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ON THE HILBERT FUNCTION AND THE MINIMAL FREE RESOLUTION OF THE $GF(q)$-POINTS OF DEL PEZZO SURFACES OF $\mathbb{P}^n$

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Here we study the Hilbert function of the points rational over a fixed finite field $GF(q)$, $q = p^e$ of some Del Pezzo surfaces and of the Veronese plane. This work is motivated by the equivalence (due to Moorhouse) between the knowledge of the Hilbert function of a finite set of a projective space over $GF(q)$ and the p-rank of its incidence matrix.

Introduction.

Let $GF(q), q = p^e$, be a finite field. Let A be the incidence matrix of points and hyperplanes of $PG(n, GF(q))$. Hence A is a square matrix of order $N := (q^{n+1} - 1)/(q - 1)$ whose entries are 0 and 1. Let S be any subset of $PG(n, GF(q))$ and denote by A_S the submatrix of A whose rows are indexed by the points of S. So, if $\mathrm{card}(S) = s$, then A_S is a $s \times N$ matrix. The main problem is to determine the p-rank of A_S, i.e. the rank of A_S in characteristic p. If $S = PG(n, GF(q))$, then Smith ([9]) proved that p-rank $(A_S) = (p + n - 1)!(n!(p - 1)!)$. Recently, G.E. Moorhouse has found a new interesting relation between the p-rank of A_S and the Hilbert function of the homogeneous ideal $I(S)$ of S. In a certain sense "if the Hilbert function of $I(S)$ is known, then the p-rank of A_S is known". For more results and

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details, see [2], [7] and [8]. See also [1] for related questions. In this paper we study the Hilbert function and the minimal free resolution of the rational points of some projective subvarieties (essentially the Del Pezzo surfaces and the Veronese plane) giving a contribution to Moorhouse approach. For any integer t let $\mathbb{O}(t)$ be the degree t line bundle of $\mathbb{P}^n$. For any closed subscheme A of a scheme X and $L \in \mathrm{Pic}\,(X)$, let $\rho(X, L, A) : H^0(X, L) \to H^0(A, L|A)$ be the restriction map. If $A \subset \mathbb{P}^n$ is a finite set we want to find the values of t such that $\rho(\mathbb{P}^n, \mathbb{O}(t), A)$ is injective and the values of t such that $\rho(\mathbb{P}^n, \mathbb{O}(t), A)$ is surjective. For any variety X defined over $GF(q)$, let $X(q)$ be the set of all points of X rational over $GF(q)$. We are interested mainly in projectively normal schemes $X \subseteq \mathbb{P}^n$, i.e. in schemes with $\rho(\mathbb{P}^n, \mathbb{O}(z), X)$ surjective for every integer $z \geq 0$. For such schemes $\rho(\mathbb{P}^n, \mathbb{O}(t), X(q))$ is surjective for every integer $z \geq 0$. For such schemes $\rho(\mathbb{P}^n, \mathbb{O}(t), X(q))$ is surjective if and only if $\rho(X, \mathbb{O}_X(t), X(q))$ is surjective and if $\rho(X, \mathbb{O}_X(t), X(q))$ is injective, then $\dim(\ker(\rho(\mathbb{P}^n, \mathbb{O}(t), X)) = \dim(\ker(\rho(\mathbb{P}^n, \mathbb{O}_X(t)) = (n+t)!(n!t!) - h^0(X, \mathbb{O}_X(t)), t \geq 0$. Now we describe the main results which will be proved in the first section.

Theorem 0.1. *Let $X \subset \mathbb{P}^d$, $3 \leq d \leq 8$, be a smooth Del Pezzo surface. Assume that there is a fibration $\pi : X \to \mathbb{P}^1$ defined over $GF(q)$ and such that a general fiber of π is a smooth rational curve. Then for all integers t with $2t \leq q$ the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is injective.*

Theorem 0.2. *Let $X \subset \mathbb{P}^8$ be the smooth Del Pezzo surface of degree 8 defined over $GF(q)$ and not isomorphic to a quadric. Then for all integers $t \geq q$ the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is surjective.*

Theorem 0.3. *Let $X \subset \mathbb{P}^5$ be the Veronese surface (defined over $GF(p)$ and hence over $GF(q)$). Then for all integers t with $t \geq q+1$ the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is surjective and for all integers t with $2t \leq q$ the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is injective.*

For the existence of a fibration $\pi : X \to \mathbb{P}^1$ defined over $GF(q)$ in the statement of Theorem 0.1, see Remark 1.5.
In the second (and last) section we will introduce a nice reducibile configuration in $\mathbb{P}^8$ union of four Del Pezzo surfaces of degree 8.

