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A UNIFYING MINIMAX THEOREM

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In the present note, a minimax theorem is given which combines the most general topological and quantitative conditions employed in the literature. This result encompasses a large part of the classes of topological, quantitative, and mixed minimax theorems, and includes several new variants.

The development of extensions of von Neumann's minimax theorem until 1995 is surveyed comprehensively in a recent paper by Simons [11]. In particular, Simons noted that newer results tend to unify the classes of topological, quantitative, and mixed theorems. In the present note, we want to confirm this point of view-as far as minimax theorems based on induction arguments are concerned-by means of a single theorem which combines the most general topological and quantitative conditions employed in the literature. In some sense, this is a complement of [10]. Our result is based on the observation that the two connectedness properties applied in the abstract minimax theorems of Kindler [1] need not be given for the same level sets. This possibility of "separating the connectedness levels" paves the way to many new combinations of topological and quantitative conditions. It leads also to a slightly simpler formulation of the topological minimax theorems of König [5] (and the more detailed results of Kindler which, for brevity, are not considered here).

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1. Preliminaries.

Only some indispensable facts about (pre)minimax equalities and the related intersection problems are stated here; we refer to [1] for more details.

We denote by $\mathcal{E}(S)$ the collection of all nonempty finite subsets of a set S. Let nonempty sets X, Y and a function $f : X \times Y \to \overline{\mathbb{R}}$ be given. The equality

$$\sup_X \inf_Y f = \inf_Y \sup_X f$$

holds if (and only if) for every $\lambda > \sup_X \inf_Y f$ the level sets $\{y \mid f(x, y) \leq \lambda\}$, $x \in X$, have nonempty intersection. In this problem, the set of all $\lambda > \sup_X \inf_Y f$ can be replaced by a smaller set Λ satisfying $\inf \Lambda = \sup_X \inf_Y f$, a so-called *border set* [5].

Roughly speaking, solutions are obtained by some compactness condition reducing the problem to nonempty intersections of finitely many sets or, equivalently, the preminimax equality $\sup_X \inf_Y f = \sup_{E \in \mathcal{E}(X)} \inf_Y \max_E f$; the latter is then established by two geometric properties: a condition A on subsets of Y such as the level sets $\{y \mid f(x, y) \leq \lambda\}$ and a condition B on subsets of X such as the level sets $\{x \mid f(x, y) \geq \lambda\}$.

However, some of the conditions of type B considered below involve properties of level sets of f itself. Therefore we shall use notation built on subsets of $X \times Y$: Given a set $F \subseteq X \times Y$, we set

$$F(x) := \{y \mid (x, y) \in F\}, \quad F^*(y) := \{x \mid (x, y) \notin F\}.$$

As we are mainly concerned with intersections of such sets, we extend this notation as follows: If A is a nonempty subset of X, then $F(A)$ denotes the intersection of all $F(x)$, $x \in A$; likewise, $F^*(B)$ is defined for every nonempty $B \subseteq Y$, and we complete the definitions by setting $F(\emptyset) := Y$ and $F^*(\emptyset) := X$. As a consequence, the following relations, ubiquitous in the proofs, hold for every $B \subseteq Y$:

$$F^*(B) = \{x \mid F(x) \cap B = \emptyset\}, \quad F^*(Y \setminus B) = \{x \mid F(x) \subseteq B\}.$$

With respect to $F(\cdot)$ only, we have to deal with intersections over a subset together with some single elements, for example $F(A) \cap F(a_1) \cap F(a_2)$; instead of forming a set we just list the arguments: $F(A, a_1, a_2)$.

Before we engage in the details, we want to show by a particular situation occuring in the proof of our theorem what is meant by "separating the connectedness levels" and what sort of arguments we are going to use.

Proposition. *Let X and Y be topological spaces. Let subsets $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4$ of $X \times Y$ be given such that all $F_1^*(y)$ are closed, all $F_2(X')$, $X' \in \mathcal{E}(X)$, are connected, all $F_3^*(Y')$, $Y' \subseteq Y$, are connected, and all $F_4(x)$ are closed. If there is a finite $A \subseteq X$ such that $F_1(A, a) \neq \emptyset$ for all $a \in X$, then $F_4(A, a_1, a_2) \neq \emptyset$ for all $a_1, a_2 \in X$.*

To prove this, let $F_4(A, a_1, a_2) = \emptyset$ for some $a_1, a_2 \in X$; we shall find some $a \in X$ such that $F_1(A, a) = \emptyset$. Let $L := Y \setminus (F_4(a_1) \cup F_4(a_2))$, and $L_1 := F_1(A) \cap F_4(a_1)$, $L_2 := F_1(A) \cap F_4(a_2)$, say. Then $a_1 \in F_3^*(L) \cap F_1^*(L_2)$ and $a_2 \in F_3^*(L) \cap F_1^*(L_1)$; since $F_3^*(L)$ is connected and $F_1^*(L_1)$, $F_1^*(L_2)$ are closed, we have $F_3^*(L) \not\subseteq F_1^*(L_1) \cup F_1^*(L_2)$ or $F_3^*(L) \cap F_1^*(L_1) \cap F_1^*(L_2) \neq \emptyset$. But for every $a \in F_3^*(L)$, $F_2(A, a) \subseteq F_4(a_1) \cup F_4(a_2)$ and $F_2(A, a) \cap F_4(a_1) \cap F_4(a_2) = \emptyset$; since $F_2(A, a)$ is connected and $F_4(a_1), F_4(a_2)$ are closed, we obtain $F_2(A, a) \cap F_4(a_1) = \emptyset$ or $F_2(A, a) \cap F_4(a_2) = \emptyset$, hence $a \in F_1^*(L_1) \cup F_1^*(L_2)$. So there exists $a \in F_3^*(L) \cap F_1^*(L_1) \cap F_1^*(L_2)$; then $F_1(A, a) \subseteq F_4(a_1) \cup F_4(a_2)$ and $F_1(A, a) \cap F_4(a_1) = \emptyset$, $F_1(A, a) \cap F_4(a_2) = \emptyset$, thus $F_1(A, a) = \emptyset$.

