|\arg z| < \frac{1}{2}\pi\Omega$, $\Omega > 0$; *and the conditions* (2.1) *and* (2.2) *are satisfied.*

Proof of (4.1). By virtue of the representation (3.1) of the $\overline{H}$-function, the l.h.s. of (4.1) denoted by $\Lambda(z; C)$ can be written as

$$\Lambda(z; C) = \sum_{K=1}^{\infty} \frac{(\sigma)_K}{K} \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \circledS (\zeta) z^{\zeta}}{(\nu)! B_h} \frac{\Gamma(\rho + C\zeta)}{\Gamma(\rho + K + C\zeta)}, \tag{4.4}$$

where $\circledS$ and ζ are respectively defined in (1.2) and (3.1).

If we invert the order of summation in (4.4) which can be shown to be permissible by absolute convergence under the conditions stated with (4.1), we find that

$$\Lambda(z; C) = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \circledS (\zeta) z^{\zeta}}{(\nu)! B_h} \sum_{K=1}^{\infty} \frac{(\sigma)_K}{K(\rho + C\zeta)_K}. \tag{4.5}$$

The K-series can be summed by means of the result (1.10) and it is seen that

$$\Lambda(z; C) = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \circledS (\zeta) z^{\zeta}}{(\nu)! B_h} \Big[\Psi(\rho + C\zeta) - \Psi(\rho - \sigma + C\zeta) \Big]$$

which is same as (4.1).

(4.3) can be established in the same way.

It is interesting to observe that (4.1) can be put in an alternative convenient form:

$$\sum_{K=1}^{\infty} \frac{(\sigma)_K}{K} \overline{H}^{m,n}_{p,q+1} \left[z \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}, (1 - \rho - K, C) \end{matrix} \right] \tag{4.6}$$

$$= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \circledS (\zeta) z^{\zeta}}{(\nu)! B_h \Gamma(\rho + C\zeta)} \Big[\Psi(\rho + C\zeta) - \Psi(\rho - \sigma + C\zeta) \Big]$$

which holds true under the same conditions as mentioned with (4.1).

5. Special cases.

On account of the most general character of the $\overline{H}$-function occurring in the main results of the preceding section, scores of special cases of the results (4.1) and (4.3) can be derived but, for the sake of brevity, a few interesting special cases will be given below:

(i) When $a_i = b_j = 1 \,\forall i$ and j in (4.1). The $\overline{H}$-function reduces to Fox's $\overline{H}$-function and we arrive at the result given by Svrivastava and Nishimoto ([17], p. 71, eq. 15), which itself is a generalization of the result due to Nishimoto and Saxena ([12], p. 135, eq. 10).

(ii) Similarly if we take $a_i = b_j = 1 \,\forall i$ and j in (4.3) then we obtain the result due to Pandey and Svrivastava ([14], p. 163, eq. 3.6).

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