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1. Smooth Del Pezzo surfaces.

We fix a prime p and let q be a power of p. Concerning the number of $GF(q)$-points of a smooth rational surface defined over $GF(q)$, the fundamental result is a theorem of A. Weil (see [6], Th. 27.1). Other very useful informations on $X(q)$ for a degree d smooth Del Pezzo X are [6], Th. 29.4, and [6], Th. 30.1 (which gives the existence of a $GF(q)$-point not on an exceptional curve if either $d = 3$, $q > 34$ or $d = 4$ and $q > 22$). For more recent results and the case of singular Del pezzo surfaces, see [4] and references therein.
We will use the following well-known elementary lemmas (the second one being completely trivial) and call them Mayer-Vietoris lemmas.

Lemma 1.1. *Let U and V be closed subschemes of the projective scheme X. Assume that $X = U \cup V$ (as schemes) and that the scheme $D := U \cap V$ is a Cartier divisor both on U and on V. Fix $L \in \mathrm{Pic}\,(X)$ and a finite set S of X. Set $S' := S \cap U$ and $S'' := S \setminus S'$. Assume that the restriction maps $\rho(U, L|U, S')$ and $\rho(V, (L|V)(-D), S'')$ are surjective. Assume the surjective of the restriction maps $H^0(X, L) \to H^0(U, L \setminus U)$ and $H^0(X, L \otimes \mathbb{I}_U) \to H^0(V, (L|V)(-D))$. Then $\rho(X, L, S)$ is surjective.*

Proof. Fix $f \in H^0(S, L|S)$ and lift $f|S'$ to $g \in H^0(U, L|U)$. Lift g to $g' \in H^0(X, L)$ and set $a := f|S \cap V - g'|S \cap V \in H^0(S \cap V, L|(S \cap V))$. By construction a induces $a' \in H^0(S'', (L|V)(-D)|S'')$. Lift a' to $a'' \in H^0(X, L)$ with $a''|U = 0$. By construction $(g' + a'')|S = f$.

Lemma 1.2. *Let U and V be closed subschemes of the projective scheme X. Assume that $X = U \cup V$ (as scheme) and that the scheme $D := U \cap V$ is a Cartier divisor both on U and on V. Fix $L \in \mathrm{Pic}\,(X)$ and a finite set S of X. Set $S' := S \cap U$ and $S'' := S \setminus S'$. Assume that the restriction maps $\rho(U, L|U, S')$ and $\rho(V, (L|V)(-D), S'')$ are injective. Then $\rho(X, L, S)$ is injective.*

We will use also the following easy and well-known lemma and call it Horace lemma.

Lemma 1.3. *Let X be a projective scheme, $D \subset X$ an effective Cartier divisor and $L \in \mathrm{Pic}\,(X)$. Let $S \subset X$ be a finite set. Set $S' := S \cap D$ and $S'' := S \setminus S'$. We have* $\dim(\ker(\rho(X, L, S))) \leq \dim(\ker(\rho(X, L(-D), S''))) + \dim(\ker(\rho(D, L|S')))$ *and* $\dim(\mathrm{coker}(\rho(X, L, S))) \leq \dim(\mathrm{coker}(\rho(X, L(-D), S''))) + \dim(\mathrm{coker}(\rho(D, L|S')))$. *In particular, if the restriction maps $\rho(X, L(-D), S'')$ and $\rho(D, L|D, S')$ are injective (resp. surjective), then $\rho(X, L, S)$ is injective (resp. surjective).*

We recall that if a smooth projective surface X is a Del Pezzo surface of degree d, $3 \leq d \leq 9$, and it is defined over $GF(q)$, then its standard degree d embedding into $\mathbb{P}^d$ is defined over $GF(q)$ because this embedding is induced by $H^0(X, \omega_X^*)$. We will need the following well-known result (see e.g. [5], Lemma 2.9 and proof of 4.1 (a); the proof there works in any characteristic).