2. The result.

Our program is as follows. We have selected two conditions of type A and eight of type B which can be combined arbitrarily. In this section we prepare for applying them in the induction step; the following sections contain the conditions and the remaining parts of the proof together with remarks and related examples. The final section sketches some variants of the theorem. We suppose throughout that X and Y are topological spaces.

Theorem. *Suppose that Y is compact and f is lower semicontinuous in y. Let Λ_A and Λ_B be border sets and*

$$\mathcal{F}_A := \{\{(x, y) \mid f(x, y) \leq \lambda\} \mid \lambda \in \Lambda_A\},$$
$$\mathcal{F}_B := \{\{(x, y) \mid f(x, y) < \lambda\} \mid \lambda \in \Lambda_B\}.$$

Then $\sup_X \min_Y f = \min_Y \sup_X f$ *holds, if one of the conditions* A1–2 *and one of the conditions* B1–8 *is satisfied.*

It is clear from the compactness and continuity assumptions that we only have to show that for every $F \in \mathcal{F}_A$ the sets $F(x)$, $x \in X$, have the finite intersection property. So let a finite set $A \subseteq X$ be given such that $F(A, a) \neq \emptyset$ for every $F \in \mathcal{F}_A$ and every $a \in X$; this is true for $A = \emptyset$ by the definition of $\mathcal{F}_A$. We assume, for contradiction, that there exist $F' \in \mathcal{F}_A$ and $a_1, a_2 \in X$ with $F'(A, a_1, a_2) = \emptyset$.

The proof is finished by choosing appropriate level sets below F' and applying the various conditions to them. However, combining all the conditions at once means dealing with a large number of sets. We fix a frame of eight sets, $F_1 \subseteq \ldots \subseteq F_8$, which offers a particularly clear view on the general argument as well as on limitations and variants of the theorem. Let

$$\nu_B < \mu_B < \kappa_A < \lambda_A < \lambda_B < \kappa_B < \mu_A < \nu_A \tag{1}$$

be chosen from Λ_A and Λ_B, as indicated by the indices, such that

$$\{(x, y) \mid f(x, y) \leq \nu_A\} \subseteq F',$$

and let $F_1, \ldots, F_8$ be the associated sets, that is,

$$F_1 = \{(x, y) \mid f(x, y) < \nu_B\} \in \mathcal{F}_B, \ldots, F_8 = \{(x, y) \mid f(x, y) \leq \nu_A\} \in \mathcal{F}_A.$$

Then for every index k we have

$$F_k(A, a) \neq \emptyset \text{ for all } a \in X \text{ and } F_k(A, a_1, a_2) = \emptyset. \tag{2}$$

Some of the conditions will require one or two other level sets from $\mathcal{F}_A$ or $\mathcal{F}_B$; we prepare for these additions by assuming that between every two of the levels in (1) we can find other members from Λ_A as well as from Λ_B.

3. Connectedness.

Although none of our conditions involves abstract connectedness, it is convenient to use this notion in the proof. Given three sets U, V, W, we say that U is connected for V, W (see [5], [1]) if

$$U \subseteq V \cup W \text{ and } U \cap V \cap W = \emptyset \text{ implies } U \cap V = \emptyset \text{ or } U \cap W = \emptyset.$$

A similar relation called pseudoconnectedness (replace $U \cap V \cap W$ by $V \cap W$) was introduced in [10], but it applies to the condition A only. However, neither of the two covers the usual quantitative conditions as these lead to relations involving more than three sets. (In [10], Lemma 2, compactness reduces the number to three.) A sufficiently general relation arises from replacing each of the sets by a pair:

$$U \subseteq V \cup W \text{ and } U \cap V' \cap W' = \emptyset \text{ implies } U' \cap V = \emptyset \text{ or } U' \cap W = \emptyset,$$

where $U' \subseteq U$, $V \subseteq V'$, $W \subseteq W'$ are given. It is easily checked that this is true if, for example, U is connected for V, W or, in general, there are sets U'' between U' and U, V'' between V and V', and W'' between W and W' such that U'' is connected for V'', W''. Such reasoning will be used now.

A1. *For every $F \in \mathcal{F}_A$, or every $F \in \mathcal{F}_B$, all $F(X')$, $X' \in \mathcal{E}(X)$, are connected.*

B1. *f is upper semicontinuous in x, and for every $F \in \mathcal{F}_B$, or every $F \in \mathcal{F}_A$, all $F^*(Y')$, $Y' \subseteq Y$, are connected.*

B2. *f is lower semicontinuous, and for every $F \in \mathcal{F}_B$, or every $F \in \mathcal{F}_A$, all $F^*(Y')$, $Y' \subseteq Y$, are connected.*

These are the conditions that constitute generalized versions of König's theorems [5]. Note that the collections $\mathcal{F}_A$, $\mathcal{F}_B$ are just examples: each of the connectedness properties in A1, B1–4 may as well hold for another collection $\mathcal{F}$ of level sets being compatible with $\mathcal{F}_A$ in the sense that every member of $\mathcal{F}$ contains a member of $\mathcal{F}_A$ and vice versa.

Other topological minimax theorems based on connectedness conditions over two arbitrary border sets were stated by Ricceri [8]; these are the only known results, where only connectedness of single level sets is required.

We apply A1 as follows: we choose a set $G_4 \in \mathcal{F}_B$ between F_4 and F_3; then, for every $a \in X$, one of the sets $F_4(A, a)$, $G_4(A, a)$ is connected. Moreover, from the continuity assumption of the theorem, the sets $F_7(a_1)$, $F_7(a_2)$ are closed. So we have a set between $F_3(A, a)$ and $F_4(A, a)$ which is connected for $F_7(a_1)$, $F_7(a_2)$, and we conclude that the following relation of type (3) holds for every $a \in X$:

$$
\begin{aligned}
&\text{if } F_4(A,a) \subseteq F_7(a_1) \cup F_7(a_2) \text{ and } F_4(A,a) \cap F_8(a_1) \cap F_8(a_2) = \emptyset,\\
&\text{then } F_3(A,a) \cap F_7(a_1) = \emptyset \text{ or } F_3(A,a) \cap F_7(a_2) = \emptyset.
\end{aligned}
\tag{4}
$$

This relation will also be deduced from A2, so that we can use it in the final step of the proof, where the conditions B1–8 are exploited. For example, from condition B1 we can deduce in a similar way the counterpart of (4):

$$
\begin{aligned}
&\text{if } F_5^*(L) \subseteq F_2^*(L_1) \cup F_2^*(L_2) \text{ and } F_5^*(L) \cap F_1^*(L_1) \cap F_1^*(L_2) = \emptyset,\\
&\text{then } F_6^*(L) \cap F_2^*(L_1) = \emptyset \text{ or } F_6^*(L) \cap F_2^*(L_2) = \emptyset,
\end{aligned}
\tag{5}
$$

where $L := Y \setminus (F_6(a_1) \cup F_6(a_2))$ and $L_i := F_3(A) \cap F_7(a_i)$, $i = 1, 2$. Let us see how combining (4) and (5) leads to a contradiction and therefore completes the proof. Clearly, $a_1, a_2 \in F_6^*(L)$ and, by (2), $a_1 \in F_3^*(L_2)$ and $a_2 \in F_3^*(L_1)$.

Now let $F_0 \in \mathcal{F}_A$, $F_0 \subseteq F_1$; of course, this set satisfies (2), too. Thus, if $a \in F_4^*(L)$, then $\emptyset \neq F_0(A, a) \subseteq F_4(a) \subseteq F_6(a_1) \cup F_6(a_2)$, hence $\emptyset \neq F_0(A, a) \cap F_6(a_i) \subseteq F_0(a) \cap L_i$ for some i, so $a \notin F_0^*(L_i)$. And if $a \in F_4^*(L)$, then $F_4(A, a) \subseteq F_7(a_1) \cup F_7(a_2)$; hence, from (2) and (4), $F_3(A, a) \cap F_7(a_i) = \emptyset$ for some i, that is, $a \in F_3^*(L_i)$. Altogether,

$$\begin{aligned} &a_1 \in F_6^*(L) \cap F_3^*(L_2), a_2 \in F_6^*(L) \cap F_3^*(L_1),\\ &F_4^*(L) \subseteq F_3^*(L_1) \cup F_3^*(L_2), F_4^*(L) \cap F_0^*(L_1) \cap F_0^*(L_2) = \emptyset. \end{aligned} \tag{6}$$

In particular,

$$\begin{aligned} &a_1 \in F_6^*(L) \cap F_2^*(L_2), a_2 \in F_6^*(L) \cap F_2^*(L_1),\\ &F_5^*(L) \subseteq F_2^*(L_1) \cup F_2^*(L_2), F_5^*(L) \cap F_1^*(L_1) \cap F_1^*(L_2) = \emptyset, \end{aligned} \tag{7}$$

contradicting (5).

Instead of establishing (5) we shall apply (7) or even (6) to obtain a contradiction from each of the conditions B1–8. Note also that, by the assumptions of the theorem, L_1 and L_2 are closed subsets of Y and therefore compact.

To complete the proof for B1 and B2, we choose $G_5 \in \mathcal{F}_A$ between F_6 and F_5; then one of the sets $F_5^*(L)$, $G_5^*(L)$ is connected, and it suffices to find closed, or open, sets between $F_1^*(L_i)$ and $F_2^*(L_i)$. But if f is upper semicontinuous in x, then $F_2^*(L_1)$ and $F_2^*(L_2)$ are closed. And if f is lower semicontinuous, we choose $G_1 \in \mathcal{F}_A$ between F_2 and F_1; then G_1 is closed, and from the compactness of L_1 and L_2 we conclude that $G_1^*(L_1)$ and $G_1^*(L_2)$ are open.

The topological argument used here is well-known: it is just the fact that a set-valued map $x \mapsto G(x)$ having compact range Y and closed graph G is upper semicontinuous, and this is equivalent to $G^*(B)$ being open for every closed $B \subseteq Y$. The reader may also recall the analogous characterization of lower semicontinuity, saying that $G^*(B)$ is closed for every open $B \subseteq Y$. Actually, there is a property behind, called quartercontinuity, which generalizes both upper and lower semicontinuity; we refer to [2] for more details, but let us sketch how it works here:

In order to exploit the connectedness of $G_5^*(L)$, say, as above it suffices to know that each $G_5^*(L) \cap F_2^*(L_i)$ is open in $G_5^*(L)$. Suppose that there is a set G_1 between F_1 and F_2 such that $x \mapsto G_1(A, x)$ is quartercontinuous, to the effect that for every x and every open $V \supseteq G_1(A, x)$ there is an open $U \subseteq X$ such that $x \in U$ and $G_1(A, u) \cap V \neq \emptyset$ for all $u \in U$. We apply this to each $x \in G_5^*(L) \cap F_2^*(L_1)$ and $V = Y \setminus F_7(a_1)$. It is readily seen that if

$u \in G_5^*(L) \cap U$, then $u \notin F_2^*(L_2)$, thus $u \in F_2^*(L_1)$. This shows that x is an interior point of $G_5^*(L) \cap F_2^*(L_1)$ relative to $G_5^*(L)$.

4. Finite intersections.

This section deals with the question, posed in [5], whether it suffices to suppose a condition B including connectedness of finite intersections only. By now several counterexamples have been found; two of them are stated below, another one was given by Naselli [7]. But there are also two quite different positive answers. The first is due to Komiya [3], who proved the minimax equality for continuous real-valued functions on compact spaces from abstract connectedness conditions involving only finite intersections of level sets. We reduce Komiya's argument that utilizes the metric structure of $\mathbb{R}$ to the underlying topological fact.