Proposition 1.4. *Let $X \subset \mathbb{P}^d$, $3 \leq d \leq 9$, be a smooth Del Pezzo surface (embedded over $GF(q)$). Then X is projectively normal, i.e. for all $t > 0$ the restriction map $(\rho(\mathbb{P}^d, \mathbb{O}(t), X) : H^0(\mathbb{P}^d, \mathbb{O}_{\mathbb{P}^d}(t)) \to H^0(X, \mathbb{O}_X(t))$ is surjective.*

Remark 1.5. The existence of the fibration π in the statement of Theorem 0.1 is always satisfied in the following cases:

(i) $d = 8$ and X is not isomorphic to a smooth quadric embedded by the degree 2 Veronese embedding of $\mathbb{P}^3$. In this case X is isomorphic to the blowing-up of $\mathbb{P}^2$ at one point of $\mathbb{P}^2(q)$. In this case every fiber of π is smooth.

(ii) There is a morphism $u : X \to \mathbb{P}^2$ defined over $GF(q)$ and birational and there are 2 point $P_i \in \mathbb{P}^2(q)$, $i = 1, 2$ with $\pi^{-1}(P_i)$ not a point. Indeed, since X is a Del Pezzo defined over $GF(q)$, u is a blowing-up of $\mathbb{P}^2$ along a finite set, T, of $9 - d$ points in sufficiently general position. By assumption there is $P_1 \in T$ defined over $GF(q)$. A choice of one of the poits of $T \cap \mathbb{P}^2(q)$ defines (over $GF(q)$) a fibration $\pi' : X \to \mathbb{P}^1$ with smooth rational curves as general fibres and embedded as conics and each of the other $8 - d$ points of T defines a singular fiber of π'. P_2 correspnds to a singular fiber defined over $GF(q)$.

(iii) (the trivial case). The degree q Frobenius map acts trivially on $\mathrm{Pic}(X)$ (or equivalently by a Theorem of A. Weil ([6], Th. 27.1) $\mathrm{card}(X)(q) = q^2 + (10 - d)q + 1$). In this case all the singular fibers of π are defined over $GF(q)$.

Proof of Theorem 0.1. We may assume $q \geq 4$. Note that $\mathbb{O}_X(-1) \cong \omega_X$. Since π is defined over $GF(q)$ we have $\pi(X(q)) \subseteq \mathbb{P}^1(q)$. Fix $P \in \mathbb{P}^1(q)$ such that $\pi^{-1}(P)$ is a smooth $\mathbb{P}^1$ embedded in $\mathbb{P}^d$ as a smooth conic by the adjunction formula. Such a point exists if $d \neq 3$ because π cannot have more that 4 singular fibers. Indeed it has exactly $8 - d$ singulars fibers. Since $\mathrm{card}(\pi^{-1}(P)(q)) = q + 1$, every $f \in \ker(\rho(X, \mathbb{O}_X(t), X(q)))$ vanishes on $\pi^{-1}(P)$. We conclude easily. In the missing case $d = 3, q = 5$ we have $t = 2$ and all the fibers of π over $\mathbb{P}^1(q)$ are the union of two smooth exceptional $\mathbb{P}^{1\prime}s$ meeting in a point defined over $GF(q)$. Hence each of these 10 $\mathbb{P}^{1\prime}s$ are defined over $GF(q)$ and have $q + 1 = 6$ $GF(q)$-points. We obtain 55 $GF(q)$ points of

X. Since the union of the 5 singular fibers of π are not contained in a quadric surface, we conclude in this case.

Proof of Theorem 0.2. We know (see e.g. [6]) that X is the blowing-up of $\mathbb{P}^2$ at a $GF(q)$-point and that there is a smooth fibration $\pi : X \to \mathbb{P}^1$ defined over $GF(q)$ with smooth rational curves as fibers. Furthermore (see e.g. [6], Th. 27.1) $\mathrm{card}(X(q)) = q^2 + 2q + 1$ and π induces a $(q + 1)$ to 1 map $X(q) \to \mathbb{P}^1(q)$. Set $A := \pi^{-1}(\mathbb{P}^1(q))$. Hence A is an effective Cartier divisor of X containing $X(q)$. We claim that $H^1(X, \mathbb{O}_X(t)(-A)) = 0$. Assume that the claim is true. By the claim the restriction map $H^0(X, \mathbb{O}_X(t)) \to H^0(A, \mathbb{O}_X(t))|A)$ is surjective. A is the disjoint union of $q + 1$ copies of $\mathbb{P}^1$ and on each of these $\mathbb{P}^1 \mathbb{O}_X(1) \cong \omega_X^*$ induces a degree 2 line bundle by the adjunction formula. Hence the restriction map $\rho(A, \mathbb{O}_X(t)|A, X(q))$ is surjective. Thus the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is surjective. Now we check the claim. Take as base of $\mathrm{Pic}(X)$ the exceptional divisor h and a fiber f of π (hence $h^2 = -1$, $h \cdot f = 1$ and $f^2 = 0$). By the adjunction formula we have $\mathbb{O}_X(1) = 2h + 3f$ (using an additive notation) and $\mathbb{O}_X(t)(-A) = 2th + (3t - q - 1)f$. Hence the claim follows from the Leray spectral sequence of π, the projection formula and the cohomology of line bundles on $\mathbb{P}^1$.

Proof of Theorem 0.3. Since $\mathbb{O}_X(1)$ has degree 2 and every $GF(q)$-line of $\mathbb{P}^2$ has $q+1$ points, the last assertion is trivial. Assume $t \geq q+1$ and fix $P \in \mathbb{P}^2(q)$. Since $\mathbb{P}^2(q)$ is contained in $q + 1$ lines belonging to the pencil of lines through P, we conclude applying $q + 1$ times Horace Lemma 1.3.

1.6. Here we assume q odd. We recall that in [3] it was described the embedding $X = A \cup B \subset \mathbb{P}^5$ (defined over $GF(q)$) of two Veronese surfaces A, B with $A \cap B$ union of three conics intersecting pairwise in one point. We have $\mathrm{card}(X(q)) = 2q^2 - q + 2$. X is the complete intersection of 3 quadric hypersurfaces and hence for all integers $t \geq 0$ the restriction map $H^0(\mathbb{P}^5, \mathbb{O}_{\mathbb{P}^5}(t)) \to H^0(X, \mathbb{O}_X(t))$ is surjective. By Theorem 0.2 and the Mayer-Vietoris Lemma 1.2 the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is injective if $2t \leq q$. Hence if $2t \leq q$, $\ker(\rho(\mathbb{P}^5, \mathbb{O}_{\mathbb{P}^5}(t), X(q))) = H^0(\mathbb{P}^5, \mathbb{I}_X(t))$ is generated by the 3 quadrics defining X, i.e. $\dim(\ker(\rho(\mathbb{P}^5, \mathbb{O}_{\mathbb{P}^5}(t), X(q)))) = h^0(\mathbb{P}^5, \mathbb{O}_{\mathbb{P}^5}(t)) - h^0(X, \mathbb{O}_X(t)) = (t+3)!40(t-2)! - (t+1)!/40(t-4)! + (t-1)!/120(t-6)!$. Using the Mayer-Vietoris Lemma h1 and Proposition h9, we see that if $2t \geq 2q+5$, i.e. $t \geq q+2$, the restriction map $(\rho(\mathbb{P}^5, \mathbb{O}_{\mathbb{P}^5}, X(q))$ is surjective.