B3. *X is compact and connected, f is continuous, and for every $F \in \mathcal{F}_B$, or every $F \in \mathcal{F}_A$, all $F^*(Y')$, $Y' \in \mathcal{E}(Y)$, are connected.*

The idea behind it is to find a finite subset of L for which (7) remains true. This is done by means of the following lemma; then applying B3 just like B1 or B2 completes this part of the proof. But, as suggested by Komiya's result, we replace the sets L_i by finite subsets, too-this enables us to employ the finite staircase condition B8.

Lemma 1. *Let $B \subseteq Y$ and $C \subseteq D \subseteq X \times Y$ be given such that B is closed, C is compact, and all $D^*(y)$ are closed. Then $D^*(B') \subseteq C^*(B)$ for some finite $B' \subseteq B$.*

To prove this, we may assume that B is nonempty. The set $K := X \setminus C^*(B)$ is compact, being the image of the compact $C \cap (X \times B)$ under the projection $(x, y) \mapsto x$. So for every $y \in B$, $D^*(y) \cap K$ is closed and compact, and $D^*(B) \cap K = \emptyset$. Hence there exists $B' \in \mathcal{E}(B)$ such that $D^*(B') \cap K = \emptyset$.

We apply Lemma 1 to L, F_4, F_5, and to L_i, F_0, F_1, observing that, by compactness of both spaces and continuity of f, these sets have the required properties. So there exist finite $M \subseteq L$ and $M_i \subseteq L_i$ such that $F_5^*(M) \subseteq F_4^*(L)$ and $F_1^*(M_i) \subseteq F_0^*(L_i)$. Combining these inclusions and (6) gives the finite version of (7):

$$\begin{aligned} &a_1 \in F_6^*(M) \cap F_2^*(M_2),\ a_2 \in F_6^*(M) \cap F_2^*(M_1), \\ &F_5^*(M) \subseteq F_2^*(M_1) \cup F_2^*(M_2),\ F_5^*(M) \cap F_1^*(M_1) \cap F_1^*(M_2) = \emptyset. \end{aligned} \tag{8}$$

Let us also note a variant of Lemma 1 that could be used here: if B is compact and C is closed, then $C^*(B)$ is open (see B2), hence K is closed, and it clearly suffices to have closed compact $D^*(y)$. We mention this because it reveals an interesting picture resembling that of the continuity assumptions in B1 and B2: for the existence of the finite M and M_i, compactness of level sets $\{(x, y) \mid f(x, y) \leq \lambda\}$ suffices as well as compactness of $\{y \mid f(x, y) \leq \lambda\}$ and $\{x \mid f(x, y) \geq \lambda\}$.

We turn to the second positive answer. It is due to König [6], who observed that connectedness of finite intersections suffices under additional assumptions concerning the topology on X only. For simplicity, we include the crucial compactness argument (see [5], Remark 2.3) in the proof. A topological space is called normal if disjoint closed sets can be separated by disjoint open sets.

B4. *X is compact, normal, and connected, f is upper semicontinuous in x, and for every $F \in \mathcal{F}_B$, or every $F \in \mathcal{F}_A$, all $F^*(Y')$, $Y' \in \mathcal{E}(Y)$, are connected.*

According to this condition, let $G_6 := F_6$ or $G_6 \in \mathcal{F}_A$ between F_6 and F_5, so that every $G_6^*(B)$, $B \in \mathcal{E}(Y)$, is connected. We observe that the sets $F_5^*(L) \cap F_2^*(L_1)$ and $F_5^*(L) \cap F_2^*(L_2)$ are closed; by (7) they are nonempty, disjoint, and cover $F_5^*(L)$. Thus $F_5^*(L)$ is disconnected, hence $F_5^*(L) \neq X$, which rules out the case $L = \emptyset$.

Since X is normal, there exist disjoint open sets $O_i \supseteq F_5^*(L) \cap F_2^*(L_i)$. For every $B \in \mathcal{E}(L)$, we see from (7) that $a_2 \in G_6^*(B) \cap O_1$ and $a_1 \in G_6^*(B) \cap O_2$; thus $G_6^*(B) \not\subseteq O_1 \cup O_2$. Hence all $F_5^*(B) \setminus (O_1 \cup O_2)$, $B \in \mathcal{E}(L)$, are nonempty, whereby they have the finite intersection property, and they are closed and compact. But their intersection is $F_5^*(L) \setminus (O_1 \cup O_2) = \emptyset$, a contradiction.

Without any compactness in X, connectedness of finite intersections is not sufficient; this was shown in [6] by an impressive example: Let H be an infinite-dimensional Hilbert space, $X := \{x \in H \mid \|x\| \geq 1\}$ endowed with the norm topology, $Y := \{y \in H \mid \|y\| \leq 1\}$ with the weak topology, and $f(x, y) := \langle x, y \rangle$, the inner product. Here Y is compact, f is continuous, and for every $\lambda \in \mathbb{R}$ arbitrary intersections of $\{y \mid f(x, y) \leq \lambda\}$ as well as finite intersections of $\{x \mid f(x, y) \geq \lambda\}$ are connected; however, the level sets in X are clearly non-compact.

We give another example that sheds more light on the matter; it is a proper example for unsymmetry even in the presence of compactness. Compared to the above, it looks rather odd-but note that the spaces must not be normal. Let $X, Y := \mathbb{Z} \setminus \{0\}$, the non-zero integers, and let f be the characteristic function of $(X \times Y) \setminus F$, where

$$F(1) := \{1\}, \quad F(-1) := \{-1\},$$
$$F(x) := \{-1, 1\} \cup \{y \mid x/y \notin \mathbb{Z} \text{ and } y/x \notin \mathbb{Z}\}, \quad \forall x \neq \pm 1.$$

We endow X and Y with the weakest topologies such that all $F(x)$ and $F^*(y)$ are closed; so f is lower semicontinuous in y and upper semicontinuous in x. All $F(x)$ are nonempty, hence $\max_X \min_Y f = 0$, and their intersection is empty, hence $\min_Y \max_X f = 1$. The space Y is compact and connected, and every $F(A)$, $A \in \mathcal{E}(X)$, is connected-all at once from the fact that the only open sets $\neq Y$ containing one of the points $1, -1$ are $Y \setminus \{-1\}$ and $Y \setminus \{1\}$. The situation in X is exactly the same, since

$$F^*(1) = \{-1\}, \quad F^*(-1) = \{1\},$$
$$F^*(y) = \{x \mid x/y \in \mathbb{Z} \text{ or } y/x \in \mathbb{Z}\} \supset \{-1, 1\} \quad \forall y \neq \pm 1.$$

5. Intervals and arcs.

In view of the conditions A1 and B1 there is no need to consider connectedness in terms of convex subsets of interval spaces unless one aims to weaken the accompanying continuity properties. Let us briefly discuss how this may be done in our framework. We recall that a topological space T together with a mapping $[\cdot, \cdot]$ from $T \times T$ into the subsets of T is an interval space if for all $t_1, t_2 \in T$ the interval $[t_1, t_2]$ is connected and contains t_1 and t_2; see [2] and the references therein.

B5. *X is an interval space, and f is upper semicontinuous on intervals and quasiconcave in x.*

We apply this to the interval $S := [a_1, a_2]$. Quasiconcavity in x means that, given $x_1, x_2 \in X$ and $y \in Y$, $f(x_1, y) \wedge f(x_2, y) \geq \lambda$ implies $f(x, y) \geq \lambda$ for all $x \in [x_1, x_2]$, and upper semicontinuity on intervals in x says that $\{x \in [x_1, x_2] \mid f(x, y) \geq \lambda\}$ is closed in $[x_1, x_2]$. Hence $a_1, a_2 \in F_5^*(L)$ gives $S \subseteq F_5^*(L)$, and we read off from (7) that S is not connected for $S \cap F_2^*(L_1)$, $S \cap F_2^*(L_2)$, though these are closed subsets of the connected S, a contradiction.

By a similar argument, starting off with two points $y_i \in F_3(A, a) \cap F_7(a_i)$, (4) can be verified if Y is an interval space and f is lower semicontinuous on intervals and quasiconvex in y. However, there are examples showing that semicontinuity on intervals does not imply a sufficient compactness property, so that we can state only a preminimax result: *If X and Y are interval spaces, f is lower semicontinuous on intervals and quasiconvex in y, and f is upper semicontinuous on intervals and quasiconcave in x, then* $\sup_X \inf_Y f = \sup_{E\in\mathcal{E}(X)} \inf_Y \max_E f$ *holds.* This is Stachó's [12] Proposition 3 without any separation or completeness assumptions.

Of course, the same holds true with upper semicontinuity on intervals in y (applied to some set from $\mathcal{F}_B$ between F_8 and F_7). On the other hand, Komornik established the following ([4], Theorem 3): *If X is a convex subset of some real topological vector space, Y is a compact interval space, f is lower semicontinuous and quasiconvex in y, and f is lower semicontinuous on intervals and quasiconcave in x, then* $\sup_X \min_Y f = \min_Y \sup_X f$ *holds.* We extend this result by means of condition B6.

With respect to connectedness, the question in [5] is still open: is it possible to replace lower semicontinuity in condition B2 by lower semicontiuity in x? Although it is hard to imagine how this could work, there is no example in sight. The following may give some useful geometric information on this problem.

B6. *X is an interval space with the property that, given a nonempty finite subset S' of an interval $[x_1, x_2]$, there exist $s_1, s_2 \in S'$ such that $s_1 \in [x_1, s]$ and $s_2 \in [s, x_2]$ for all $s \in S'$, and f is lower semicontinuous on intervals and quasiconcave in x.*

As before let $S := [a_1, a_2]$; then $S \subseteq F_5^*(L)$. From (6) we see that S is not connected for $S \cap F_3^*(L_1)$, $S \cap F_3^*(L_2)$. We show that $S \cap F_3^*(L_2)$ is open in S; likewise, $S \cap F_3^*(L_1)$ is open in S, contradicting the connectedness of S. The sets $F_3(x) \cap L_2$, $x \in S \cap F_3^*(L_1)$, are closed subsets of the compact L_2, and they have the finite intersection property. To see the latter, let $S' \in \mathcal{E}(S \cap F_3^*(L_1))$ be given, and let $s_1 \in S'$ satisfy $s_1 \in [a_1, s]$ for all $s \in S'$. By quasiconcavity, $F_2(s_1) \subseteq F_2(a_1) \cup F_2(s)$ for all $s \in S'$, so $F_1(s_1) \cap L_2 \subseteq (F_2(a_1) \cap L_2) \cup (F_3(S') \cap L_2)$. But $F_2(a_1) \cap L_2 = \emptyset$ and $s_1 \notin F_1^*(L_2)$, since $s_1 \in F_5^*(L) \cap F_1^*(L_1)$; hence $\emptyset \neq F_1(s_1) \cap L_2 \subseteq F_3(S') \cap L_2$.

It follows that the sets $F_3(x) \cap L_2$, $x \in S \cap F_3^*(L_1)$, have nonempty intersection, that is, $S \cap F_3^*(L_1) \cap F_3^*(y) = \emptyset$ for some $y \in L_2$. But $S \subseteq F_3^*(L_1) \cup F_3^*(L_2)$, and $F_3^*(y)$ includes $F_3^*(L_2)$, hence $S \cap F_3^*(L_2) = S \cap F_3^*(y)$, which is open in S.

There is another result in this direction, Tuy's [13] Theorem 1$'$, and it is his argument we analysed here. Suppose that, given a level set F and $x_1, x_2 \in X$, there exists an arc φ joining x_1 and x_2 such that

$$F(\varphi(q)) \subseteq F(\varphi(p)) \cup F(\varphi(r)) \quad \text{for all } 0 \leq p < q < r \leq 1. \tag{9}$$

For $F \in \mathcal{F}_A$ this is the condition employed by Tuy. Now let $S := \varphi([0, 1])$; then S is connected, contains x_1, x_2, and $F(x) \subseteq F(x_1) \cup F(x_2)$ holds for all

$x \in S$. Given $S' \in \mathcal{E}(S)$, we can choose a finite $Q \subseteq [0, 1]$ with $\varphi(Q) = S'$. Then $s_1 := \varphi(\min Q)$ satisfies $F(s_1) \cup F(x_1) \cup F(s)$ for all $s \in S'$; likewise, $s_2 := \varphi(\max Q)$ satisfies $F(s_2) \subseteq F(s) \cup F(x_2)$ for all $s \in S'$.