1.7. Set $X = \mathbb{P}^2$ with as $\mathbb{O}(1)$ the degree 1 line bundle. Set $M := \ker(\rho(X, \mathbb{O}(q + 1), X(q)))$. We will check that $\dim(M) = 3$. Fix 3 not collinear points of $X(q)$, say P_i, $1 \leq i \leq 3$, and let T_i, $1 \leq i \leq 3$, be the union of the $q+1$ lines through P_i defined over $GF(q)$ (hence T_i contains $X(q)$ and has a point of multiplicity $q+1$ at P_i). Note that T_1, T_2 and T_3 are linearly independent because the pencil generated by T_1 and T_2 is formed by the reducible curves containing the line D_{12} spanned by P_1 and P_2 (and hence does not contain T_3). Let M' be the linear span of the cirves T_1, T_2 and T_3 (i.e. a net). We want to check that $M' = M$. Assume $M' \neq M$ and hence $\dim(M) \geq 4$. Choose a point A_3 on $D_{12} \setminus X(q)$. By Bezout theorem every $f \in M$ with $f(P) = 0$ vanishes on D_{12}. Taking a point A_2 (resp. A_1) on the line D_{13} (resp. D_{23}) containing P_1 and P_3 (resp. P_2 and P_3) we find $g \in M$, $g \neq 0$ and g vanishing on $D_{12} \cup D_{23} \cup D_{13}$. Since we see easily that the $q^2 - 2q + 1$ points of $X(q) \setminus (D_{12} \cup D_{23} \cup D_{13})(q)$ are not contained in a curve of degree $q-2$ (e.g. using Bezout to show that this curve contains $q-1$ lines) we obtain a contradiction. Note also that for every $P \in X(q)$ we may find $f \in M'$ formed by the $q+1$ lines defined over $GF(q)$ and passing through P.

Proposition 1.8. *Let $X \subset \mathbb{P}^d$, $3 \leq d \leq 7$, be a smooth Del Pezzo surface. Assume that there is a fibration $\pi : X \to \mathbb{P}^1$ defined over $GF(q)$ and such that a general fiber of π is a smooth rational curve. Assume that each singular fiber of π defined is over $GF(q)$. Then for all integers $t < q$ the restriction map $\rho(X, \mathbb{O}_X(t), X(q))$ is not surjective. More, precisely, if $q/2 < t < q$ we have* $\dim(\ker(\rho(X, \mathbb{O}_X(t), X(q)))) \geq 2(8-d)(q-t)$.

Proof. Since $d < 8$ we know that X has $8-d$ singular fiber, each of them consisting of two smooth $\mathbb{P}^1{}'s$ defined over $GF(q)$ (by assumption) and each of these $\mathbb{P}^1$ is embedded as a line and each singular fiber is embedded as a reducible conic. Hence if $t \leq q$ each of these $8-d$ reducible conics, D, contains at least $2(q-t)$ points of $D(q)(q-t$ of them on each line) such that every plane curve of degree t passing through the remaining $2q+1-2(q-t)$ points of $D(q)$ contains all points of $D(q)$.

1.9. Here we consider a Del Pezzo surface X of degree 7 defined over $GF(q)$ such that there is a morphism $\pi : X \to \mathbb{P}^2$ over $GF(q)$ with π blowing-up at two points, P and Q, of $\mathbb{P}^2$. This implies that the line containing P and Q is defined over $GF(q)$ and its strict transform is an exceptional curve E of X (embedded as line of $\mathbb{P}^d$). We distinguish two subcases

(a) Here we assume $P \notin \mathbb{P}^2(q)$. Hence $E(q)$ is the unique $GF(q)$-line of $X(q) \subset \mathbb{P}^7$. Thus (as in 1.8) if $t < q$ we have $\dim(\ker(\rho(X, \mathbb{O}_X(t), X(q)))) \geq q-t$.

(b) Here we assume $P \in \mathbb{P}^2(q)$. Hence $E(q), \pi^{-1}(P)$ and $\pi^{-1}(Q)$ are the unique $GF(q)$-line of $X(q) \subset \mathbb{P}^7$. Thus (as in 1.8) if $t < q$ we have $\dim(\ker(\rho(X, \mathbb{O}_X(t), X(q)))) \geq 3(q-t)$.