Altogether, we may proceed as in the above proof-if we can apply (9) to both F_5 and F_2 (or another pair of sets between F_6 and F_5, respectively F_2 and F_1), that is, if the arcs given for these sets coincide. In other words, Tuy's condition requires a condition A that allows to set $F_2 = \ldots = F_5 \in \mathcal{F}_A$. For example, we may suppose connectedness of all $F(X')$, $F \in \mathcal{F}_A$, $X' \in \mathcal{E}(X)$; however, this results in the original.

6. Staircases.

The conditions we present in this section are modifications of those given in [10]. They are equally elementary: having defined the level sets corresponding to the inequalities we need only invoke the following combinatorial lemma which yields precisely the abstract connectedness relation (3). But they also include the more complicated quantitative conditions weakened by some ε.

Lemma 2. *Let $m, n > 0$ and sets $U, U_0, \ldots, U_{m+n}, V, V_0, \ldots, V_m, W, W_0, \ldots, W_n$ be given such that $U_0 \subseteq \ldots \subseteq U_{m+n} \subseteq U$, $V \subseteq V_m$, $W \subseteq W_n$, and $U_0 \cap V \cap W_0 \neq \emptyset$, $U_0 \cap W \cap V_0 \neq \emptyset$. Suppose that, for all $j < m + n$, $k < m$, $l < n$,*

$$(10) \qquad \text{if } U_j \cap V \cap W_l \neq \emptyset \text{ and } U_j \cap W \cap V_k \neq \emptyset, \quad \text{then } U_{j+1} \cap V_{k+1} \cap W_{l+1} \neq \emptyset.$$

Then $U \cap V_m \cap W_n \neq \emptyset$ or $U \not\subseteq V \cup W$.

To prove this, just apply (10), starting off with $j = k = l = 0$; the procedure will stop after less than $m + n$ steps: Given $U_j \cap V \cap W_l \neq \emptyset$ and $U_j \cap W \cap V_k \neq \emptyset$, pick $u \in U_{j+1} \cap V_{k+1} \cap W_{l+1}$. If $u \notin V \cup W$, then $U \not\subseteq V \cup W$. If $u \in V$ and $l + 1 = n$, or $u \in W$ and $k + 1 = m$, then $U \cap V_m \cap W_n \neq \emptyset$. If $u \in V$ and $l+1 < n$, increase j and l by one; likewise, if $u \in W$ and $k+1 < m$, increase j and k by one.

B7. *For all levels $\kappa > \lambda > \mu > \nu$ from Λ_B, all $\sigma, \tau < \nu$ from $f(X \times Y)$, and $Y_0 := Y$, the following holds:*

$$
\begin{aligned}
&\exists \sigma_0, \ldots, \sigma_m, \tau_0, \ldots, \tau_n, \lambda_0, \ldots, \lambda_{m+n} \in \overline{\mathbb{R}}: \\
&\sigma_0 \leq \sigma,\ \sigma_m = \nu,\ \tau_0 \leq \tau,\ \tau_n = \nu,\ \kappa = \lambda_0 \geq \ldots \geq \lambda_{m+n} = \lambda; \\
&\forall j < m+n,\ k < m,\ l < n,\ x_1, x_2 \in X : \exists x \in X : \forall y \in Y_0 : \\
&\textit{if } f(x_1, y) \geq \lambda_j \textit{ and } f(x_2, y) \geq \lambda_j \textit{ then } f(x, y) \geq \lambda_{j+1}; \\
&\textit{if } f(x_1, y) \geq \mu \textit{ and } f(x_2, y) \geq \sigma_k \textit{ then } f(x, y) \geq \sigma_{k+1}; \\
&\textit{if } f(x_1, y) \geq \tau_l \textit{ and } f(x_2, y) \geq \mu \textit{ then } f(x, y) \geq \tau_{l+1}.
\end{aligned} \tag{11}
$$

A few remarks are in order here. First, we note that dealing with two unordered sequences, (σ_k) and (τ_l), instead of a single ordered one, as in (12), does not lead to anything new; but we do not want to dwell on how to order and merge staircases in the example on mean functions below.

Second, conditions of this kind are usually accompanied by some boundedness assumption such as $\inf_{y \in Y} f(x, y) > -\infty$, $\forall x \in X$, the reason being that one needs finite levels σ, τ when the condition involves arithmetic operations. This is not the case here, but it will be in the examples, where f has to be real-valued. However, the compactness and continuity conditions of our theorem include a suitable assumption as they ensure that $\min_{y \in B} f(x, y)$ exists for every nonempty compact $B \subseteq Y$ (in particular for L_1, L_2); this allows us even to restrict σ, τ to $f(X \times Y)$.

We apply B7 to $(\kappa, \lambda, \mu, \nu) = (\kappa_B, \lambda_B, \mu_B, \nu_B)$, so that

$$
\begin{aligned}
F_6^*(y) &= \{x \mid f(x, y) \geq \kappa\}, \quad & F_5^*(y) &= \{x \mid f(x, y) \geq \lambda\}, \\
F_2^*(y) &= \{x \mid f(x, y) \geq \mu\}, \quad & F_1^*(y) &= \{x \mid f(x, y) \geq \nu\}.
\end{aligned}
$$

Let $\sigma := \min_{y \in L_1} f(a_1, y)$, $\tau := \min_{y \in L_2} f(a_2, y)$. Then $\sigma, \tau < \nu$, since $a_1 \notin F_1^*(L_1)$ and $a_2 \notin F_1^*(L_2)$ by (11). Let $\lambda_j, \sigma_k, \tau_l$ as given by (7). We define

$$
\begin{aligned}
U &:= F_5^*(L), \quad & U_j &:= \{x \mid f(x, y) \geq \lambda_j \ \forall y \in L\}, \\
V &:= F_2^*(L_1), \quad & V_k &:= \{x \mid f(x, y) \geq \sigma_k \ \forall y \in L_1\}, \\
W &:= F_2^*(L_2), \quad & W_l &:= \{x \mid f(x, y) \geq \tau_l \ \forall y \in L_2\}.
\end{aligned}
$$

Then $F_6^*(L) = U_0 \subseteq \ldots \subseteq U_{m+n} = U$, $V \subseteq F_1^*(L_1) = V_m$, $W \subseteq F_1^*(L_2) = W_n$; from (7), $a_1 \in U_0 \cap W \cap V_0$, $a_2 \in U_0 \cap V \cap W_0$; and (10) follows from (11). Now Lemma 2 yields $U \cap V_m \cap W_n \neq \emptyset$ or $U \not\subseteq V \cup W$, which contradicts (7).

Replacing L, L_i by M, M_i, setting $Y_0 := M \cup M_1 \cup M_2$, and refering to (8) instead of (7) gives the appropriate argument for the finite variant:

B8. *X is compact, f is continuous, and for all levels $\kappa > \lambda > \mu > \nu$ from Λ_B, all $\sigma, \tau < \nu$ from $f(X \times Y)$, and every $Y_0 \in \mathcal{E}(Y)$,* (11) *holds.*

After what has been said about the lower bound assumption, the second condition of the main result in [9], where f is supposed to be real-valued, reads as follows: for all $\gamma < \beta < \alpha$ in $\mathbb{R}$ with $\gamma \in f(X \times Y)$,

$$(12)\quad \begin{aligned} &\exists \gamma_0, \ldots, \gamma_n \colon \gamma = \gamma_0 < \ldots < \gamma_n = \beta;\ \forall \varepsilon > 0,\ x_1, x_2 \in X : \exists x \in X : \\ &\forall k < n,\ y \in Y : f(x, y) \geq (f(x_1, y) \wedge f(x_2, y)) - \varepsilon; \\ &\text{if } f(x_1, y) \vee f(x_2, y) > \alpha > \beta > f(x_1, y) \wedge f(x_2, y) \geq \gamma_k, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{then } f(x, y) \geq \gamma_{k+1}. \end{aligned}$$

Let us see how (11) follows from this. Given $\kappa > \lambda > \mu > \nu$ and $\sigma, \tau < \nu$ from $f(X \times Y)$, we choose $\nu < \beta < \alpha < \mu$ and set $\gamma := \sigma \wedge \tau$. Now let m be the least index for which $\gamma_m \geq \nu$, and let $\sigma_k := \gamma_k$ for $k < m$, $\sigma_m := \nu$; that is, we cut the staircase beyond ν, and we use (σ_k) in place of (τ_l). Further, let $\kappa = \lambda_0 > \ldots > \lambda_{2m} = \lambda$ and $0 < \varepsilon \leq \beta - \nu$ satisfy $\lambda_j - \varepsilon \geq \lambda_{j+1}$ for all j; so the first implication of (11) holds. For the second, let $f(x_1, y) \geq \mu > \alpha$ and $f(x_2, y) \geq \sigma_k = \gamma_k$. If $f(x_2, y) < \beta$, then $f(x, y) \geq \gamma_{k+1} \geq \sigma_{k+1}$; if $f(x_2, y) \geq \beta$, then $f(x, y) \geq (f(x_1, y) \wedge f(x_2, y)) - \varepsilon \geq \beta - \varepsilon \geq \nu \geq \sigma_{k+1}$. Likewise, $f(x_1, y) \geq \sigma_k$ and $f(x_2, y) \geq \mu$ implies $f(x, y) \geq \sigma_{k+1}$.

Before we complete our theorem by the staircase condition of type A, let us turn to another technical detail from [9]. Let $R \subseteq \mathbb{R}$ be an interval including $f(X \times Y)$ and Λ_B. We denote by $\Phi(R)$ the set of all functions $\phi : R \times R \to R$ satisfying $\phi(\alpha, \beta) \geq \alpha \wedge \beta$ for all $\alpha, \beta \in R$. Then the condition of type B in [9], Theorem 13, may be formulated as follows: there exists a monotone $\phi \in \Phi(R)$ such that

$$\begin{aligned} &\exists m, n > 0 : \phi(\mu, \cdot)^m(\sigma) \geq \nu;\ \phi(\cdot, \mu)^n(\tau) \geq \nu; \\ &\forall x_1, x_2 \in X : \exists x \in X : \forall y \in Y_0 : f(x, y) \geq \phi(f(x_1, y), f(x_2, y)). \end{aligned}$$

Is it possible to add $-\varepsilon$ here? We have no complete answer, but we have found natural assumptions that allow this weakening: Suppose that R contains some $\rho < \sigma \wedge \tau$ and there exists a strictly monotone $\phi \in \Phi(R)$ such that

$$\begin{aligned} &\exists m, n > 0 : \phi(\mu, \cdot)^m(\rho) \geq \nu;\ \phi(\cdot, \mu)^n(\rho) \geq \nu; \\ &\forall \varepsilon > 0,\ x_1, x_2 \in X : \exists x \in X : \forall y \in Y_0 : f(x, y) \geq \phi(f(x_1, y), f(x_2, y)) - \varepsilon. \end{aligned}$$

Then (11) can be deduced as follows: Choose $\alpha_k \in R$ with $\sigma = \alpha_0 > \ldots > \alpha_m = \rho$; let $\sigma_k := \phi(\mu, \cdot)^k(\alpha_k)$ for $k < m$ and $\sigma_m := \nu \le \phi(\mu, \cdot)^m(\alpha_m)$. Since ϕ is strictly monotone, it follows for $k < m$ that $\phi(\mu, \sigma_k) = \phi(\mu, \cdot)^{k+1}(\alpha_k) > \phi(\mu, \cdot)^{k+1}(\alpha_{k+1}) \ge \sigma_{k+1}$. Similarly, there exist $\tau_l \in R$ such that $\tau_0 = \tau$, $\tau_n = \nu$, and $\phi(\tau_l, \mu) > \tau_{l+1}$ for all $l < n$. Further, choose $\kappa = \lambda_0 > \ldots > \lambda_{m+n} = \lambda$. Then there exists $\varepsilon > 0$ such that $\phi(\lambda_j, \lambda_j) \ge \lambda_j \ge \lambda_{j+1} + \varepsilon$ for all $j < m+n$, $\phi(\mu, \sigma_k) \ge \sigma_{k+1} + \varepsilon$ for all $k < m$, and $\phi(\tau_l, \mu) \ge \tau_{l+1} + \varepsilon$ for all $l < n$; (11) is immediate from these inequalities and the monotonicity of ϕ.