2. A reducible surface of degree 32 in $\mathbb{P}^8$.

Here we will construct in $\mathbb{P}^8$ (over any base field $\mathbb{K}$ with at least 5 elements) the following configuration W which is a projective reducible algebraic surface of degree 32 embedded (over $\mathbb{K}$) into $\mathbb{P}^8$. The construction is motivated by the fact that we have partial but interesting informations on the Hilbert functions of union of sets (Mayer-Vietoris Lemmas 1.1 and 1.2 and Horace Lemma 1.3) and the use of the union of two Veronese surfaces of $\mathbb{P}^5$ made in [3] (see 1.6). W is the union of 4 Del Pezzo varieties X_1, X_2, X_3, X_4 of degree 8 embedded in $\mathbb{P}^8$. We will see a Del pezzo degree 8 surface as the embedding of a plane blowing-up at a point P (defined over $\mathbb{K}$) by the system of cubics through P. First we consider 3 planes $\prod(1), \prod(2), \prod(3)$ contained in $\mathbb{P}^8$ and such that for all $i \neq j$ the projective space $\prod(i, j)$ spanned by $\prod(i)$ and $\prod(j)$ have dimension 5 (i.e. $\prod(i) \cap \prod(j) = \emptyset$) but the projective space $\prod(1, 2, 3)$ spanned by them has only dimension 7, i.e. for each permutation (ijk) of (123) $\prod(i, j) \cap \prod(k)$ is a point. Now in each $\prod(i)$ we take a smooth conic $C(i)$. Note that there is always a Del Pezzo surface containing $C(1), C(2), C(3)$ and for which the conics $C(1), C(2)$ and $C(3)$ are the image of 3 plane lines through the blown-up point. We note that any two such configurations are projectively equivalent. Since $\mathrm{card}(\mathbb{K}) \geq 5$, we may find one such configuration defined over $\mathbb{K}$. Now we will take 4 such configurations and call $\prod(i; y), \prod(i, j; y), \prod(1, 2, 3; y), C(i; y)$, $1 \leq y \leq 4$, the corresponding objects. We have a priori 12 conics $C(i; y)$, $1 \leq i \leq 3$, $1 \leq y \leq 4$. We will impose that these conics are exactly 6. More precisely, we will take 4 Del Pezzo degree 8 surfaces X_y, $1 \leq y \leq 4$, with X_y containing all $C(i; y)$, $1 \leq i \leq 3$, and we will assume that if $y \neq z X_y \cap X_z$ is one of the conics $C(i; z)$, $1 \leq i \leq 3$, and also one of the conics $C(j, y)$, $1 \leq j \leq 3$. We take $C(1; 2) := C(2; 1), C(1; 3) := C(3; 1), C(2; 3) := C(3; 2), C(1; 4) := C(1; 1), C(2; 4) := C(2; 2)$ and $C(3; 4) := C(3; 3)$. Hence we have $\prod(1; 2) = \prod(2; 1), \prod(1; 3) = \prod(3; 1), \prod(2; 3) = \prod(3; 2), \prod(1; 4) = \prod(1; 1), \prod(2; 4) = \prod(2; 2)$ and $\prod(3; 4) = \prod(3; 3)$. We start with a hyperplane $\prod(1, 2, 3; 1)$ of $\mathbb{P}^8$ and take three general planes $\prod(1; 1), \prod(2; 1)$ and $\prod(3; 1)$ of $\prod(1, 2, 3; 1)$. Then we take another general hyperplane $\prod(1, 2, 3; 2)$ containing $\prod(2; 1)$ and two general planes $\prod(2; 2)$ and $\prod(3; 2)$ of $\prod(1, 2, 3; 2)$. It is sufficient to takes as $\prod(3; 3)$ a general plane intersecting both the linear span of $\prod(3; 1) \cup \prod(3; 2)$ and the linear span of $\prod(1; 1) \cup \prod(2; 2)$.

Another nice reducible degree 24 surface in $\mathbb{P}^7$ is obtained taking 3 such Del Pezzo surfaces with $\prod(i; y) = \prod(i; z)$ for all i, y and z and choosing $C(i; y)$ general with the restriction $C(1; 1) = C(1; 2), C(2; 1) = C(2; 3)$ and $C(3; 2) = C(3; 3)$.

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