A2. *For all $\kappa < \lambda < \mu < \nu$ from Λ_A, all $\sigma, \tau > \nu$ from $f(X \times Y)$, and every $X_0 \in \mathcal{E}(X)$, the following holds:*

$$
\begin{aligned}
&\exists \sigma_0, \ldots, \sigma_m, \tau_0, \ldots, \tau_n, \lambda_0, \ldots, \lambda_{m+n} \in \overline{\mathbb{R}}: \\
&\sigma_0 \ge \sigma,\ \sigma_m = \nu,\ \tau_0 \ge \tau,\ \tau_n = \nu,\ \kappa = \lambda_0 \le \ldots \le \lambda_{m+n} = \lambda; \\
&\forall j < m+n,\ k < m,\ l < n,\ y_1, y_2 \in Y : \exists y \in Y : \forall x \in X_0: \\
&\textit{if } f(x, y_1) \le \lambda_j \textit{ and } f(x, y_2) \le \lambda_j \textit{ then } f(x, y) \le \lambda_{j+1}; \\
&\textit{if } f(x, y_1) \le \mu \textit{ and } f(x, y_2) \le \sigma_k \textit{ then } f(x, y) \le \sigma_{k+1}; \\
&\textit{if } f(x, y_1) \le \tau_l \textit{ and } f(x, y_2) \le \mu \textit{ then } f(x, y) \le \tau_{l+1}.
\end{aligned} \tag{13}
$$

We apply this to $(\kappa, \lambda, \mu, \nu) = (\kappa_A, \lambda_A, \mu_A, \nu_A)$, so that

$$
\begin{aligned}
F_3(x) &= \{y \mid f(x, y) \le \kappa\}, \quad F_4(x) = \{y \mid f(x, y) \le \lambda\}, \\
F_7(x) &= \{y \mid f(x, y) \le \mu\}, \quad F_8(x) = \{y \mid f(x, y) \le \nu\}.
\end{aligned}
$$

In order to prove (4) for every $a \in X$ we assume that there exist $b_1 \in F_3(A, a) \cap F_7(a_2)$ and $b_2 \in F_3(A, a) \cap F_7(a_1)$. Let $\sigma := f(a_1, b_1)$ and $\tau := f(a_2, b_2)$. Then $\sigma, \tau > \nu$, since from (2) $b_1 \notin F_8(a_1)$ and $b_2 \notin F_8(a_2)$. Let $X_0 := A \cup \{a, a_1, a_2\}$ and $\lambda_j, \sigma_k, \tau_l$ as given by (13). We define

$$
\begin{aligned}
U &:= F_4(A, a), \quad U_j := \{y \mid f(x, y) \le \lambda_j \ \forall x \in A \cup \{a\}\}, \\
V &:= F_7(a_1), \quad V_k := \{y \mid f(a_1, y) \le \sigma_k\}, \\
W &:= F_7(a_2), \quad W_l := \{y \mid f(a_2, y) \le \tau_l\}.
\end{aligned}
$$

Then $F_3(A, a) = U_0 \subseteq \ldots \subseteq U_{m+n} = U$, $V \subseteq F_8(a_1) = V_m$, $W \subseteq F_8(a_2) = W_n$, $b_1 \in U_0 \cap W \cap V_0$, $b_2 \in U_0 \cap V \cap W_0$, and (10) follows from (13). Lemma 2 yields $U \cap V_m \cap W_n \neq \emptyset$ or $U \not\subseteq V \cup W$, which proves (4).

7. Some remarks on variants.

As mentioned above, our staircase conditions are modelled after those in [10]; however, they do not include them. The essential difference between (13) and condition (2.2) in [10] is that the latter is formulated for levels $\kappa = \lambda = \mu < \nu$. Following the arguments given above, a condition A for $\lambda_A = \mu_A$ can not be combined with the conditions B3, B4, B7, B8; also, B1 and B2 need to be specialized to $\mathcal{F}_A$. After all, one attains to a different theorem built around the staircase condition for $\kappa_A < \lambda_A = \mu_A < \nu_A$ which is weaker than A2. Similarly, another theorem including the weaker staircase condition for $\kappa_B > \lambda_B = \mu_B > \nu_B$ (or Tuy's condition modified for sets from $\mathcal{F}_B$) may be stated.

There are also variants of the theorem which include conditions on the level set $F_* := \{(x, y) \mid f(x, y) \leq \sup_X \inf_Y f\}$. They can be obtained by inserting a compactness argument into the induction step: if $F(A, a) \neq \emptyset$ for all $F \in \mathcal{F}_A$ and $a \in X$, and if all these sets are closed and compact (as they are by the assumptions of the theorem), then $F_*(A, a) \neq \emptyset$ for all $a \in X$. This enables to work with $F_8, \dots, F_{k+1}$ defined as above and $F_k = \dots = F_1 = F_*$ for some k. For example, we may replace lower semicontinuity in B2 by the weaker closedness of F_* (set $k = 2$), and we may combine condition B2, B3 (note that L_1, L_2 need not be replaced by finite sets there), or B6 with connectedness of all $F_*(X')$, $X' \in \mathcal{E}(X)$, or with the staircase condition for $\sup_X \inf_Y f = \kappa_A = \lambda_A < \mu_A < \nu_A$ (set $k = 4$).

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